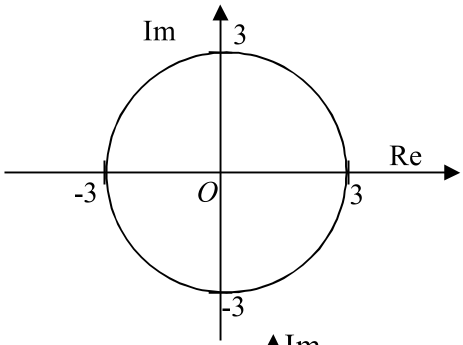
**National Junior College**
**2016 – 2017 H2 Further Mathematics**
**Topic F8: Further Complex Numbers (Tutorial Set 2) Solutions**

## Basic Mastery Questions

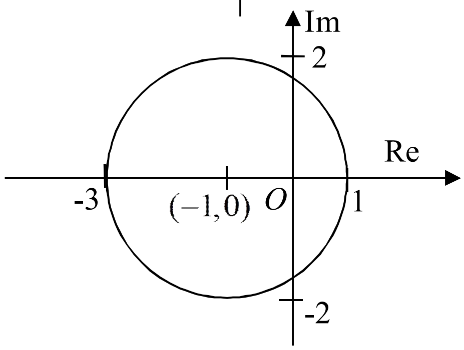
1 (a) $|z|=3 \quad \Rightarrow \quad |z-0|=3$

Locus of $z$ is a circle with centre at (0,0) and radius 3 units.




(b) $|z+1|=2 \quad \Rightarrow \quad |z-(-1)|=2$

Locus of $z$ is a circle with centre at $(-1,0)$ and radius 2 units




(c) $|4-2\mathrm{i}z|=6$

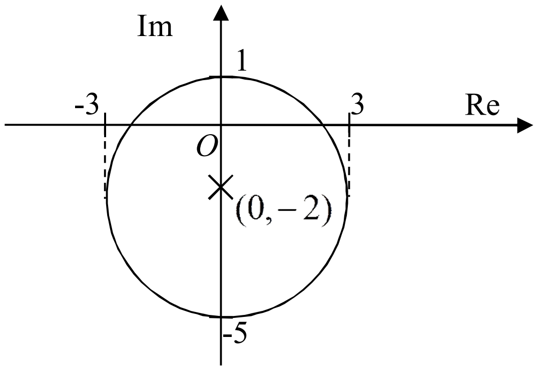
$\Rightarrow |(-2\mathrm{i})(z+2\mathrm{i})|=6$

$\Rightarrow |-2\mathrm{i}||z-(-2\mathrm{i})|=6$

$\Rightarrow 2|z-(-2\mathrm{i})|=6$

$\therefore |z-(-2\mathrm{i})|=3$

Locus of $z$ is a circle centered at $(0,-2)$ and radius 3 units.

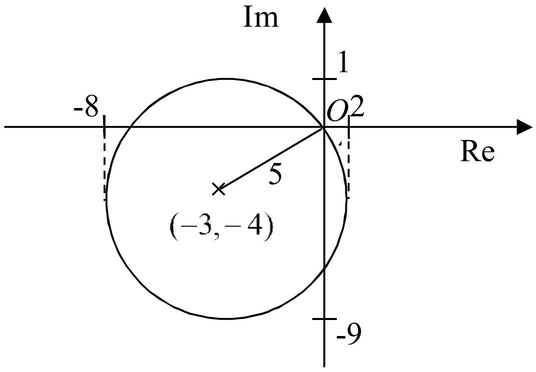



(d) $|z+3+4\mathrm{i}|=|3-4\mathrm{i}|$

$\Rightarrow |z-(-3-4\mathrm{i})|=\sqrt{3^2+(-4)^2}$

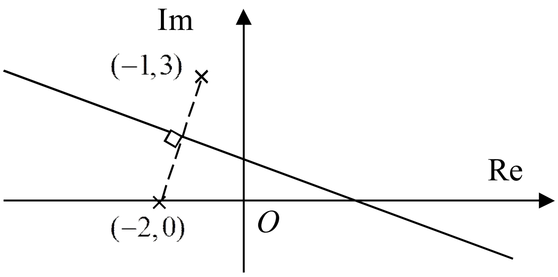
$\Rightarrow |z-(-3-4\mathrm{i})|=5$

Locus of $z$ is a circle centered at $(-3,-4)$ and radius 5 units.




(e) $|z-(-2)|=|z-(-1+3\mathrm{i})|$

Locus of $z$ is a perpendicular bisector of the line joining $(-2,0)$ and $(-1,3)$.





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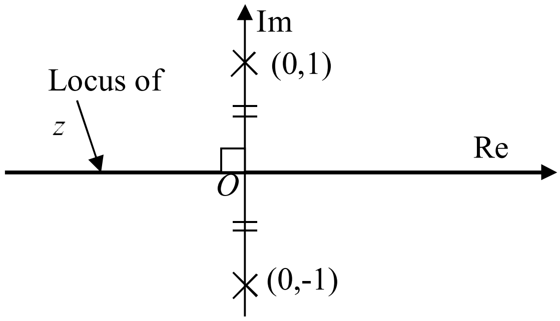
(f) $|z-\mathrm{i}|=|z+\mathrm{i}|$

$\Rightarrow |z-\mathrm{i}|=|z-(-\mathrm{i})|$

Locus of $z$ is a perpendicular bisector of line joining $(0,1)$ and $(0,-1)$.

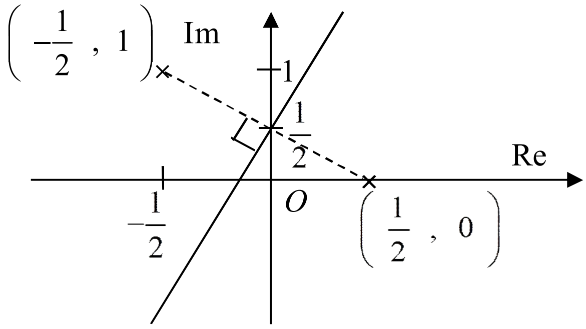



(g) $|2z+1-2\mathrm{i}|=|1-2z|$

$\Rightarrow \left|2\left(z+\frac{1}{2}-\mathrm{i}\right)\right|=\left|(-2)\left(z-\frac{1}{2}\right)\right|$

$\Rightarrow |2|\left|z-\left(-\frac{1}{2}+\mathrm{i}\right)\right|=|-2|\left|z-\frac{1}{2}\right|$

$\Rightarrow \left|z-\left(-\frac{1}{2}+\mathrm{i}\right)\right|=\left|z-\frac{1}{2}\right|$




Locus of $z$ is a perpendicular bisector of the line joining $\left(-\frac{1}{2},1\right)$ and $\left(\frac{1}{2},0\right)$.

(h) $\left|5(z+1)^2\right|=150$

$|5||z+1|^2=150$

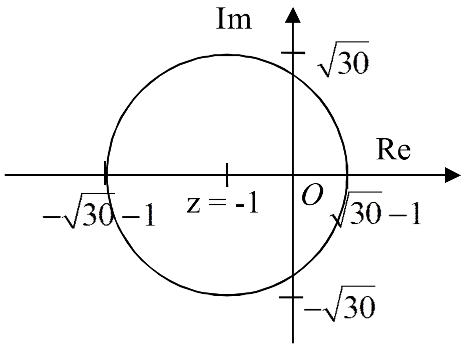
$|z+1|^2=30$

$|z+1|=\sqrt{30}$ or $|z+1|=-\sqrt{30}$

(Rejected $\because$ radius $>0$)

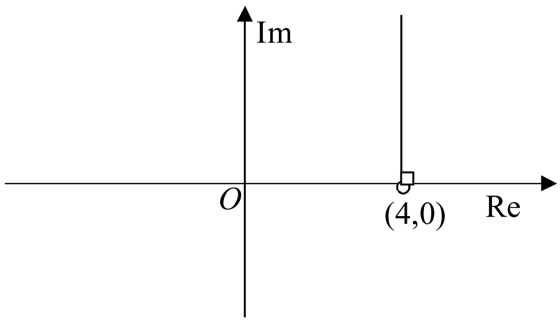
$\therefore |z-(-1)|=\sqrt{30}$

Locus of $z$ is a circle centered at $(-1,0)$ with radius $\sqrt{30}$ units.




(i) $\arg(z-4)=\frac{\pi}{2}$

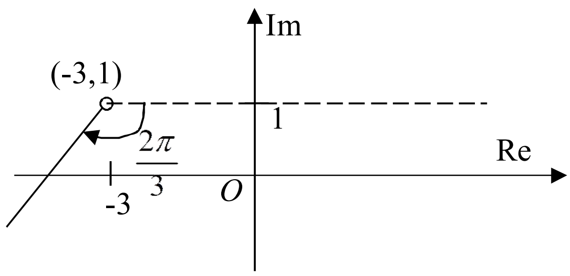
Locus of $z$ is a half-line at (4,0) that makes an angle of $\frac{\pi}{2}$ with the horizontal in the positive real axis direction.




(j) $\arg\sqrt{z+3-\mathrm{i}}=-\frac{\pi}{3}$

$\Rightarrow \frac{1}{2}\arg\left(z-(-3+\mathrm{i})\right)=-\frac{\pi}{3}$

$\Rightarrow \arg\left(z-(-3+\mathrm{i})\right)=-\frac{2\pi}{3}$




Locus of $z$ is a half-line at $(-3,1)$ that makes an angle of $-\frac{2\pi}{3}$ with the horizontal in the positive real axis direction.



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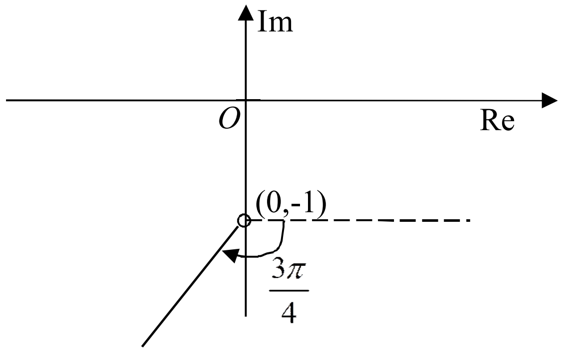
(k) $\arg(1-\mathrm{i}z)=\dfrac{3\pi}{4}$




$\arg(-\mathrm{i})(z+\mathrm{i})=\dfrac{3\pi}{4}$

$\arg(-\mathrm{i})+\arg\left(z-(-\mathrm{i})\right)=\dfrac{3\pi}{4}$

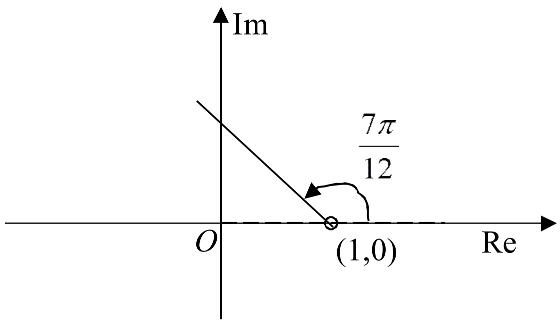
$-\dfrac{\pi}{2}+\arg\left(z-(-\mathrm{i})\right)=\dfrac{3\pi}{4}$

$\therefore \arg\left(z-(-\mathrm{i})\right)=\dfrac{5\pi}{4}\equiv-\dfrac{3\pi}{4}$

Locus of $z$ is a half-line at (0,1) that makes an angle of $-\dfrac{3\pi}{4}$ with the horizontal in the positive real axis direction.

(l) $\arg\left(\dfrac{1-z}{1-\mathrm{i}}\right)=-\dfrac{\pi}{6}\Rightarrow \arg(1-z)-\arg(1-\mathrm{i})=-\dfrac{\pi}{6}$

$\Rightarrow \arg(1-z)-\left(-\dfrac{\pi}{4}\right)=-\dfrac{\pi}{6}$




$\Rightarrow \arg(-1)(z-1)=-\dfrac{5\pi}{12}$

$\therefore \arg(-1)+\arg(z-1)=-\dfrac{5\pi}{12}$

$\arg(z-1)=-\dfrac{17\pi}{12}\equiv\dfrac{7\pi}{12}$

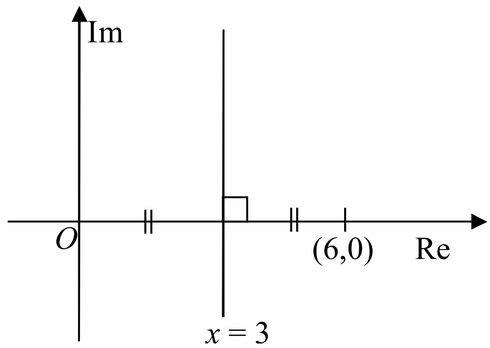
Locus of $z$ is a half line at (1,0) that makes an angle of $\dfrac{7\pi}{12}$ with the horizontal in the positive real axis direction.

(m) $\sqrt{zz^*}=|z-6|$

$\Rightarrow \sqrt{|z|^2}=|z-6|$

$\therefore |z|=|z-6|$

Locus of $z$ is a perpendicular bisector of the line joining $(0,0)$ and $(6,0)$.





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(n) $\left|\dfrac{2z-\text{i}}{z-2\text{i}}\right|=1$

$\Rightarrow |2z-\text{i}|=|z-2\text{i}|$

Let $z=x+\text{i}y$

$|2x+\text{i}2y-\text{i}|=|x+\text{i}y-2\text{i}|$

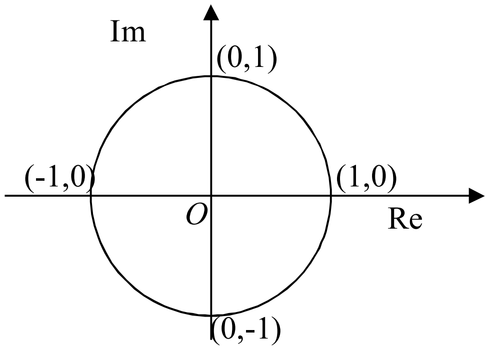
$\Rightarrow |2x+\text{i}(2y-1)|=|x+\text{i}(y-2)|$

$\Rightarrow \sqrt{(2x)^2+(2y-1)^2}=\sqrt{x^2+(y-2)^2}$

$\Rightarrow 4x^2+4y^2-4y+1=x^2+y^2-4y+4$

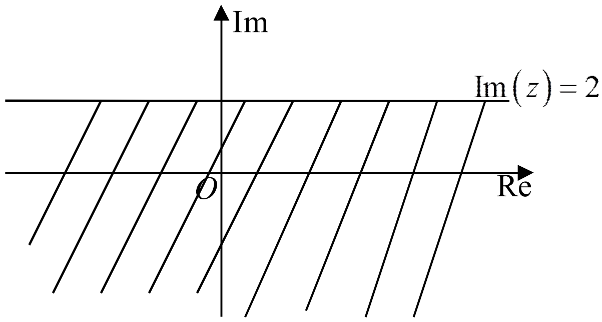
$\therefore 3x^2+3y^2=3$

$\therefore x^2+y^2=1$

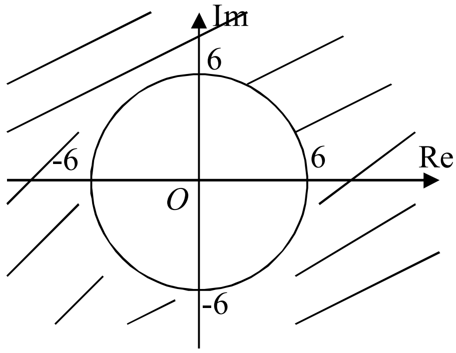



Locus of $z$ is a circle with centre at $(0,0)$ and radius 1 unit.

2 (a) $\text{Im}(z)\le 2$




(b) $|z^2|\ge 36 \Rightarrow |z|^2\ge 36 \Rightarrow |z|\ge 6$

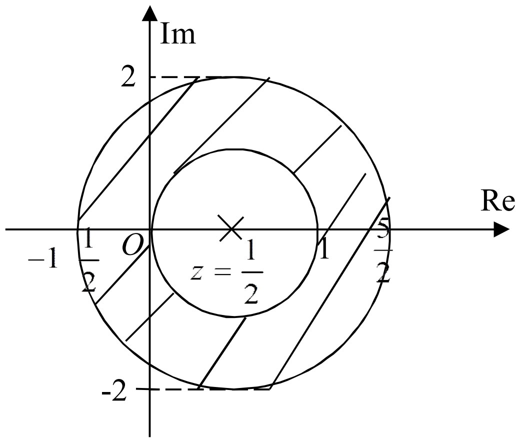



(c) $1\le |2z-1|\le |\sqrt{15}\text{i}+1|$

$\Rightarrow 1\le \left|(2)\left(z-\frac{1}{2}\right)\right|\le \sqrt{\left(\sqrt{15}\right)^2+1^2}$

$\Rightarrow 1\le |2|\left|z-\frac{1}{2}\right|\le 4$

$\Rightarrow \frac{1}{2}\le \left|z-\frac{1}{2}\right|\le 2$





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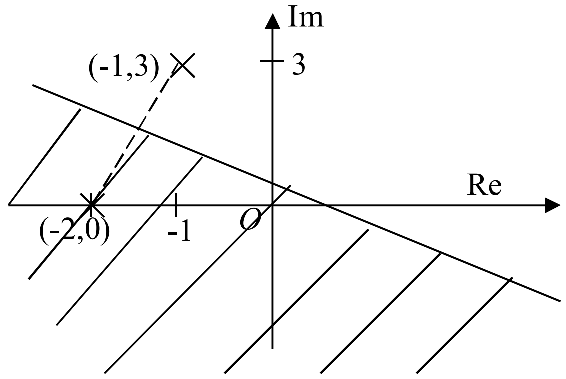
(d) $|\, z+2 \,| \le |\, z+1-3\mathrm{i} \,|$

$\Rightarrow |z-(-2)| \le |z-(-1+3\mathrm{i})|$

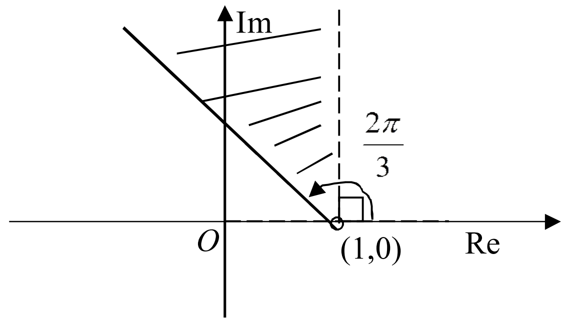



(e) $\frac{\pi}{2} < \arg(2z-2) \le \frac{2\pi}{3}$

$\Rightarrow \frac{\pi}{2} < \arg(2)(z-1) \le \frac{2\pi}{3}$

$\Rightarrow \frac{\pi}{2} < \arg(2)+\arg(z-1) \le \frac{2\pi}{3}$

$\therefore \frac{\pi}{2} < \arg(z-1) \le \frac{2\pi}{3}$




(f) $\arg(z-1-\mathrm{i})^2 \le -\frac{\pi}{3}$

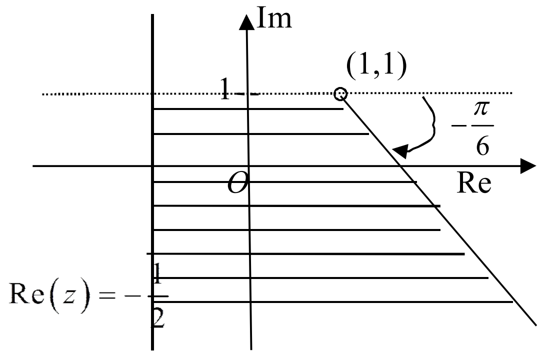
$\Rightarrow 2\arg(z-1-\mathrm{i}) \le -\frac{\pi}{3}$

$\therefore \arg(z-(1+\mathrm{i})) \le -\frac{\pi}{6}$

$z+z^* \ge -1$

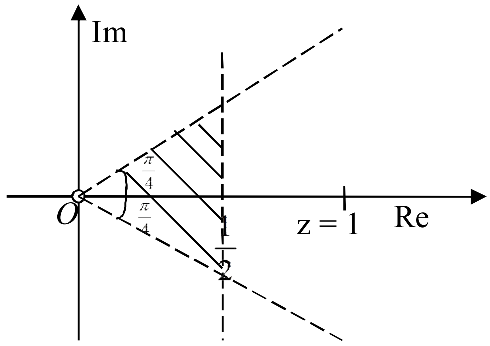
$\Rightarrow 2\mathrm{Re}(z) \ge -1$

$\therefore \mathrm{Re}(z) \ge -\frac{1}{2}$




(g) $|\, z \,| < |\, z-1 \,|$

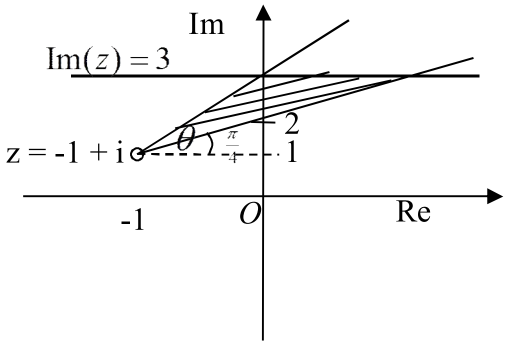
$-\frac{\pi}{4} < \arg(z) < \frac{\pi}{4}$




(h) $\mathrm{Im}(z) \le 3$

$\frac{\pi}{4} \le \arg(z-(-1+\mathrm{i})) \le \tan^{-1}2$

where $\theta = \tan^{-1}2 \Rightarrow \tan\theta = 2$





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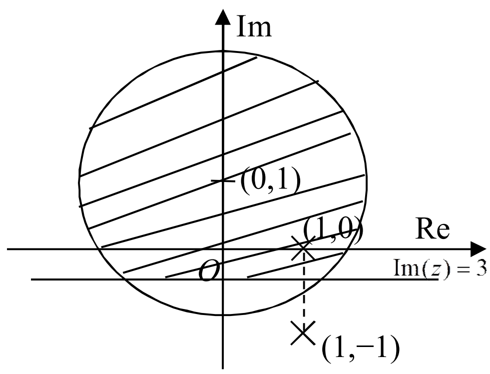
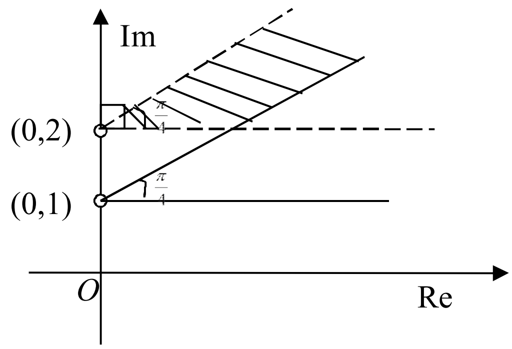
(i) $|z-\mathrm{i}| \le 2$

$|z-1| \le |z-(1-\mathrm{i})|$

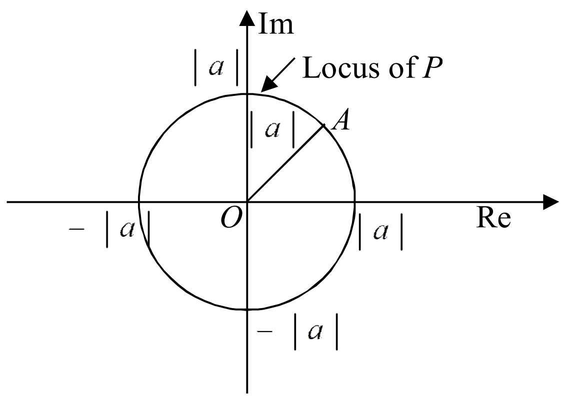



(j) $\dfrac{\pi}{4} \le \arg(z-\mathrm{i}) < \dfrac{\pi}{2}$

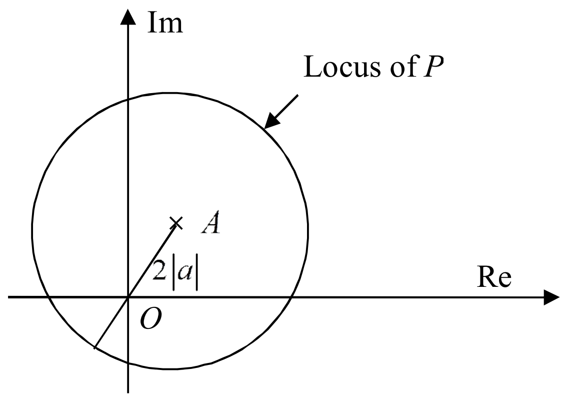
$0 < \arg\,(z-2\mathrm{i}) < \dfrac{\pi}{4}$



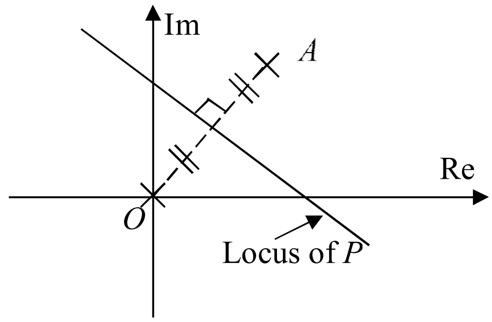

3 (i) $|z| = |a|$



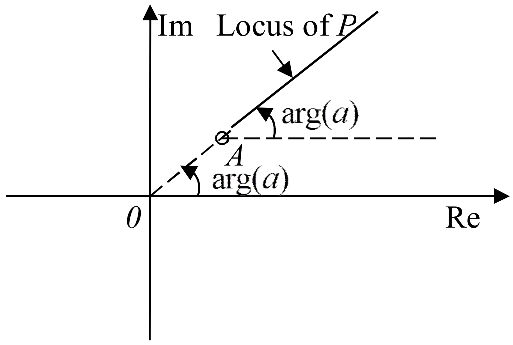

(ii) $|z-a| = 2|a|$



(iii) $|z-a|=|z|$




(iv) $\arg(z-a)=\arg(a)$




**Practice Questions**

1 (i) $\arg(z_1)=\dfrac{\pi}{3}, |z_1|=2$ while $\arg(z_2)=-\dfrac{3\pi}{4}, |z_2|=\sqrt{2}$. Hence

$$z_1=2\left(\cos\frac{\pi}{3}+\mathrm{i}\sin\frac{\pi}{3}\right),$$

$$z_2=\sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right)+\mathrm{i}\sin\left(-\frac{3\pi}{4}\right)\right)$$

(ii)

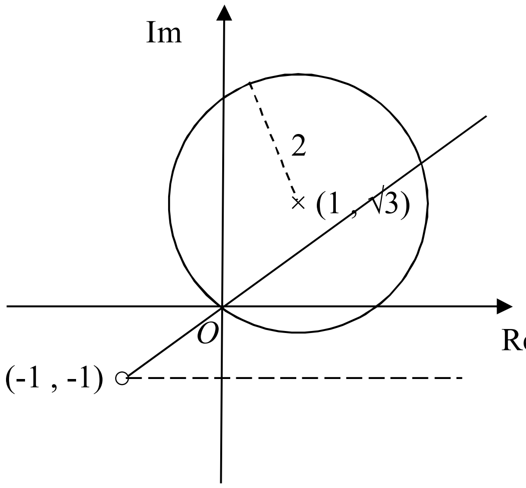
$$\left(\frac{z_1}{z_2}\right)^*=\left[\frac{2e^{\mathrm{i}\frac{\pi}{3}}}{\sqrt{2}e^{\mathrm{i}\left(-\frac{3\pi}{4}\right)}}\right]^*$$

$$=\left(\sqrt{2}e^{\mathrm{i}\frac{13\pi}{12}}\right)^*=\left(\sqrt{2}e^{\mathrm{i}\left(\frac{-13\pi}{12}\right)}\right)$$

$$=\sqrt{2}e^{\mathrm{i}\left(\frac{11\pi}{12}\right)}$$

$$=\sqrt{2}\left(\cos\left(\frac{11\pi}{12}\right)+i\sin\left(\frac{11\pi}{12}\right)\right)$$

(iii)




(iv) Since the circle $|z-z_1|=2$ is symmetrical about the line $\mathrm{Re}(z)=1$ and passes through $(0,0)$, it will also passes through $(2,0)$ where it meets the positive real axis.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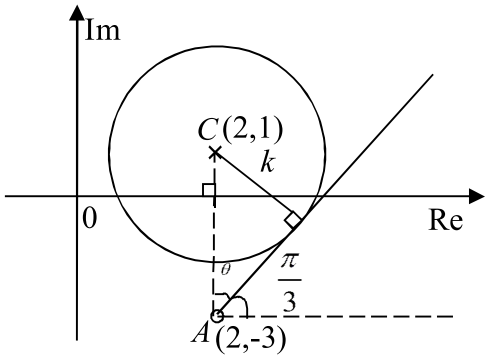
2 $\arg\left(z-\left(2-3\mathrm{i}\right)\right)=\dfrac{\pi}{3}$

$\left|z-\left(2+\mathrm{i}\right)\right|=k$ is a circle centered at (2, 1) with radius k units.




(i) $\theta=\dfrac{\pi}{2}-\dfrac{\pi}{3}=\dfrac{\pi}{6}$

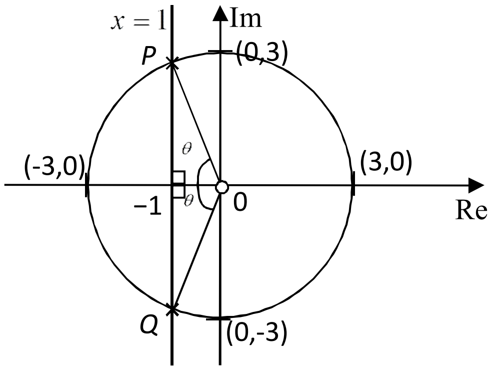
$\sin\theta=\dfrac{k}{4}$

$k=4\sin\dfrac{\pi}{6}=4\left(\dfrac{1}{2}\right)$

$k=2\ (shown)$

(ii) From the Argand diagram, observed that $2<k<4$ such that the two loci intersect at two points.

3 (i)




Let $z=x+\mathrm{i}y$

$|z|=|z+2|$

$\Rightarrow|x+\mathrm{i}y|=|x+2+\mathrm{i}y|$

$\Rightarrow\sqrt{x^2+y^2}=\sqrt{(x+2)^2+y^2}$

$\Rightarrow x^2+y^2=x^2+4x+4+y^2$

$\Rightarrow 4x+4=0$

$\Rightarrow x=-1\ (shown)$

(ii)

$\cos\theta=\dfrac{1}{3}$

$\theta=1.23096$

Let $P$ and $Q$ represent complex numbers $z_1$ and $z_2$ respectively.

$\arg(z_1)=\pi-\theta=1.91$

$\arg(z_2)=-\left(\pi-\theta\right)=-1.91$

(iii)

$|\ z-a\ |=b$ is a circle centered at a with radius b units, passing through $P$ and $Q$.

$PQ$ is the diameter of the circle such that b is least.

Least $b=\sqrt{3^2-1^2}=\sqrt{8}=2\sqrt{2}$

$\therefore a=-1$ (Note: $a$ is real.)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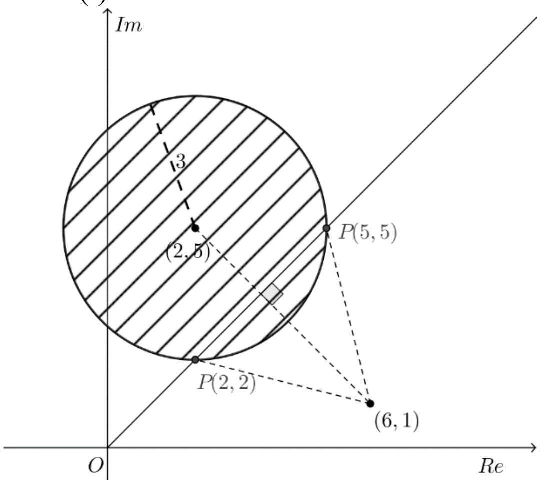
4 (i)




(ii) Maximum value of $|z| = \sqrt{2^2+5^2}+3 = \sqrt{29}+3$

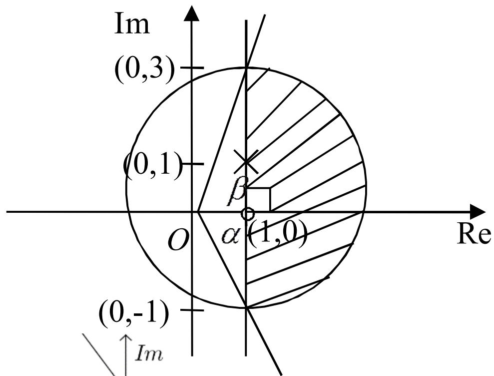
Minimum value of $|z| = \sqrt{2^2+5^2}-3 = \sqrt{29}-3$

(iii) The line joining the centre of the circle to the point 6+i makes right angle with the line $\theta = \dfrac{\pi}{4}$. Hence, the two points marked *P* correspond to the maximum distance.

By the intersection of the line $y = x$ and circle $(x-2)^2+(y-5)^2 = 3^2$, we find that the two points are (2, 2) and (5, 5). The max distance is $\sqrt{(6-2)^2+(1-2)^2} = \sqrt{17}$.

5 $\left|z-(1+\mathrm{i})\right| \le 2$

$-\dfrac{\pi}{2} \le \arg(z-1) \le \dfrac{\pi}{2}$




Least $\tan(\arg z) = \tan\alpha = -1$

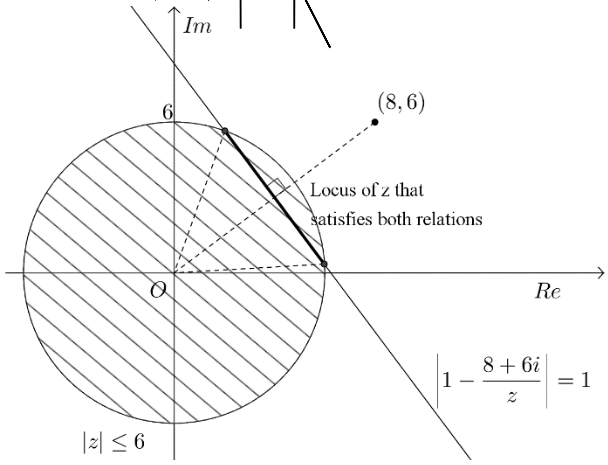
Greatest $\tan(\arg z) = \tan\beta = \dfrac{3}{1} = 3$

6 (a) $\left|1-\dfrac{8+6\mathrm{i}}{z}\right| = 1$

$\left|\dfrac{z-(8+6\mathrm{i})}{z}\right| = 1$

$\dfrac{\left|z-(8+6\mathrm{i})\right|}{|z|} = 1$

$\left|z-(8+6\mathrm{i})\right| = |z-0|$




(b) Consider the right angled triangle *OAB*

$\angle AOB = \cos^{-1}\left(\dfrac{5}{6}\right)$

Least arg z $= \tan^{-1}\left(\dfrac{6}{8}\right) - \cos^{-1}\left(\dfrac{5}{6}\right) = 0.058$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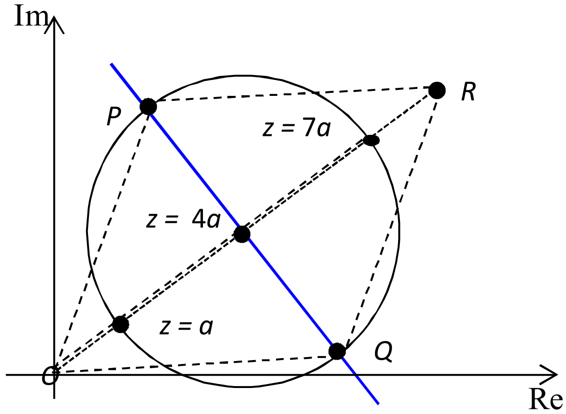
Greatest arg z $= \tan^{-1}\left(\frac{6}{8}\right)+\cos^{-1}\left(\frac{5}{6}\right)=1.229$

7 If $\arg(p)>\arg(q)$ ,

$$\arg\left(\frac{p}{q}\right)=\arg(p)-\arg(q)$$
$$=2\tan^{-1}\frac{|3a|}{|4a|}$$
$$=2\tan^{-1}\frac{3}{4}=1.29$$



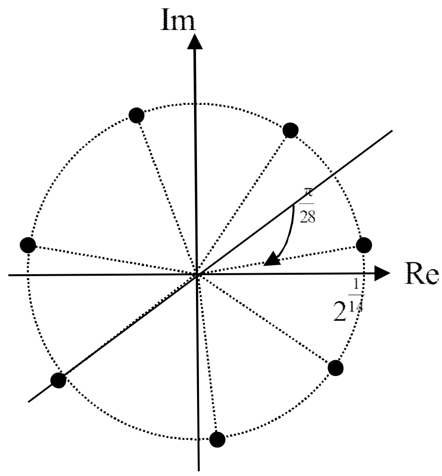

If $\arg(p)<\arg(q)$ ,

$$\arg\left(\frac{p}{q}\right)=\arg(p)-\arg(q)$$
$$=-2\tan^{-1}\frac{|3a|}{|4a|}=-2\tan^{-1}\frac{3}{4}=-1.29$$

$$|p+q|=\left|\overrightarrow{OP}+\overrightarrow{OQ}\right|=\left|\overrightarrow{OR}\right|$$
$$=2\times(\text{distance between } O \text{ and centre of the circle})$$
$$=2|4a|=8|a|$$

8 (i) $z^7=(1+\mathrm{i})=\sqrt{2}\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}=\sqrt{2}\mathrm{e}^{\mathrm{i}\pi\frac{8k+1}{4}}$

$z=2^{\frac{1}{14}}\mathrm{e}^{\mathrm{i}\pi\frac{8k+1}{28}},k=0,\pm1,\pm2,\pm3$




(iii) Explanation 1. $|z-z_1|=|z-z_2|$ defines a set of points equidistant from $z_1$ and $z_2$. Since both $z_1$ and $z_2$ are equidistant from the origin as they lie on the circumference of a circle centred at the origin, the origin must be a point on the locus.

Alternatively, we can show that the point (0, 0) satisfies the equation given, arguing that $|z_1|=|z_2|$ since they are the radii of a circle.

OR

Explanation 2. The points $z_1$, $z_2$ and the origin defines an isosceles triangle with $OZ_1 = OZ_2$. A perpendicular bisector of the line joining $z_1$ and $z_2$ (base of the triangle) will pass through the vertex (the origin) of the triangle.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OR

Explanation 3. Line joining $z_1$ and $z_2$ is a chord of a circle centred at the origin. A perpendicular bisector of the chord will pass through the centre of the circle (which is the origin). [This is related to a property of circles students learnt in sec 3/4.]

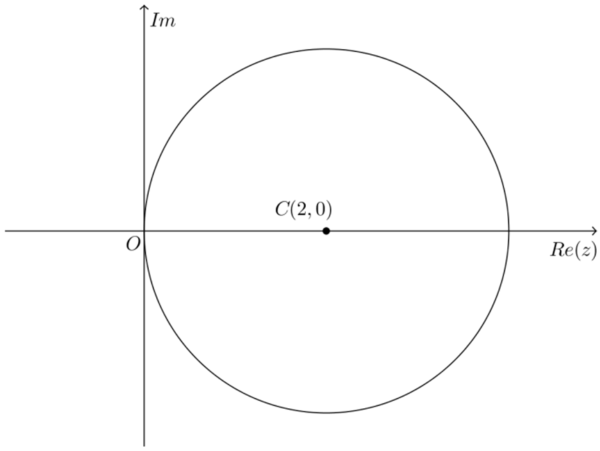
Angle with which bisector makes with positive real axis

$$=\frac{\pi}{28}+\frac{2\pi}{7}\times\frac{1}{2}=\frac{5\pi}{28}$$

Hence, the locus is $y = x\tan\frac{5\pi}{28}$

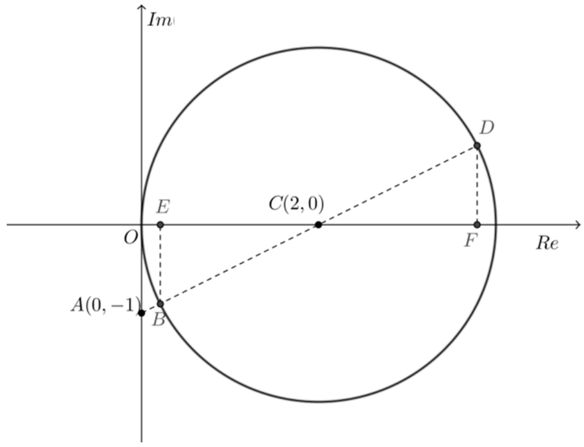
9

$$\begin{aligned}
z &= 2\left(1+\cos\theta+\mathrm{i}\sin\theta\right)\\
z-2 &= 2\cos\theta+\mathrm{i}2\sin\theta\\
|z-2| &= |2\cos\theta+\mathrm{i}2\sin\theta|\\
&= \sqrt{4\cos^2\theta+4\sin^2\theta}\\
&= \sqrt{4\left(\sin^2\theta+\cos^2\theta\right)}\\
&= \sqrt{4(1)}=2\\
|z-2| &= 2
\end{aligned}$$




Locus of $z$ is a circle centred at (2,0) with radius 2 units.

$$\begin{aligned}
\max|z-(-\mathrm{i})| &= AD\\
&= AC+2\\
&= \sqrt{2^2+(0-(-1)^2)}+2\\
&= \sqrt{5}+2
\end{aligned}$$




Since triangle $CDF$ and triangle $CAO$ are similar, $\frac{DF}{AO}=\frac{FC}{OC}=\frac{CD}{CA}$

$$\frac{DF}{1}=\frac{FC}{2}=\frac{2}{\sqrt{5}}\Rightarrow DF=\frac{2\sqrt{5}}{5} \text{ and } FC=\frac{4\sqrt{5}}{5}$$

The coordinates of $D$ are $\left(2+\frac{4\sqrt{5}}{5},\frac{2\sqrt{5}}{5}\right)$, the corresponding value of $z$ is

$\left(2+\frac{4\sqrt{5}}{5}\right)+\left(\frac{2\sqrt{5}}{5}\right)\mathrm{i}$.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$$\min|z-(-\mathrm{i})| = AB$$
$$= AC-2$$
$$= \sqrt{5}-2$$

Since triangle $CFD$ and triangle $CEB$ are congruent, $BE = DF = \dfrac{2\sqrt{5}}{5}$ and $EC = FC = \dfrac{4\sqrt{5}}{5}$.

The coordinates of $D$ are $\left(2-\dfrac{4\sqrt{5}}{5}, -\dfrac{2\sqrt{5}}{5}\right)$, the corresponding value of $z$ is

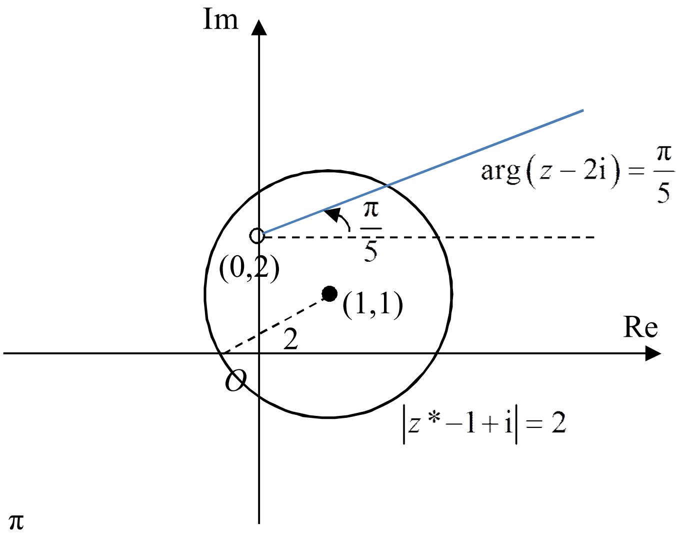
$\left(2-\dfrac{4\sqrt{5}}{5}\right)+\left(-\dfrac{2\sqrt{5}}{5}\right)\mathrm{i}$.

10 Let $z = x+\mathrm{i}y$.

From $|z^*-1+\mathrm{i}| = 2$

$\Rightarrow |x-\mathrm{i}y-1+\mathrm{i}| = 2 \Rightarrow |(x-1)-\mathrm{i}(y-1)| = 2$

$\Rightarrow (x-1)^2+(y-1)^2 = 4$ --------- (1)




$\arg(z-2\mathrm{i}) = \dfrac{\pi}{5}$

$\Rightarrow \tan\dfrac{\pi}{5} = \dfrac{y-2}{x} \Rightarrow y = x\tan\dfrac{\pi}{5}+2$ where $y>2$, $x>0$ --------- (2)

Sub (2) into (1): $(x-1)^2+\left(x\tan\dfrac{\pi}{5}+1\right)^2 = 4 \Rightarrow \left(1+\tan^2\dfrac{\pi}{5}\right)x^2+\left(2\tan\dfrac{\pi}{5}-2\right)x-2 = 0$

Using GC, $x \approx 2.7145$ or $x \approx -0.737$ (N.A. $\because x>0$).

Thus, $y \approx (2.7145)\tan\dfrac{\pi}{5}+2 = 3.97$ (3s.f.) $\Rightarrow z = 2.71+3.97\mathrm{i}$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11 $z^5 + 32 = 0$

$z^5 = -32 = 32e^{(\pi + 2k\pi)i}$

$z = 2e^{(\frac{\pi}{5} + \frac{2k\pi}{5})i}, \quad k = 0, \pm 1, \pm 2$

$= 2e^{-\frac{3\pi}{5}i}, 2e^{-\frac{\pi}{5}i}, 2e^{\frac{\pi}{5}i}, 2e^{\frac{3\pi}{5}i}, 2e^{\pi i}$

(i) Method 1

$$\left(\frac{z_1}{z_2 *}\right)^n = \frac{2^n e^{\left(\frac{n\pi}{5}\right)i}}{2^n e^{\left(-\frac{3n\pi}{5}\right)i}} = e^{\left(\frac{4n\pi}{5}\right)i}$$

For $\left(\frac{z_1}{z_2 *}\right)^n$ to be real and positive, smallest $n = 5$

Method 2

$$\arg\left(\frac{z_1}{z_2 *}\right)^n = n\left[\arg(z_1) - \arg(z_2 *)\right]$$

$$= n\left[\arg(z_1) - \left(-\arg(z_2)\right)\right]$$

$$= n\left[\frac{\pi}{5} + \frac{3\pi}{5}\right]$$

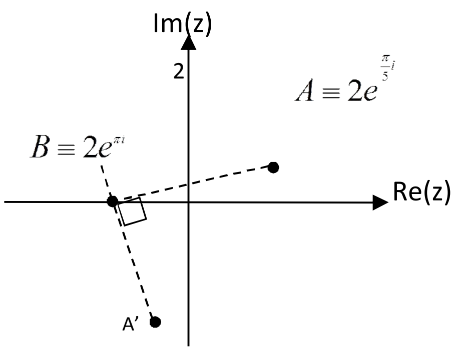
$$= \frac{4n\pi}{5}$$

For $\left(\frac{z_1}{z_2 *}\right)^n$ to be real and positive,

$\frac{4n\pi}{5} = 2k\pi, \quad k \in \mathbb{Z}$

$n = \frac{5}{2}k, \quad k \in \mathbb{Z}$, so smallest $n = 5$

(ii)





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Let the complex number represented by A' be $x + \mathrm{i}y$

$BA$ rotates 90° clockwise about $B$ to get $BA'$:

$$(x+\mathrm{i}y)-2\mathrm{e}^{\mathrm{i}\pi}=(-i)\left(2\mathrm{e}^{\mathrm{i}\frac{\pi}{5}}-2\mathrm{e}^{\mathrm{i}\pi}\right)$$

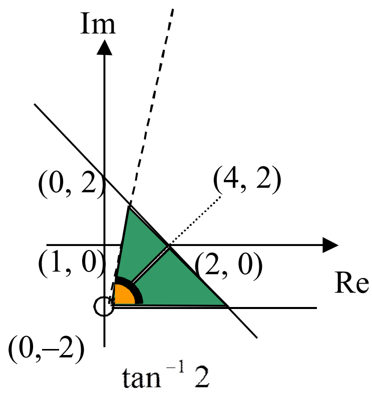
$$x+\mathrm{i}y=(-2)-\mathrm{i}\left[2\mathrm{e}^{\frac{1}{5}\pi\mathrm{i}}-(-2)\right] \quad \text{since } \mathrm{e}^{\mathrm{i}\pi}=-1$$

$$x+\mathrm{i}y=-2-\mathrm{i}\left(2\cos\frac{\pi}{5}+2\mathrm{i}\sin\frac{\pi}{5}+2\right)$$

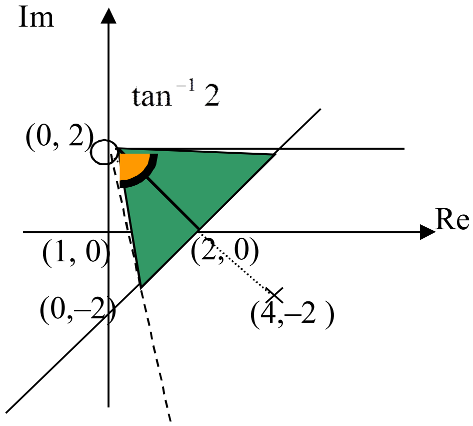
Real part = $-2-2\mathrm{i}^2\sin\frac{\pi}{5}=-2+2\sin\frac{\pi}{5}$

12 (i) The greatest value of $|z-4\mathrm{i}|$ satisfying the aforementioned conditions is the largest distance from the point (0,4) to the shaded region, which is the distance between (0,4) and (4, –2),

i.e $\sqrt{(0-4)^2+(4-(-2))^2}=\sqrt{52}=2\sqrt{13}$ units.




(ii)





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