JURONG PIONEER JUNIOR COLLEGE

JC2 Preliminary Examination 2019

MATHEMATICS **9758/01**
Higher 2 **18 September 2019**

Paper 1 **3 hours**

1 **(i)** The first four terms of a sequence are given by $u_1 = -13$, $u_2 = -12.8$, $u_3 = 1.8$ and $u_4 = 38$. Given that u_m is a cubic polynomial in *m*, find u_m in terms of *m*. [3]

(ii) Find the range of values of *m* for which u_m is greater than 2000. [2]

2 **(a)** Express $y = \dfrac{3x-1}{x-2}$ in the form $y = A + \dfrac{B}{x-2}$, where A and B are constants to be found. Hence, state a sequence of transformations that transforms the graph of $y = \dfrac{1}{x}$ to the graph of $y = \dfrac{3x-1}{x-2}$. [4]

(b) It is given that $\text{g}(x) = x^2 - 2x + 2$. Sketch the graph of $y = \text{g}(|x|)$, stating clearly the coordinates of any turning points and axial intercepts. Find numerically, the volume of revolution when the region bounded by the curve $y = \text{g}(|x|)$ and the line $y = 5$ is rotated completely about the *x*-axis. [5]

3 Referred to an origin *O*, the points *A* and *B* are such that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. The point *C* is such that *OACB* is a parallelogram. The point *D* is on *BC* such that $\overrightarrow{BD} = \lambda\overrightarrow{BC}$ and the point *E* is on *AC* such that $\overrightarrow{AE} = \mu\overrightarrow{AC}$, where λ and μ are positive constants. The area of triangle *ODE* is *k* times the area of triangle *OCE*.

(i) By finding the area of triangle *ODE* and *OCE* in terms of **a** and **b**, find *k* in terms of μ and λ. [6]

(ii) The point *F* is on *OC* and *ED* such that $OF : FC = 6:1$ and $DF : FE = 3:4$. By finding the values of λ and μ, calculate the value of *k*. [3]

4 The curve C has equation $y = \dfrac{x^2+5}{x-2}$.

(i) Prove, using an algebraic method, that *C* cannot lie between two values to be determined. [4]

(ii) Sketch *C*, showing clearly the equations of any asymptotes and coordinates of any turning points and axial intercepts. [4]

(iii) By adding a suitable graph to your sketch in **(ii)**, deduce the range of values of *h* for which the equation

$$\left(x^2+5\right)^2 + (x+1)^2(x-2)^2 = h^2(x-2)^2$$

has at least one positive real root. [3]

[Turn over

5 **Do not use a graphing calculator in answering this question.**

(a) **(i)** It is given that $w_1 = -3 + \sqrt{5}\,\mathrm{i}$. Find the value of w_1^3, showing clearly how you obtain your answer. [2]

(ii) Given that $-3 + \sqrt{5}\,\mathrm{i}$ is a root of the equation

$$4w^3 + pw^2 + qw - 14 = 0,$$

using your result in **(i)** , find the values of the real numbers p and q. [3]

(iii) For these values of p and q, find the other two roots of the equation in part **(ii)**. [3]

(b) It is given that $z = -1 - \sqrt{3}\ \mathrm{i}$.

Find the set of values of n for which $\dfrac{z^*}{z^n}$ is purely imaginary. [4]

6 The function f is defined for all real x by

$$\mathrm{f}(x) = \mathrm{e}^{2x} - 9\mathrm{e}^{-2x}.$$

(i) Show that $\mathrm{f}'(x) > 0$ for all x. [2]

(ii) Show that the set of values of x for which the graph $y = \mathrm{f}(x)$ is concave upward is the same as the set of values of x for which $\mathrm{f}(x) > 0$, and find this set of values of x, in the form of $k\ln 3$, where k is a constant to be found. [3]

(iii) Sketch the graph of $y = \mathrm{f}(x)$, showing clearly any points of intersections with the axes. [2]

(iv) Hence, find the exact value of $\int_0^2 \left|\mathrm{e}^{2x} - 9\mathrm{e}^{-2x}\right| \mathrm{d}x$. [4]

7 It is given that

$$\mathrm{f}(x) = \begin{cases} 4a^2 - x^2, & \text{for } 0 < x \le 2a, \\ 2a(x - 2a), & \text{for } 2a < x \le 4a, \end{cases}$$

and that $\mathrm{f}(x) = \mathrm{f}(x + 4a)$ for all real values of x, where a is a positive real constant.

(i) Evaluate $\mathrm{f}(2019a)$ in terms of a. [1]

(ii) Sketch the graph of $y = \mathrm{f}(x)$ for $-3a \le x \le 5a$. [3]

The function g is defined by

$$\mathrm{g} : x \mapsto \sqrt{4a^2 - (x - 2a)^2}, \qquad 2a < x < 4a.$$

(iii) Determine whether the composite function gf exists, justifying your answer. [1]

(iv) Give, in terms of a, a definition of fg. [2]

(v) Given that $(\mathrm{fg})^{-1}(27) = \dfrac{7}{2}a$, find the exact value of a. [2]

8

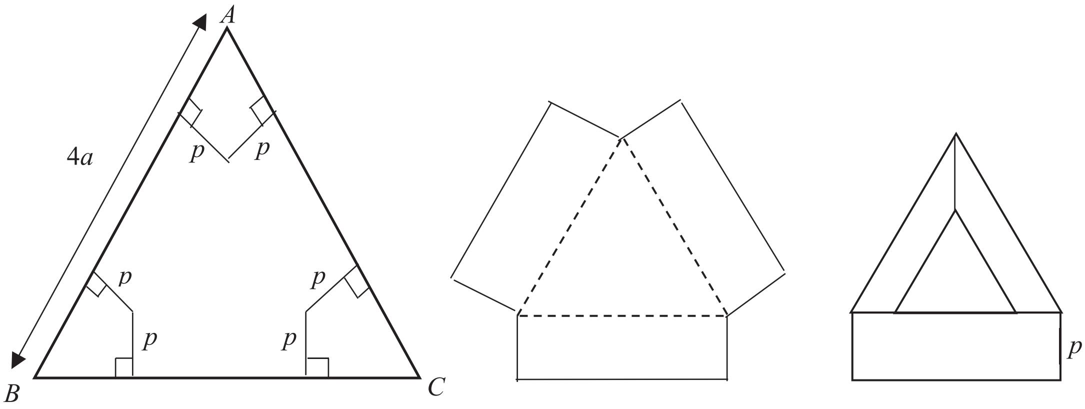


Fig.1 **Fig.2** **Fig.3**

Fig. 1 shows a metal sheet, ABC, in the form of an equilateral triangle of side $4a$ cm. A kite shape is cut from each corner, to give the shape as shown in Fig. 2. The remaining metal sheet shown in Fig. 2 is bent along the dotted lines, to form an open triangular prism of height p cm shown in Fig. 3.

(i) Show that the volume of the prism is given by $V = \sqrt{3}p\left(2a - \sqrt{3}p\right)^2 \text{ cm}^3$. [3]

(ii) Without using a calculator, find in terms of a, the exact value of p that gives a stationary value of V, and explain why there is only one answer. [6]

(iii) The prism is used by a housewife as a mould for making a dessert. To make the dessert, The housewife has to fill up $\frac{3}{4}$ of the mould with coconut milk. The cost of coconut milk is 0.4 cents per cm^3. What is the exact maximum cost in terms of a she needs to pay for the coconut milk? [3]

9 Find

(a) $\int \frac{e^{\frac{1}{x}}}{x^2} \mathrm{d}x$, [2]

(b) $\int \cos kx \cos\left(k+2\right)x \, \mathrm{d}x$, where k is a positive constant, [2]

(c) $\int x \tan^{-1}\left(3x\right) \mathrm{d}x$. [6]

10 **(a)** The Deep Space spacecraft launched in October 1998 used an ion engine to travel from Earth to the Comet Borrelly. The average speed of the spacecraft in October 1998 was 44 000 km/hr. The monthly average speed, v_n of the spacecraft in month n based on its first 5 months of operation was given by:

Month, n	1	2	3	4	5
Average speed, v_n	44 000	44335	44 670	45 005	45 340

Assume that v_n follows the same increment for the rest of its flight.

(i) State a general formula for v_n in terms of n. [1]

(ii) In which month and year did the average speed of the spacecraft first exceed 53 500 km/hr ? [3]

[Turn over

(iii) Assume that there are 30 days per month. It is known that the total distance travelled by the spacecraft from Earth is given by $\sum_{r=1}^{n}\left(v_r T\right)$ where T is the time taken, in hours, by the spacecraft to travel in one month. Given that the spacecraft travelled from Earth continuously for 3 years to reach Comet Borrelly, find the total distance that it travelled. [3]

(b) Dermontt's Law is an empirical formula for the orbital period of major satellites orbiting planets in the solar system. It is represented by the equation $T_n = T_0 C^n$, where T_n is the orbital period, in days, of the $(n+1)^{\text{th}}$ satellite and C is a constant associated with the satellite system in question. It is known that the planet Jupiter has 67 satellites. The orbital period of its first satellite is 0.44 days and $C = 2.03$.

(i) Find the longest orbital period of a satellite of Jupiter. [2]

(ii) Find the largest value of n for which the total orbital periods of the first n satellites of Jupiter is within 5×10^6 days of the orbital period of the 20^{th} satellite of Jupiter. [3]

JURONG PIONEER JUNIOR COLLEGE

JC2 Preliminary Examination 2019

MATHEMATICS 9758/02

Section A : Pure Mathematics [40 Marks]

1 A sequence a_0, a_1, a_2, … is given by $a_0 = \frac{3}{5}$ and $a_{n+1} = a_n + 3^n - n$ for $n \geq 0$. By considering $\sum_{r=0}^{n-1}\left(a_{r+1} - a_r\right)$, find a formula for a_n in terms of *n*. [5]

2 **In this question, you may use expansions from the List of Formula (MF26).**

(a) **(i)** Find the Maclaurin expansion of ln(cos 3*x*) in ascending powers of *x*, up to and including the term in x^6. [5]

(ii) Hence, state the Maclaurin expansion of tan 3*x*, up to and including the term in x^5. [2]

(b) Given that *x* is sufficiently small, find the series expansion of $\frac{e^{\tan x}}{\left(2+x\right)^2}$ in ascending powers of *x*, up to and including the term in x^2. [3]

3 A curve *C* has parametric equations $x = 4\sin 2t$, $y = 4\cos 2t$, where $0 \leq t \leq \frac{\pi}{4}$.

(i) Sketch *C*. [2]

(ii) State the exact value(s) of *t* at the point(s) where the tangent(s) to *C* are parallel to the
(a) *y*-axis,
(b) *x*-axis. [2]

(iii) *P* is a point on *C* where the normal at *P* is parallel to the line $y = \sqrt{3}x - 2$. Find the equation of the tangent at *P*, in the form of $y = \frac{a}{3}\left(b - x\right)$, where *a* and *b* are the constants to be determined. [3]

(iv) Find the exact area bounded by *C*, the tangent at *P* and the axes. [2]

4 A differential equation is given by $2u^2\frac{\mathrm{d}^2x}{\mathrm{d}u^2} + 4u\frac{\mathrm{d}x}{\mathrm{d}u} = 15u + 12$ where $x = 0$ and $\frac{\mathrm{d}x}{\mathrm{d}u} = 1$ when $u = 1$. By differentiating $u^2\frac{\mathrm{d}x}{\mathrm{d}u}$ with respect to u, show that the solution of the differential equation is given by $x = au + \frac{b}{u} + c\mathrm{f}(u) + d$, where *a*, *b*, *c* and *d* are constants to be determined and f(*u*) is a function in *u* to be found. [6]

5 The plane p has equation $\mathbf{r}=\begin{pmatrix}1\\0\\-3\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\0\end{pmatrix}+\mu\begin{pmatrix}2\\3\\-4\end{pmatrix}$, and the line l has equation $\mathbf{r}=\begin{pmatrix}-10\\0\\4\end{pmatrix}+t\begin{pmatrix}8\\-4\\1\end{pmatrix}$, where λ, μ and t are parameters.

(i) Show that l is perpendicular to p and find the values of λ, μ and t which give the coordinates of the point at which l and p intersect. [5]

(ii) Find the cartesian equations of the planes such that the perpendicular distance from each plane to p is 2. [5]

Section B : Statistics [60 Marks]

6 On average, 35% of the boxes of Brand A cereal contain a voucher. Brandon buys one box of cereal each week. The number of vouchers he obtains is denoted by X.

(i) State, in context, two conditions needed for X to be well modelled by a binomial distribution. [2]

Assume now that X has a binomial distribution. In order to claim a free gift, 8 vouchers are required. Find the probability that Brandon

(ii) obtains at most 3 vouchers in 9 weeks, [1]

(iii) will be able to claim a free gift only in the 10^{th} week. [2]

$100p$ % of the boxes of Brand B cereal contain a voucher. Brandon also buys a box of Brand B cereal each week for 10 weeks. Given that the probability that Brandon obtains at most 1 voucher in ten weeks is 0.4845, write down an equation in terms of p and hence, find the value of p. [2]

7 A company sells peanut butter in jars. Each jar is labelled as containing m grams of peanut butter on average. A consumer group suspects that the average mass of peanut butter in a jar is overstated. To test this suspicion, the consumer group checks a random sample of 50 jars and the mass of peanut butter per jar, x grams, are summarized by

$$\sum(x-390)=120 \qquad \text{and} \qquad \sum(x-390)^2=3100.$$

The consumer group uses the above data to carry out a test at the 2% level of significance. The result leads the consumer group to conclude that the company has overstated the average mass of peanut butter in a jar.

(i) Explain why the consumer group is able to carry out a hypothesis test without knowing anything about the distribution of the mass of the peanut butter of the jars. [1]

(ii) Find the least possible value of m, to the nearest gram that leads to the result of the hypothesis test as stated above. [7]

8 **(a)** Find the number of different 6-digit numbers that can be formed from the digits 0, 2, 4, 5, 6 and 8, if no digit is repeated and the numbers formed are divisible by 5. [2]

(b) Find how many 4-letter code words can be formed from the letters of the word ***DIFFERENT***. [4]

9 In this question you should state the parameters of any distributions that you use.

The masses of Grade *A* and Grade *B* strawberries are normally distributed with mean 18 grams and 12 grams respectively and standard deviation 3 grams and 2 grams respectively.

(i) Find the probability that a randomly chosen Grade *A* strawberry has mass between 17 and 20 grams. [1]

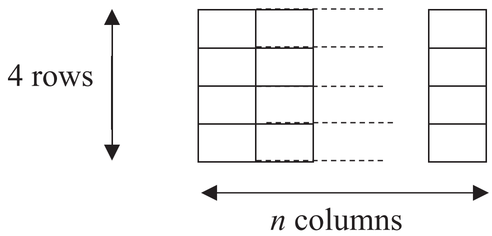
(ii) Any Grade *B* strawberry that weighs less than *m* grams will be downgraded to Grade *C*. Given that there is a probability of at least 0.955 that a Grade *B* strawberry will not be downgraded to Grade *C*, find the greatest value of *m*. [2]

Grade *A* strawberries are packed into bags of 12 while Grade *B* strawberries are packed into bags of 15.

(iii) Find the probability that a bag of Grade *A* strawberries weighs more than a bag of Grade *B* strawberries. [3]

(iv) State an assumption needed for your calculation in part **(iii)**. [1]

10 At a particular booth in a funfair, Kathryn is given some boxes which are arranged in the layout as shown below.



Each box contains a numbered card. One card is numbered '4', three cards are numbered '2', and the rest of the cards are numbered '1'. All the boxes are closed initially and Kathryn is required to open 2 different boxes. Her score is the sum of the numbers obtained.

If she scores more than 4, she wins \$10. If she scores less than 4, she loses \$2. If she scores 4, she does not win anything. The random variable *W* is her winnings after one game.

(i) Show that $\mathrm{P}(W=10)=\dfrac{1}{2n}$. [2]

(ii) Given that $\mathrm{P}(W=10)=\dfrac{1}{8}$, find the value of *n*. Hence, find the probability distribution of *W*. [4]

(iii) Find E(*W*) and Var(*W*). Explain whether Kathryn should play the game. [4]

50 other participants play the game. Find the probability that the mean winnings is at most \$1. [2]

11 **[Leave your answers in fraction]**

The events A and B are such that $P(A|B) = \frac{7}{10}$, $P(B|A) = \frac{4}{15}$ and $P(A \cup B) = \frac{3}{5}$.

Find the exact values of

(i) $P(A \cap B)$, [3]

(ii) $P(A' \cap B)$. [2]

For a third event C, it is given that $P(C) = \frac{3}{10}$ and that A and C are independent.

(iii) Find $P(A' \cap C)$. [2]

(iv) Hence state an inequality satisfied by $P(A' \cap B \cap C)$. [1]

12 As part of a medical research on diabetes, a team of researchers conducts a study to investigate the amount of glucose y, measured to the nearest 0.5 mg/dl, present in human bodies at different age x, measured in years. The results are given in the table.

x	20	28	36	44	52	60	68	76
y	86.0	90.5	94.5	97.5	100.0	105.0	103.5	104.0

(i) Calculate the product moment correlation coefficient between x and y and explain whether your answer suggests that a linear model is appropriate. [2]

(ii) Draw a scatter diagram for the data, labelling the axes clearly. [1

One of the values of y appears to be incorrect.

(iii) Circle the corresponding point on your diagram and label it P. [1]

For part **(iv)** and **(v)** of this question, you should omit P.

(iv) Explain from your scatter diagram why the relationship between x and y should not be modelled by an equation of the form $y = ax + b$. [1]

(v) Suppose that the relationship between x and y is modelled by an equation of the form $y = c + d \ln x$, where c and d are constants. Find the product moment correlation coefficient between y and $\ln x$ and the constants c and d. [2]

Assume that the value of x at P is correct.

(vi) Use the model $y = c + d \ln x$, with the values of c and d found in **(v)** to estimate the correct value of y at P, giving your answer to the nearest 0.5 mg/dl. Explain why you would expect this estimate to be reliable. [2]

(vii) If the correct value of y at P is used, the 8 data points may be fitted by the model $y = 43.942 + 14.079 \ln x$. Find the correct value of y at P, giving your answer to the nearest 0.5 mg/dl. [3]

Jurong Pioneer Junior College
H2 Mathematics Preliminary Exam P1 Solution

Q1

(i)

Let $u_m = am^3 + bm^2 + cm + d$

$u_1 = a + b + c + d = -13$

$u_2 = 8a + 4b + 2c + d = -12.8$

$u_3 = 27a + 9b + 3c + d = 1.8$

$u_4 = 64a + 16b + 4c + d = 38$

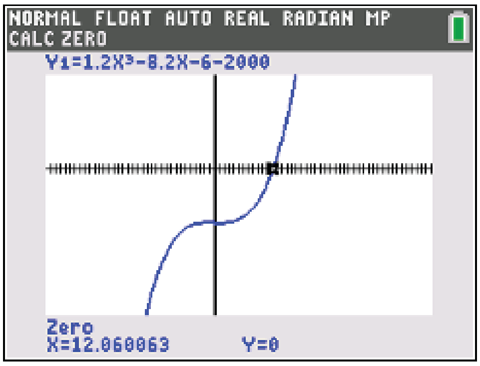
Using GC, $a = 1.2$, $b = 0$, $c = -8.2$, $d = -6$

$u_m = 1.2m^3 - 8.2m - 6$

(ii)

$u_m > 2000$

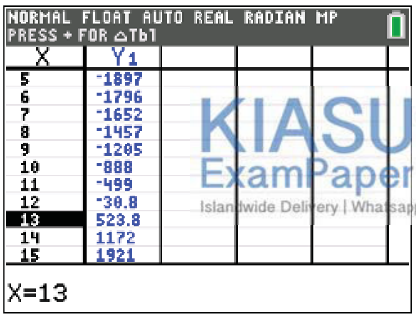
Let $y = 1.2m^3 - 8.2m - 6 - 2000$



From GC graphing, $m > 12.06$

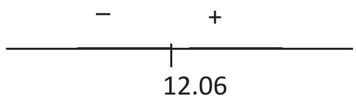
$m \geq 13$ where m is an integer

Or : From GC table,



$m \geq 13$ where m is an integer

Or : GC poly root finder



$m > 12.06$
$m \geq 13$ where m is an integer

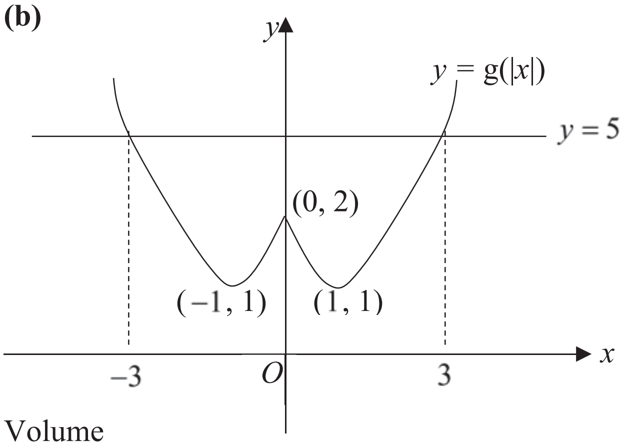
Q2

(a)

$$y = \frac{3x-1}{x-2} = 3 + \frac{5}{x-2}$$

- Translation of 2 units in the positive x direction.
- Scaling parallel to the y-axis by a scale factor of 5.
- Translation of 3 units in the positive y direction.

(b)



Volume

$$= 2\left[\pi(5)^2(3) - \pi\int_0^3 \left(x^2 - 2x + 2\right)^2 \, dx\right]$$

$$= 373 \text{(3sf)}$$

or

Volume

$$= \pi(5)^2(6) - \pi\int_{-3}^{3} \left(|x|^2 - 2|x| + 2\right)^2 \, dx$$

$$= 373 \text{(3sf)}$$

Note :

Cannot use

$$\pi(5)^2(6) - \pi\int_{-3}^{3} \left(x^2 - 2x + 2\right)^2 \, dx$$

Cannot use

$$\pi(5)^2(6) - \pi\int_{-3}^{3} \left(|x|^2 - 2|x| + 2\right)^2 \, dx$$ if question ask for exact.

Q3

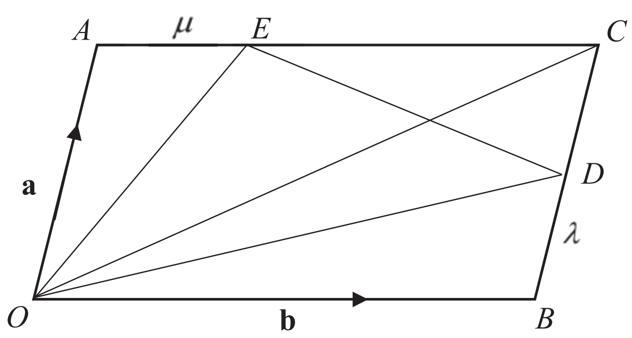
(i)

$\overrightarrow{OD} = \mathbf{b} + \lambda\mathbf{a}$

$\overrightarrow{OE} = \mathbf{a} + \mu\mathbf{b}$

area of triangle OCE

$$= \frac{1}{2}\left|\overrightarrow{OC} \times \overrightarrow{EC}\right|$$



$= \frac{1}{2}\left|(\mathbf{a}+\mathbf{b})\times(1-\mu)\mathbf{b}\right|$

$= \frac{1-\mu}{2}\left|\mathbf{a}\times\mathbf{b}\right|$ since $0<\mu<1$

area of triangle *ODE*

$= \frac{1}{2}\left|\overrightarrow{OD}\times\overrightarrow{OE}\right|$

$= \frac{1}{2}\left|(\mathbf{b}+\lambda\mathbf{a})\times(\mathbf{a}+\mu\mathbf{b})\right|$

$= \frac{1}{2}\left|(\mathbf{b}\times\mathbf{a})+\lambda\mu(\mathbf{a}\times\mathbf{b})\right|$

$= \frac{1}{2}\left|(\mathbf{b}\times\mathbf{a})-\lambda\mu(\mathbf{b}\times\mathbf{a})\right|$

$= \frac{1-\lambda\mu}{2}\left|\mathbf{b}\times\mathbf{a}\right|$

$= \frac{1-\lambda\mu}{2}\left|\mathbf{a}\times\mathbf{b}\right|$ since $0<\mu<1$ and $0<\lambda<1$

area of triangle *ODE* = *k* (area of triangle *OCE*)

$\frac{1-\lambda\mu}{2}\left|\mathbf{a}\times\mathbf{b}\right| = k\left(\frac{1-\mu}{2}\right)\left|\mathbf{a}\times\mathbf{b}\right|$

$1-\lambda\mu = k(1-\mu)$

$k = \frac{1-\lambda\mu}{1-\mu}$

Note :

- explanation $0<\mu<1$ and $0<\lambda<1$ should be seen.

- For area, must have modulus if written as $\lambda\mu-1$ and $\mu-1$. i.e $|\lambda\mu-1|$, $|\mu-1|$.

- For *k*, if written as $k = \left|\frac{1-\lambda\mu}{1-\mu}\right|$, must proceed to

$k = \frac{1-\lambda\mu}{1-\mu}$

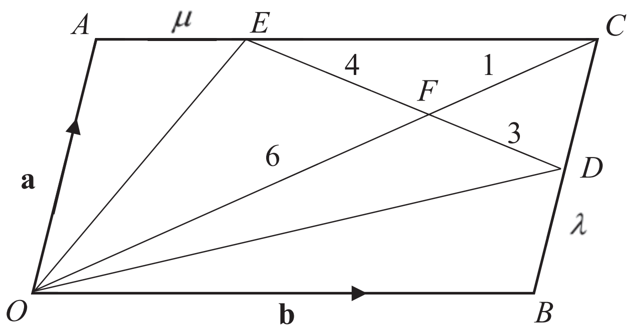
or

$k = -\left(\frac{1-\lambda\mu}{1-\mu}\right)$ (rejected as $k>0$)

(ii)

Given $OF:FC = 6:1$ and $DF:FE = 3:4$.

By Ratio Theorem,



$\overrightarrow{OF} = \frac{3\overrightarrow{OE}+4\overrightarrow{OD}}{7}$

$\frac{6}{7}(\mathbf{a}+\mathbf{b}) = \frac{1}{7}\left[3(\mathbf{a}+\mu\mathbf{b})\right]+\frac{1}{7}\left[4(\mathbf{b}+\lambda\mathbf{a})\right]$

a : $6 = 3+4\lambda \Rightarrow \lambda = \frac{3}{4}$

b : $6 = 3\mu+4 \Rightarrow \mu = \frac{2}{3}$

$k = \frac{1-\left(\frac{3}{4}\right)\left(\frac{2}{3}\right)}{1-\left(\frac{2}{3}\right)} = \frac{\frac{1}{2}}{\frac{1}{3}} = \frac{3}{2}$

Q4

(i)

Consider $y=\dfrac{x^2+5}{x-2}$ and the line $y=k$.

$k=\dfrac{x^2+5}{x-2}$

$kx-2k=x^2+5$

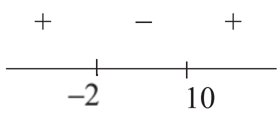
$x^2-kx+2k+5=0$

For $y=\dfrac{x^2+5}{x-2}$ and the line $y=k$ not to intersect,

$k^2-4(2k+5)<0$

$k^2-8k-20<0$

$(k-10)(k+2)<0$



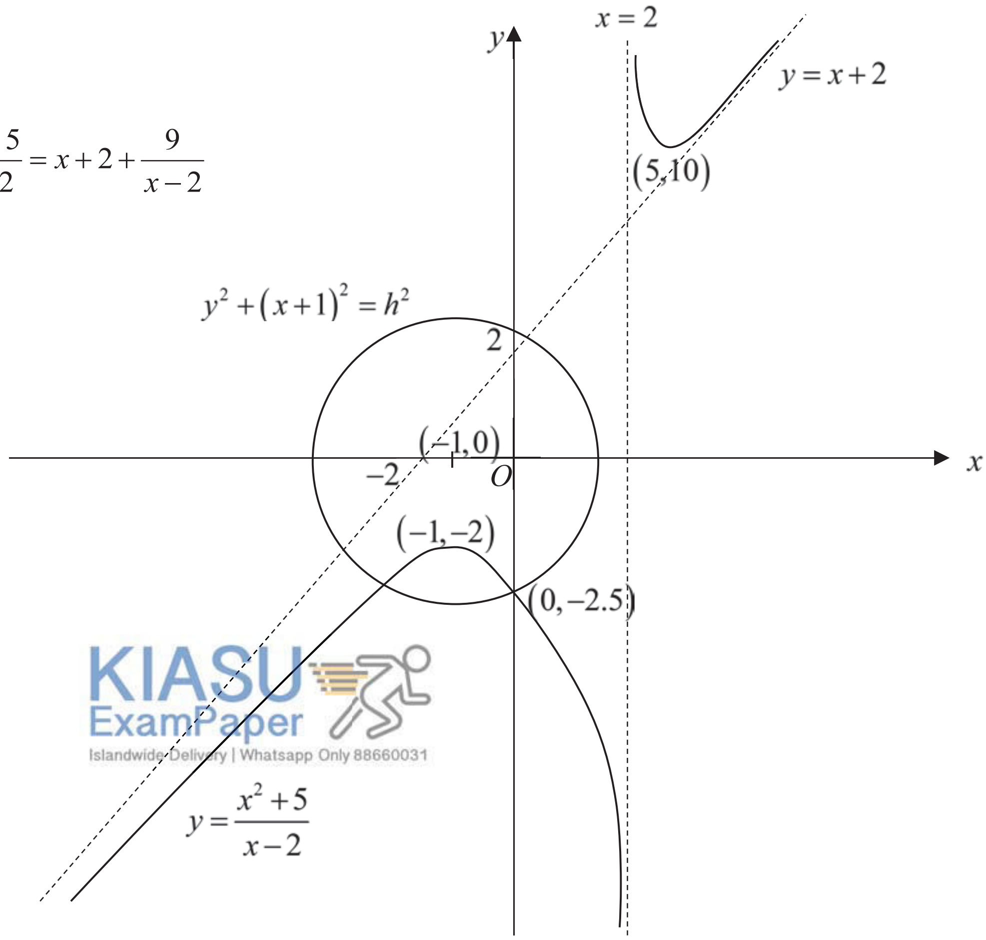
$-2<k<10$

$-2<y<10$

C cannot lie between -2 and 10.

(ii)

$y=\dfrac{x^2+5}{x-2}=x+2+\dfrac{9}{x-2}$



(iii)

$$\left(x^2+5\right)^2+(x+1)^2(x-2)^2=h^2(x-2)^2$$

$$\frac{\left(x^2+5\right)^2}{(x-2)^2}+(x+1)^2=h^2$$

$$y^2+(x+1)^2=h^2$$

Consider distance from centre of circle to y-intercept of C

$|h|=\sqrt{(-1-0)^2+(0-(-2.5))^2}=\frac{1}{2}\sqrt{29}$ or 2.69

$$h^2>\frac{29}{4}$$

$h<-\frac{1}{2}\sqrt{29}$ or $h>\frac{1}{2}\sqrt{29}$

Q5

(a)(i)

$$w_1^{\ 3}=\left(-3+\sqrt{5}\,\mathrm{i}\right)^3$$

$$=(-3)^3+3(-3)^2\left(\left(\sqrt{5}\right)\mathrm{i}\right)+3(-3)\left(\left(\sqrt{5}\right)\mathrm{i}\right)^2+\left(\left(\sqrt{5}\right)\mathrm{i}\right)^3$$

$$=-27+27\sqrt{5}\,\mathrm{i}+45-5\sqrt{5}\,\mathrm{i}$$

$$=18+22\sqrt{5}\,\mathrm{i}$$

(ii)

Since $w_1=-3+\sqrt{5}\,\mathrm{i}$ is a root,

$$4w_1^{\ 3}+pw_1^{\ 2}+qw_1-14=0$$

$$4(18+22\sqrt{5}\,\mathrm{i})+p\left(-3+\sqrt{5}\,\mathrm{i}\right)^2+q\left(-3+\sqrt{5}\,\mathrm{i}\right)-14=0$$

$$4(18+22\sqrt{5}\,\mathrm{i})+p\left(9-6\sqrt{5}\,\mathrm{i}-5\right)+q\left(-3+\sqrt{5}\,\mathrm{i}\right)-14=0$$

$$72+4p-3q-14+\left(88\sqrt{5}-6\sqrt{5}p+\sqrt{5}q\right)\mathrm{i}=0$$

Comparing real and imaginary parts,

$4p-3q+58=0$ ---------(1)

$88\sqrt{5}-6\sqrt{5}p+\sqrt{5}q=0$

$\therefore 88-6p+q=0$------(2)

$(2)\times 3: 264-18p+3q=0$ ------(3)

Solving (1) and (3) , $14p=322$

$p=23$, $q=50$.

(iii)

Since $w_1 = -3+\sqrt{5}\,\mathrm{i}$ is a root, and polynomial equation has real coeffcients, $w_1^* = -3-\sqrt{5}\,\mathrm{i}$ is also a root.

$$\begin{aligned}4w^3+23w^2+50w-14 &= \left(w-\left(-3+\sqrt{5}\,\mathrm{i}\right)\right)\left(w-\left(-3-\sqrt{5}\,\mathrm{i}\right)\right)g(w)\\ &= \left((w+3)+\sqrt{5}\,\mathrm{i}\right)\left((w+3)-\sqrt{5}\,\mathrm{i}\right)g(w)\\ &= \left((w+3)^2-\left(\sqrt{5}\,\mathrm{i}\right)^2\right)g(w)\\ &= \left(w^2+6w+14\right)(4w-1)\end{aligned}$$

When $4w^3+23w^2+50w-14=0$,

Therefore, other two roots are $-3-\sqrt{5}\,\mathrm{i}$ and $\dfrac{1}{4}$.

(b)

$$z=-1-\sqrt{3}\ \mathrm{i}=2\mathrm{e}^{-\frac{2}{3}\pi\mathrm{i}}$$

$$\frac{z^*}{z^n}=\frac{2\mathrm{e}^{\frac{2}{3}\pi\mathrm{i}}}{2^n\mathrm{e}^{-\frac{2}{3}n\pi\mathrm{i}}}=2^{1-n}\mathrm{e}^{\frac{2}{3}\pi(n+1)\mathrm{i}}=2^{1-n}\left(\cos\left(\frac{2}{3}\pi(n+1)\right)+\mathrm{i}\sin\left(\frac{2}{3}\pi(n+1)\right)\right)$$

$\dfrac{z^*}{z^n}$ is imaginary: $\cos\left(\dfrac{2}{3}\pi(n+1)\right)=0$

$$\frac{2}{3}\pi(n+1)=\frac{(2k+1)}{2}\pi,\ \text{where } k\in\mathbb{Z}$$

$$n=\frac{3k}{2}-\frac{1}{4}$$

Note :

Cosine is zero when

$\theta=\pm\dfrac{\pi}{2},\pm\dfrac{3\pi}{2},\pm\dfrac{5\pi}{2}$......

$\pm1,\pm2,\pm3,$...... odd number

Q6

(i)

$$\mathrm{f}(x)=\mathrm{e}^{2x}-9\mathrm{e}^{-2x}$$

$$\mathrm{f}'(x)=2\mathrm{e}^{2x}+18\mathrm{e}^{-2x}$$

Since $\mathrm{e}^{2x}>0$ and $\mathrm{e}^{-2x}>0$ for all x, $\mathrm{f}'(x)=2\mathrm{e}^{2x}+18\mathrm{e}^{-2x}>0$ for all x.

(ii)

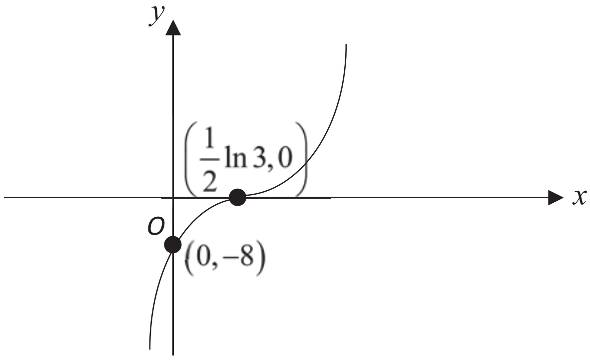
$$\mathrm{f}''(x)=4\mathrm{e}^{2x}-36\mathrm{e}^{-2x}$$

For $y=\mathrm{f}(x)$ to be concave upward, $\mathrm{f}''(x)=4\mathrm{e}^{2x}-36\mathrm{e}^{-2x}>0\Rightarrow \mathrm{e}^{2x}-9\mathrm{e}^{-2x}>0$ -(1)

For $\mathrm{f}(x)>0$, $\mathrm{e}^{2x}-9\mathrm{e}^{-2x}>0$ which is the same as (1) (Shown)

$$\begin{aligned}\mathrm{e}^{2x}-9\mathrm{e}^{-2x}&>0\\ \mathrm{e}^{4x}-9&>0\\ \mathrm{e}^{4x}&>9\\ x&>\frac{1}{4}\ln 9=\frac{1}{2}\ln 3\end{aligned}$$

(iii)



(iv)

$$\int_0^2\left|\mathrm{e}^{2x}-9\mathrm{e}^{-2x}\right|\mathrm{d}x$$

$$=-\int_0^{\frac{1}{2}\ln 3}\mathrm{e}^{2x}-9\mathrm{e}^{-2x}\,\mathrm{d}x+\int_{\frac{1}{2}\ln 3}^{2}\mathrm{e}^{2x}-9\mathrm{e}^{-2x}\,\mathrm{d}x$$

$$=-\left[\frac{1}{2}\mathrm{e}^{2x}+\frac{9}{2}\mathrm{e}^{-2x}\right]_0^{\frac{1}{2}\ln 3}+\left[\frac{1}{2}\mathrm{e}^{2x}+\frac{9}{2}\mathrm{e}^{-2x}\right]_{\frac{1}{2}\ln 3}^{2}$$

$$=-\left[\left(\frac{1}{2}\mathrm{e}^{2\left(\frac{1}{2}\ln 3\right)}+\frac{9}{2}\mathrm{e}^{-2\left(\frac{1}{2}\ln 3\right)}\right)-\left(\frac{1}{2}\mathrm{e}^{2(0)}+\frac{9}{2}\mathrm{e}^{-2(0)}\right)\right]+\left[\left(\frac{1}{2}\mathrm{e}^{2(2)}+\frac{9}{2}\mathrm{e}^{-2(2)}\right)-\left(\frac{1}{2}\mathrm{e}^{2\left(\frac{1}{2}\ln 3\right)}+\frac{9}{2}\mathrm{e}^{-2\left(\frac{1}{2}\ln 3\right)}\right)\right]$$

$$=-\left[\frac{3}{2}+\frac{9}{2}\left(\frac{1}{3}\right)-\frac{1}{2}-\frac{9}{2}\right]+\left[\frac{1}{2}\mathrm{e}^{4}+\frac{9}{2}\mathrm{e}^{-4}-\frac{3}{2}-\frac{9}{2}\left(\frac{1}{3}\right)\right]$$

$$=-1+\frac{1}{2}\mathrm{e}^{4}+\frac{9}{2}\mathrm{e}^{-4}$$

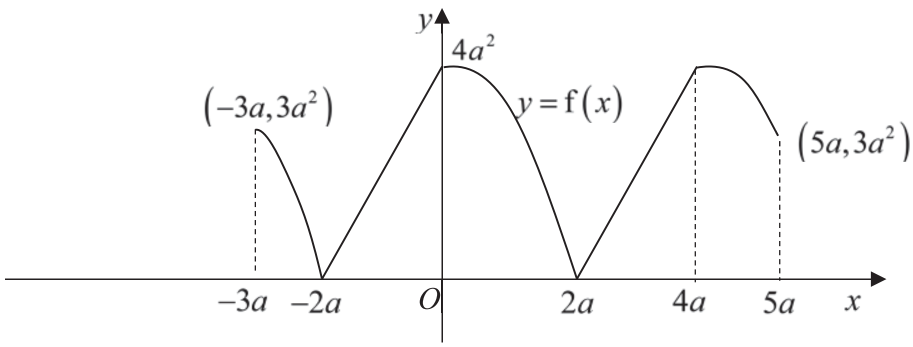
Q7

(i)

Since $\mathrm{f}(x)=\mathrm{f}(x+4a)$,

$\mathrm{f}(2019a)=\mathrm{f}(3a)=2a(3a-2a)=2a^2$

(ii)



(iii)

Since $\mathrm{R_f} = [0, 4a^2] \not\subset \mathrm{D_g} = (2a, 4a)$, gf does not exist.

(iv) Please note that g(2*a*) = 0, g(4*a*) = 2*a*, so f takes 0 to 2*a*

$\mathrm{fg}(x) = 4a^2 - \left(\sqrt{4a^2 - (x-2a)^2}\right)^2 = (x-2a)^2$

$\mathrm{fg}: x \mapsto (x-2a)^2, \quad 2a < x < 4a,$

(v)

$(\mathrm{fg})^{-1}(27) = \frac{7}{2}a$

$\mathrm{fg}\left(\frac{7}{2}a\right) = 27$

$\left(\frac{7}{2}a - 2a\right)^2 = 27$

$\frac{9}{4}a^2 = 27$

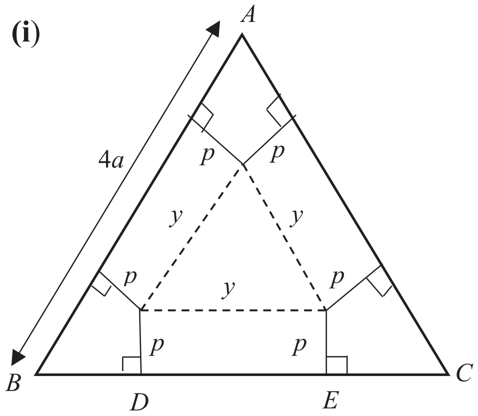
$a^2 = 12$

$a = \pm 2\sqrt{3}$

Since $a > 0$, $a = 2\sqrt{3}$

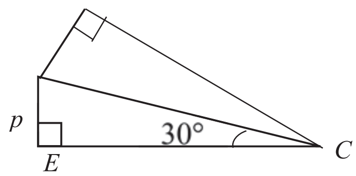
Q8

(i)



Since ABC is an equilateral triangle, $\angle ABC = \angle BCA = \angle CAB = 60^0$

length of BD = length of $CE = \dfrac{p}{\tan 30^0} = \sqrt{3}p$



Thus $y = 4a - 2\sqrt{3}p$

Volume of the prism, $V = \left(\dfrac{1}{2}y^2 \sin 60^0\right)p$

$$= \frac{1}{2}\left(4a - 2\sqrt{3}p\right)^2 \times \frac{\sqrt{3}}{2} \times p$$

$$= \frac{\sqrt{3}}{4}p\left(4a - 2\sqrt{3}p\right)^2$$

$$= \frac{\sqrt{3}}{4}p(2)^2\left(2a - \sqrt{3}p\right)^2$$

$$= \sqrt{3}p\left(2a - \sqrt{3}p\right)^2 \text{ cm}^3 \text{ (Shown)}$$

(ii)

$$\frac{dV}{dp} = \sqrt{3}\left(2a - \sqrt{3}p\right)^2 + 2\sqrt{3}p\left(2a - \sqrt{3}p\right)\left(-\sqrt{3}\right)$$

$$= 4\sqrt{3}a^2 - 12ap + 3\sqrt{3}p^2 - 12ap + 6\sqrt{3}p^2$$

$$= 9\sqrt{3}p^2 - 24ap + 4\sqrt{3}a^2$$

$$\frac{dV}{dp} = 0$$

$$9\sqrt{3}p^2 - 24ap + 4\sqrt{3}a^2 = 0$$

$$p = \frac{-(-24a) \pm \sqrt{(-24a)^2 - 4(9\sqrt{3})\left(4\sqrt{3}a^2\right)}}{2(9\sqrt{3})}$$

$$p = \frac{24a \pm \sqrt{144a^2}}{18\sqrt{3}} = \frac{2a}{3\sqrt{3}} \text{ or } \frac{2a}{\sqrt{3}}$$

When $p=\frac{2a}{3\sqrt{3}}$, $y=4a-2\sqrt{3}\left(\frac{2a}{3\sqrt{3}}\right)=4a-\frac{4}{3}a=\frac{8}{3}a>0$

When $p=\frac{2a}{\sqrt{3}}$, $y=4a-2\sqrt{3}\left(\frac{2a}{\sqrt{3}}\right)=4a-4a=0$ (NA as $y\neq 0$)

Or

When $p=\frac{2a}{3\sqrt{3}}$, $V=\sqrt{3}\left(\frac{2a}{3\sqrt{3}}\right)\left(2a-\sqrt{3}\left(\frac{2a}{3\sqrt{3}}\right)\right)^2=\frac{2a}{3}\left(2a-\frac{2a}{3}\right)^2=\frac{32}{27}a^3>0$

When $p=\frac{2a}{\sqrt{3}}$, $V=\sqrt{3}\left(\frac{2a}{\sqrt{3}}\right)\left(2a-\sqrt{3}\left(\frac{2a}{\sqrt{3}}\right)\right)^2=2a\left(2a-2a\right)^2=0$ (NA as $V\neq 0$)

Therefore, $p=\frac{2a}{3\sqrt{3}}=\frac{2\sqrt{3}a}{9}$ cm

(iii)

$\frac{\mathrm{d}V}{\mathrm{d}p}=9\sqrt{3}p^2-24ap+4\sqrt{3}a^2$

$\frac{\mathrm{d}^2V}{\mathrm{d}p^2}=18\sqrt{3}p-24a$

When $p=\frac{2a}{3\sqrt{3}}=\frac{2\sqrt{3}a}{9}$, $\frac{\mathrm{d}^2V}{\mathrm{d}p^2}=18\sqrt{3}\left(\frac{2a}{3\sqrt{3}}\right)-24a=12a-24a=-12a<0$ (max)

Hence, V is maximum when $p=\frac{2a}{3\sqrt{3}}=\frac{2\sqrt{3}a}{9}$

Maximum volume of the prism = $\frac{\sqrt{3}}{4}p\left(4a-2\sqrt{3}p\right)^2=\frac{32}{27}a^3$ cm^3

$\frac{3}{4}$ of the maximum volume = $\frac{3}{4}\times\frac{32}{27}\times a^2=\frac{8}{9}a^3$ cm^3

Maximum cost that the housewife has to pay = $\frac{8}{9}a^3\times 0.4=\frac{16}{45}a^3$ cents.

9

(a)

$$\int \frac{e^{\frac{1}{x}}}{x^2}\,dx=-\int -\frac{1}{x^2}e^{\frac{1}{x}}\,dx$$
$$=-e^{\frac{1}{x}}+C$$

(b)

$$\int \cos kx \cos(k+2)x\,dx = \frac{1}{2}\int 2\cos(k+2)x\cos kx\,dx$$
$$= \frac{1}{2}\int \left[\cos(2k+2)x + \cos 2x\right]dx$$
$$= \frac{1}{2}\left[\frac{\sin(2k+2)x}{2k+2} + \frac{\sin 2x}{2}\right] + C$$

(c)

$$u = \tan^{-1}(3x) \qquad \frac{dv}{dx} = x$$
$$\frac{du}{dx} = \frac{3}{1+9x^2} \qquad v = \frac{1}{2}x^2$$

$$\int x\tan^{-1}(3x)\,dx$$
$$= \frac{1}{2}x^2\tan^{-1}(3x) - \frac{3}{2}\int \frac{x^2}{1+9x^2}dx$$
$$= \frac{1}{2}x^2\tan^{-1}(3x) - \frac{1}{6}\int 1 - \frac{1}{1+9x^2}dx$$
$$= \frac{1}{2}x^2\tan^{-1}(3x) - \frac{1}{6}\int 1\,dx + \frac{1}{6}\int \frac{1}{9\left(\frac{1}{9}+x^2\right)}dx$$
$$= \frac{1}{2}x^2\tan^{-1}(3x) - \frac{1}{6}x + \frac{1}{18}\tan^{-1}(3x) + C$$

Q10

(a)(i)

Using AP, $v_n = 44\ 000 + (n-1)(335) = 43665 + 335n$

(ii)

Using $v_n > 53\ 500$

$43665 + 335n > 53\ 500$

$n > 29.358$

least $n = 30$

Average speed of 53 500 km/hr was reached in March 2001.

(iii)

$T =$ 1 month = 30(24) hours = 720 hours , 3 years = 36 months

Distance = $720\left[\frac{36}{2}(2(44000)+35(335))\right]$ = 1 292 436 000 km

(b)(i)

GP : 0.44, 0.44(2.03) , $0.44(2.03)^2$, ...

longest orbital period = $0.44(2.03)^{66} = 8.67 \times 10^{19}$ days

(ii)

$$\left|S_n - 0.44(2.03)^{19}\right| < 5\times10^6$$

$$\left|\frac{0.44(2.03^n - 1)}{2.03 - 1} - 0.44(2.03)^{19}\right| < 5\times10^6$$

Using GC (table) , largest $n = 23$

OR:

$$-5\times10^6 < \frac{0.44(2.03^n - 1)}{2.03 - 1} - 0.44(2.03)^{19} < 5\times10^6$$

$$-5\times10^6 + 306110.3422 < \frac{0.44\left(2.03^n\right)}{1.03} < 5\times10^6 + 306110.3422$$

Since $\frac{0.44\left(2.03^n\right)}{1.03}$ is always positive, $\frac{0.44\left(2.03^n\right)}{1.03} < 5\times10^6 + 306110.3422$

Solving, $n < 23.0707$

largest $n = 23$

Jurong Pioneer Junior College

H2 Mathematics **JC2-2019**
Preliminary Exam P2 Solution

Q1

$a_{n+1} = a_n + 3^n - n$

$$\sum_{r=0}^{n-1}\left(a_{r+1} - a_r\right) = \sum_{r=0}^{n-1}\left(3^r - r\right)$$

$$= \sum_{r=0}^{n-1} 3^r - \sum_{r=0}^{n-1} r$$

$$\begin{aligned}
&a_1 - a_0 \\
&+a_2 - a_1 \\
&+a_3 - a_2 \\
&+ \\
&\quad\vdots \\
&+a_{n-2} - a_{n-3} = \frac{1(3^n - 1)}{3-1} - \frac{(n-1)n}{2} \\
&+a_{n-1} - a_{n-2} \\
&+a_n - a_{n-1}
\end{aligned}$$

$$a_n - a_0 = \frac{(3^n - 1)}{2} - \frac{(n-1)n}{2}$$

$$a_n = a_0 + \frac{(3^n - 1)}{2} - \frac{(n-1)n}{2}$$

$$= \frac{3}{5} - \frac{1}{2} + \frac{3^n}{2} - \frac{(n-1)n}{2}$$

$$= \frac{1}{10} + \frac{3^n}{2} - \frac{(n-1)n}{2}$$

Q2
(a)(i)

$$\ln(\cos 3x) \approx \ln\left(1 - \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!} - \frac{(3x)^6}{6!}\right) = \ln\left(1 + \left(-\frac{9}{2}x^2 + \frac{27}{8}x^4 - \frac{81}{80}x^6\right)\right)$$

$$\approx \left(-\frac{9}{2}x^2 + \frac{27}{8}x^4 - \frac{81}{80}x^6\right) - \frac{\left(-\frac{9}{2}x^2 + \frac{27}{8}x^4 - \frac{81}{80}x^6\right)^2}{2} + \frac{\left(-\frac{9}{2}x^2 + \frac{27}{8}x^4 - \frac{81}{80}x^6\right)^3}{3}$$

$$\approx \left(-\frac{9}{2}x^2 + \frac{27}{8}x^4 - \frac{81}{80}x^6\right) - \frac{1}{2}\left(\frac{81}{4}x^4 - \frac{243}{8}x^6\right) + \frac{1}{3}\left(-\frac{729}{8}x^6\right)$$

$$= -\frac{9}{2}x^2 - \frac{27}{4}x^4 - \frac{81}{5}x^6$$

(a)(ii)

Differentiating wrt x,

$$\frac{-3\sin 3x}{\cos 3x} \approx -9x - 27x^3 - \frac{486}{5}x^5$$

$$-3\tan 3x \approx -9x - 27x^3 - \frac{486}{5}x^5$$

$$\tan 3x \approx 3x + 9x^3 + \frac{162}{5}x^5$$

(b)

$$\frac{e^{\tan x}}{(2+x)^2}$$

$$= e^{\tan x}(2+x)^{-2}$$

$$= \frac{1}{4}e^{\tan x}\left(1+\frac{x}{2}\right)^{-2}$$

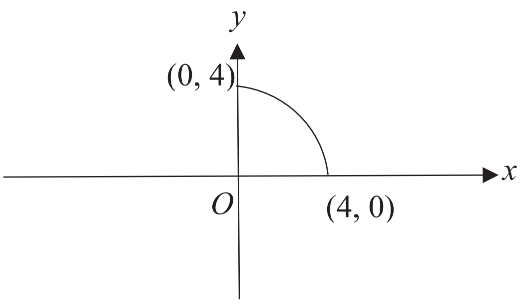
$$\approx \frac{1}{4}e^{x}\left(1+\frac{x}{2}\right)^{-2}$$

$$\approx \frac{1}{4}\left(1+x+\frac{1}{2}x^2\right)\left(1-x+\frac{3}{4}x^2\right)$$

$$= \frac{1}{4}\left(1+\frac{1}{4}x^2\right)$$

Q3

(i)



(ii)

$x = 4\sin 2t$ $\qquad$ $y = 4\cos 2t$

$\frac{dx}{dt} = 8\cos 2t$ $\qquad$ $\frac{dy}{dt} = -8\sin 2t$,

$\frac{dy}{dx} = -\tan 2t$

(a) For tangents parallel to y-axis ($\frac{dy}{dx}$ is undefined): $2t = \frac{\pi}{2} \Rightarrow t = \frac{\pi}{4}$

(b) For tangent parallel to x-axis ($\frac{dy}{dx} = 0$): $2t = 0 \Rightarrow t = 0$

(iii)

Given that gradient of tangent = $-\dfrac{1}{\sqrt{3}}$

$-\tan 2t = -\dfrac{1}{\sqrt{3}}$,

$2t = \dfrac{\pi}{6}$

$\therefore\ t = \dfrac{\pi}{12}$

When $t = \dfrac{\pi}{12}$, $x = 4\sin 2\left(\dfrac{\pi}{12}\right) = 2$, $y = 4\cos 2\left(\dfrac{\pi}{12}\right) = 2\sqrt{3}$

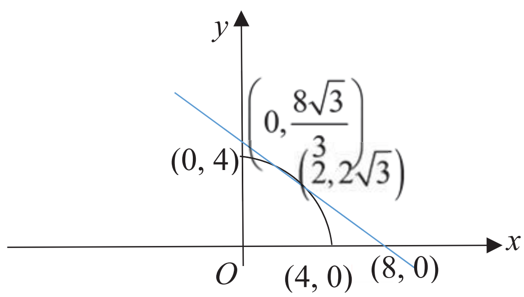
Equation of tangent at $P(2,\ 2\sqrt{3})$:

$y - 2\sqrt{3} = -\dfrac{1}{\sqrt{3}}(x-2)$

$y = -\dfrac{\sqrt{3}}{3}x + \dfrac{8}{3}\sqrt{3} = \dfrac{\sqrt{3}}{3}(8-x)$, where $a = \sqrt{3}$, $b = 8$.

(iv)

When $x = 0$, $y = \dfrac{8\sqrt{3}}{3}$. When $y = 0$, $x = 8$



Required area

= Area of the triangle – Area of the quadrant of the circle

$= \int_0^8 \dfrac{\sqrt{3}}{3}(8-x)\ \mathrm{d}x - \dfrac{1}{4}\pi(4)^2$

$= \left(\dfrac{1}{2}\times 8 \times \dfrac{8\sqrt{3}}{3}\right) - \dfrac{1}{4}\pi(4)^2 = \left(\dfrac{32\sqrt{3}}{3} - 4\pi\right)$ units^2

Q4

$\dfrac{\mathrm{d}}{\mathrm{d}u}\left[u^2\dfrac{\mathrm{d}x}{\mathrm{d}u}\right] = u^2\dfrac{\mathrm{d}^2x}{\mathrm{d}u^2} + 2u\dfrac{\mathrm{d}x}{\mathrm{d}u}$

$2u^2\dfrac{\mathrm{d}^2x}{\mathrm{d}u^2} + 4u\dfrac{\mathrm{d}x}{\mathrm{d}u} = 15u + 12$ becomes $2\dfrac{\mathrm{d}}{\mathrm{d}u}\left[u^2\dfrac{\mathrm{d}x}{\mathrm{d}u}\right] = 15u + 12$

Thus $u^2\dfrac{\mathrm{d}x}{\mathrm{d}u} = \dfrac{1}{2}\int 15u + 12\ \mathrm{d}u$

$u^2\dfrac{\mathrm{d}x}{\mathrm{d}u} = \dfrac{1}{2}\left[\dfrac{15u^2}{2} + 12u\right] + C$

$x = 0$ and $\dfrac{\mathrm{d}x}{\mathrm{d}u} = 1$ when $u = 1$

$C = -\dfrac{35}{4}$

$u^2\dfrac{\mathrm{d}x}{\mathrm{d}u} = \dfrac{1}{2}\left[\dfrac{15u^2}{2} + 12u\right] - \dfrac{35}{4}$,

$$\frac{dx}{du}=\frac{1}{2}\left[\frac{15}{2}+\frac{12}{u}\right]-\frac{35}{4u^2}$$

$$x=\int\frac{1}{2}\left[\frac{15}{2}+\frac{12}{u}\right]-\frac{35}{4u^2}\ du$$

$$=\ \frac{15}{4}u+6\ln|u|+\frac{35}{4u}+D$$

$x=0$ when $u=1$

$\therefore\ D=-\frac{25}{2}$

$$x=\frac{15}{4}u+6\ln|u|+\frac{35}{4u}-\frac{25}{2}$$

Q5

(i)

Method 1

$$\begin{pmatrix}8\\-4\\1\end{pmatrix}\bullet\begin{pmatrix}1\\2\\0\end{pmatrix}=8-8+0=0$$

$$\begin{pmatrix}8\\-4\\1\end{pmatrix}\bullet\begin{pmatrix}2\\3\\-4\end{pmatrix}=16-12-4=0$$

Since direction vector of l is perpendicular to two vectors parallel to p, l is perpendicular to p.

Method 2

$$\begin{pmatrix}1\\2\\0\end{pmatrix}\times\begin{pmatrix}2\\3\\-4\end{pmatrix}=\begin{pmatrix}-8\\4\\-1\end{pmatrix}=-\begin{pmatrix}8\\-4\\1\end{pmatrix}$$ which is parallel to l

l is perpendicular to p.

$$\begin{pmatrix}1\\0\\-3\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\0\end{pmatrix}+\mu\begin{pmatrix}2\\3\\-4\end{pmatrix}=\begin{pmatrix}-10\\0\\4\end{pmatrix}+t\begin{pmatrix}8\\-4\\1\end{pmatrix}$$

$x:\quad 1+\lambda+2\mu=-10+8t\Rightarrow\lambda+2\mu-8t=-11\qquad(1)$

$y:\quad 2\lambda+3\mu=-4t\Rightarrow 2\lambda+3\mu+4t=0\qquad(2)$

$z:\quad -3-4\mu=4+t\Rightarrow -4\mu-t=7\qquad(3)$

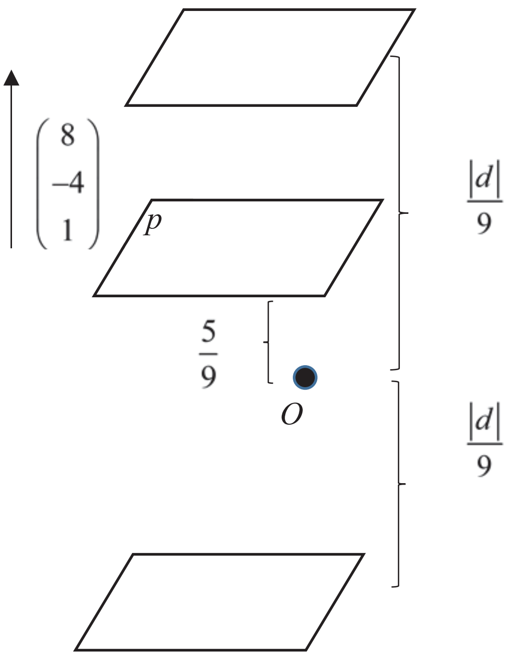
$\lambda=1,\quad \mu=-2,\quad t=1$

(ii)

Method 1

Equation of p is $\mathbf{r}\bullet\begin{pmatrix}8\\-4\\1\end{pmatrix}=\begin{pmatrix}1\\0\\-3\end{pmatrix}\bullet\begin{pmatrix}8\\-4\\1\end{pmatrix}=5$

Let the equations of the required planes be $\mathbf{r}\bullet\begin{pmatrix}8\\-4\\1\end{pmatrix}=d$



Given distance between p and planes = 2

$$\frac{|d-5|}{9}=2$$
$$|d-5|=18$$
$d-5=18$ or $d-5=-18$
$d=23$ or $d=-13$

Cartesian equations of the required planes are
$8x-4y+z=23$ and $8x-4y+z=-13$

Method 2

Equation of p is $\mathbf{r}\bullet\begin{pmatrix}8\\-4\\1\end{pmatrix}=\begin{pmatrix}1\\0\\-3\end{pmatrix}\bullet\begin{pmatrix}8\\-4\\1\end{pmatrix}=5$

$$\mathbf{r}\bullet\frac{1}{9}\begin{pmatrix}8\\-4\\1\end{pmatrix}=\frac{5}{9}$$

Equations of the required planes are

$\mathbf{r}\bullet\frac{1}{9}\begin{pmatrix}8\\-4\\1\end{pmatrix}=\frac{5}{9}+2$ and $\mathbf{r}\bullet\frac{1}{9}\begin{pmatrix}8\\-4\\1\end{pmatrix}=\frac{5}{9}-2$

$\mathbf{r}\bullet\begin{pmatrix}8\\-4\\1\end{pmatrix}=23$ and $\mathbf{r}\bullet\begin{pmatrix}8\\-4\\1\end{pmatrix}=-13$

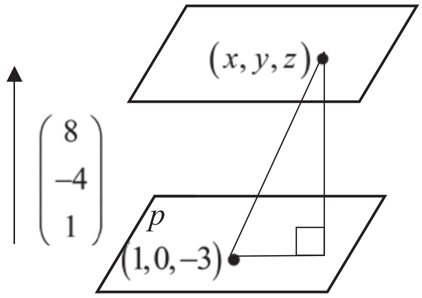
Cartesian equations of the required planes are
$8x-4y+z=23$ and $8x-4y+z=-13$

Method 3

Let a point on the required planes be (x, y, z) .

Consider point $(1, 0, -3)$ on *p*.

Given distance between *p* and planes = 2



$$\frac{1}{9}\left|\left[\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}\right] \bullet \begin{pmatrix} 8 \\ -4 \\ 1 \end{pmatrix}\right| = 2$$

$$\frac{1}{9}\left|\begin{pmatrix} x-1 \\ y \\ z+3 \end{pmatrix} \bullet \begin{pmatrix} 8 \\ -4 \\ 1 \end{pmatrix}\right| = 2$$

$$|8x - 8 - 4y + z + 3| = 18$$

$$|8x - 4y + z - 5| = 18$$

$8x - 4y + z - 5 = 18$ or $8x - 4y + z - 5 = -18$

Cartesian equations of the required planes are
$8x - 4y + z = 23$ and $8x - 4y + z = -13$

Q6
(i)
Whether a box contains voucher is independent of any other boxes.
The probability of a box containing a voucher is constant.

(ii)
Let *X* be the number of vouchers obtained out of 9 boxes of Brand *A* cereal

$$X \sim \mathrm{B}(9, 0.35)$$

$$\mathrm{P}(X \le 3) = 0.60889 \approx 0.609$$

(iii)
Required probability = $\mathrm{P}(X = 7) \times 0.35 = 0.0034251 \approx 0.00343$

Alternative

Required probability = ${}^{9}C_{7}(0.35)^{8}(0.65)^{2} = 0.0034251 \approx 0.00343$

Let *W* be the number of vouchers obtained out of 10 boxes of Brand *B* cereal

$$W \sim \mathrm{B}(10, p)$$

$$\mathrm{P}(W \le 1) = 0.4845$$

$$\binom{10}{0} p^{0}(1-p)^{10} + \binom{10}{1} p^{1}(1-p)^{9} = 0.4845$$

$$(1-p)^{10} + 10p(1-p)^{9} = 0.4845$$

$$p = 0.16667 \approx 0.167$$

Q7

(i)
Central Limit Theorem states that sample means will follow a normal distribution approximately when sample size is big enough.
(ii)
$\sum(x-390)=120$ and $\sum(x-390)^2=3100$

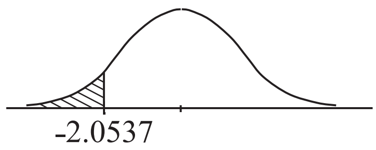
$\bar{x}=\frac{120}{50}+390=392.4$

$s^2=\frac{1}{50-1}\left[3100-\frac{120^2}{50}\right]=57.3877551\approx 57.388$

$H_0 : \mu = m$ vs $H_1 : \mu < m$

Since n = 50 is large, by Central Limit Theorem, $\overline{X} \sim N(m, \frac{57.388}{50})$ approximately

Level of significance: 2%
Critical region: Reject H_0 when $z < -2.0537$



Consider $\frac{392.4-m}{\sqrt{\frac{57.388}{50}}} < -2.0537$

$m > 394.6$ (5 s.f.)
The least possible value of m is 395grams.

Q8
(a)

Case 1: Last digit is 5

{2,4,6,8}					{5}

Number of different 6-digit numbers = $4\times 4! = 96$

Case 2: Last digit is 0

{2,4,5,6,8}					{0}

Number of different 6-digit numbers = $5! = 120$

$\therefore$ Required number of different 6-digit numbers = 96 + 120 = 216

(b)
Case 1: All letters are different.
Number of 4-letter code words = ${}^7C_4 \times 4! = 840$
Case 2: One pair of repeated letters.
Number of 4-letter code words = ${}^2C_1 \times {}^6C_2 \times \frac{4!}{2!} = 360$
Case 3: Two pairs of repeated letters.
Number of 4-letter code words = $\frac{4!}{2!2!} = 6$
$\therefore$ Total number of 4-letter code words = 840 + 360 + 6 = 1206

Q9
(i)
Let A be the mass of a Grade A strawberry

$$A \sim \mathrm{N}\left(18, 3^2\right)$$

$$\mathrm{P}\left(17 < A < 20\right) = 0.37806 \approx 0.378$$

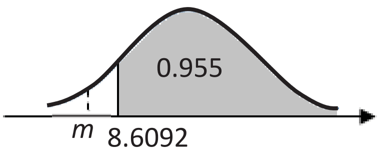
(ii)
Let B be the mass of a Grade B strawberry

$$B \sim \mathrm{N}\left(12, 2^2\right)$$

$$\mathrm{P}\left(B > m\right) \geq 0.955$$

$$m \leq 8.6092$$

Greatest value of m = 8.60



(iii)
Let $X = \left(A_1 + ... + A_{12}\right) - \left(B_1 + ... + B_{15}\right)$

$$\mathrm{E}\left(X\right) = 12\left(18\right) - 15\left(12\right) = 36$$

$$\mathrm{Var}\left(X\right) = 12\left(3^2\right) + 15\left(2^2\right) = 168$$

$$X \sim \mathrm{N}\left(36, 168\right)$$

$$\mathrm{P}\left(X > 0\right) = 0.99726 \approx 0.997$$

(iv)
The masses of strawberries are independent of each other.

Q10
(i)

	$4n$-4 Cards	3 Cards	1 Card
Score (+)	**1**	**2**	**4**
1	2	3	5
2	3	4	6
4	5	6	Nil

$$\mathrm{P}\left(W = 10\right)$$

$$= \mathrm{P}\left(\text{Score} > 4\right)$$

$$= \mathrm{P}\left(1,4\right) + \mathrm{P}\left(4,1\right) + \mathrm{P}\left(2,4\right) + \mathrm{P}\left(4,2\right)$$

$$= \left(\frac{4n-4}{4n}\right)\left(\frac{1}{4n-1}\right) \times 2 + \left(\frac{3}{4n}\right)\left(\frac{1}{4n-1}\right) \times 2$$

$$= \frac{8n - 8 + 6}{4n\left(4n-1\right)}$$

$$= \frac{1}{2n}$$

(ii)

$$\frac{1}{2n} = \frac{1}{8}$$
$$n = 4$$

$$\text{P}(W = -2)$$
$$= \text{P}(\text{Score} < 4)$$
$$= \text{P}(1,1) + \text{P}(1,2) + \text{P}(2,1)$$
$$= \left(\frac{12}{16}\right)\left(\frac{11}{15}\right) + \left(\frac{12}{16}\right)\left(\frac{3}{15}\right) \times 2$$
$$= \frac{17}{20}$$

$$\text{P}(W = 0)$$
$$= \text{P}(\text{Score} = 4)$$
$$= \text{P}(2,2)$$
$$= \left(\frac{3}{16}\right)\left(\frac{2}{15}\right)$$
$$= \frac{1}{40}$$

Hence, the probability distribution of W is

W	-2	0	10
$\text{P}(W = w)$	$\frac{17}{20}$	$\frac{1}{40}$	$\frac{1}{8}$

(iii)

$$\text{E}(W) = (-2)\left(\frac{17}{20}\right) + (0)\left(\frac{1}{40}\right) + (10)\left(\frac{1}{8}\right)$$
$$= -\frac{9}{20}$$

$$\text{E}(W^2) = (-2)^2\left(\frac{17}{20}\right) + (0)^2\left(\frac{1}{40}\right) + (10)^2\left(\frac{1}{8}\right)$$
$$= \frac{159}{10}$$

$$\text{Var}(W) = \text{E}(W^2) - [\text{E}(W)]^2$$
$$= \frac{159}{10} - \left(-\frac{9}{20}\right)^2$$
$$= \frac{6279}{400} \text{ or } 15.6975 \text{ (exact)}$$

Since $\text{E}(W) = -\frac{9}{20} < 0$, it is expected that she will lose money. Hence, Kathryn should not play the game.

(iv)

Since $n = 50$ is large, by Central Limit Theorem, $\overline{W} \sim \mathrm{N}\left(-\frac{9}{20}, \frac{15.6975}{50}\right)$ approximately

$\mathrm{P}\left(\overline{W} \le 1\right) = 0.99517 \approx 0.995$

Q11

(i)

Given $\mathrm{P}(A|B) = \frac{7}{10}$

$$\frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} = \frac{7}{10}$$

$\therefore\ \mathrm{P}(B) = \frac{10}{7}\mathrm{P}(A \cap B)$ -----(1)

Given $\mathrm{P}(B|A) = \frac{4}{15}$

$$\frac{\mathrm{P}(A \cap B)}{\mathrm{P}(A)} = \frac{4}{15}$$

$\therefore\ \mathrm{P}(A) = \frac{15}{4}\mathrm{P}(A \cap B)$ -----(2)

Given that $\mathrm{P}(A \cup B) = \frac{3}{5}$

$\mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B) = \frac{3}{5}$ -----(3)

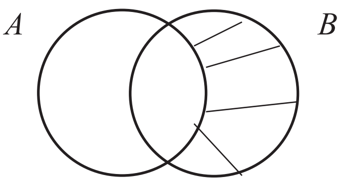
Substitute (1) and (2) into (3):

$$\frac{15}{4}\mathrm{P}(A \cap B) + \frac{10}{7}\mathrm{P}(A \cap B) - \mathrm{P}(A \cap B) = \frac{3}{5}$$

$\therefore\ \mathrm{P}(A \cap B) = \frac{28}{195}$

(ii)

$$\mathrm{P}(A' \cap B) = \mathrm{P}(B) - \mathrm{P}(A \cap B)$$
$$= \frac{10}{7}\left(\frac{28}{195}\right) - \frac{28}{195} = \frac{4}{65}$$



Alternative Method 1:

$$\mathrm{P}(A' \cap B) = \mathrm{P}(A \cup B) - \mathrm{P}(A) = \frac{3}{5} - \left(\frac{15}{4} \times \frac{28}{195}\right) = \frac{4}{65}$$

Alternative Method 2:

$$\mathrm{P}(A'|B) = 1 - \mathrm{P}(A|B) = 1 - \frac{7}{10} = \frac{3}{10}$$

$$\therefore\ \mathrm{P}(A' \cap B) = \mathrm{P}(A'|B) \times \mathrm{P}(B) = \frac{3}{10} \times \left(\frac{10}{7} \times \frac{28}{195}\right) = \frac{4}{65}$$

(iii)

$\text{P}(A) = \frac{15}{4}\left(\frac{28}{195}\right) = \frac{7}{13}$

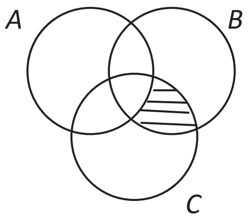
Since events A and C are independent,

$\text{P}(A' \cap C) = \text{P}(A') \times \text{P}(C) = \left(1 - \frac{7}{13}\right) \times \frac{3}{10} = \frac{9}{65}$

(iv)

$\text{P}(A' \cap B \cap C) \leq \text{P}(A' \cap C)$

$\therefore\ 0 \leq \text{P}(A' \cap B \cap C) \leq \frac{9}{65}$

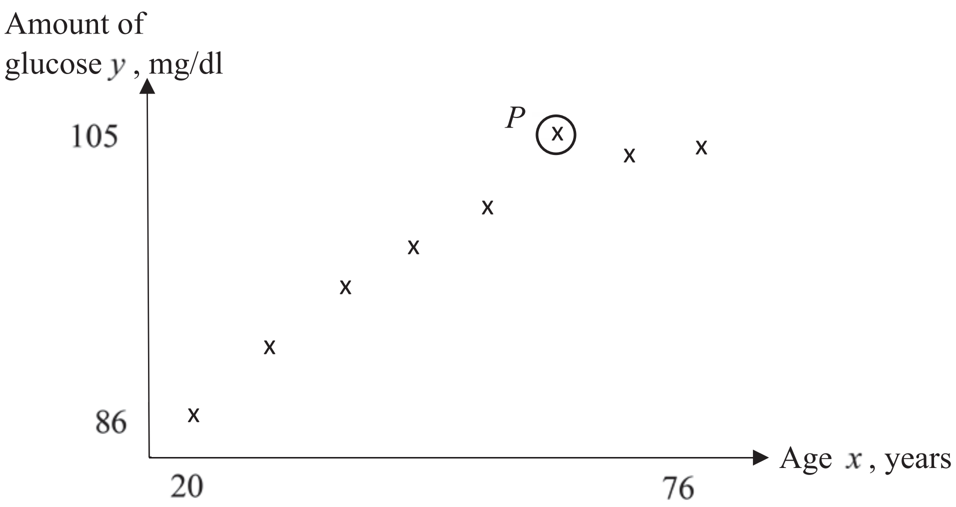


Q12

(i)

$r = 0.954$. Since the r value is close to 1, there is a strong positive linear correlation between x and y. A linear model may be appropriate.

(ii) and (iii)



(iv)

From the scatter diagram, excluding P, it can be observed that as x increases, y also increases but at a decreasing rate, hence a linear model $y = ax + b$ may not be appropriate.

(v)

From GC,

$r = 0.998$

$y = 44.281 + 13.972 \ln x$

$c = 44.3 \qquad d = 14.0$

(vi)

When $x = 60$, $y = 44.281 + 13.972 \ln 60 = 101.5$ (nearest 0.5)

$r = 0.998$ is close to 1. $x = 60$ lies within the given data range, hence interpolation is being done. The linear model $y = a + b \ln x$ still holds, hence, the estimate is reliable.

(vii)

From GC, mean of $\ln x$ = 3.7864

$\overline{y} = 43.942 + 14.079(3.7864) = 97.251$

$\overline{y} = 97.251 = \frac{1}{8}(86.0 + 90.5 + 94.5 + 97.5 + 100 + 103.5 + 104 + y)$

$y = 102.0$ (nearest 0.5)