RAFFLES JUNIOR COLLEGE
JC1 Promotion Examination 2008

MATHEMATICS **9740**

Higher 2 **September 2008**

3 hours

Additional materials : Answer Paper
List of Formulae (MF15)

READ THESE INSTRUCTIONS FIRST

Write your name, exam number and CT group on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

Can do the **whole** question.	✓
Can do **part** of question only.	✓.

1 Find

(i) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(3^x\right)$, [1]

(ii) $\int 3^x \,\mathrm{d}x$. [1]

Hence find $\int x3^{x+1}\,\mathrm{d}x$. [3]

2 The functions f and g are defined by

$$\mathrm{f}: x \to \frac{x-2}{x+2}, \quad x \in \mathbb{R},\ x \neq -2,$$

$$\mathrm{g}: x \to -x^2 \quad, \ x \in \mathbb{R},\ x < a.$$

(i) Sketch the graph of f. [1]

(ii) Find f^{-1}, the inverse function of f. [3]

(iii) Find the largest value of a so that the composite function fg exists. [2]

3 Two concentric circles have radii R and r, where $R > r$.
R increases at a constant rate of 1 cm per minute and the area between the two circles remains constant at 25cm^2. At the instant when $R = 5\,\text{cm}$, find the exact rate of increase of

(i) the area of the smaller circle, [2]

(ii) r. [3]

4 State precisely a sequence of 2 geometric transformations on the graph of $y = \mathrm{g}(x)$ that would result in the graph of $y = \mathrm{g}(4 - x)$. [2]

The graph of $y = \mathrm{g}(x)$ has asymptotes $x = 2$ and $y = 4 - x$. It also has a maximum point whose x-coordinate is 3. Given that $\mathrm{g}(x) = \mathrm{g}(4 - x)$ for all real values of x in the domain of g, sketch one possible graph of $y = \mathrm{g}(x)$.
[*Your sketch should demonstrate clearly all the appropriate features of the graph that show that the given conditions are satisfied.*] [3]

5 (a) Find $\int_{-1}^{1} |x| e^{x^3}\,\mathrm{d}x$. [1]

(b) Find $\int x\cos(x^2)\,\mathrm{d}x$. [2]

(c) Use the substitution $u = \sqrt{x+1}$ to find the exact value of $\int_3^8 \frac{1}{x\sqrt{x+1}}\,\mathrm{d}x$. [4]

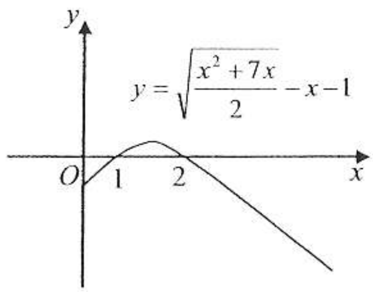
6 A sequence of real numbers $x_1, x_2, x_3, \ldots$ satisfies the recurrence relation

$$x_{n+1} = \sqrt{\frac{x_n^2 + 7x_n}{2}} - 1.$$

(i) Prove algebraically that, if the sequence converges, then it converges to either 1 or 2. **[2]**

(ii) Determine the range of the function $f(x) = \sqrt{\frac{x^2 + 7x}{2}} - 1,\ 1 \le x \le 2$. **[1]**

(iii) The diagram below shows a portion of the graph of $y = \sqrt{\frac{x^2 + 7x}{2}} - x - 1$, for $x \ge 0$.



When $x_1 = 1.001$, by considering $x_{n+1} - x_n$ and using the graph given above, show that $x_{n+1} > x_n$ for all integers $n \ge 1$. **[2]**

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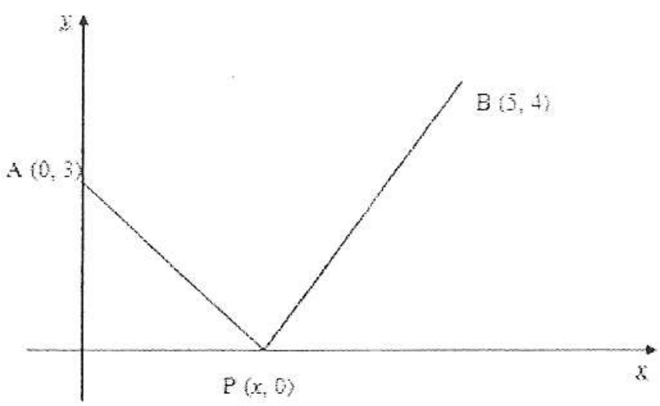


The above diagram shows the points $A(0,\ 3)$ and $B(5,\ 4)$. P is a variable point with coordinates $(x,\ 0)$, where $x > 0$. Let $f(x)$ be the length of the path APB (i.e. $AP + PB$).

Show that

$$f(x) = \sqrt{x^2 + 9} + \sqrt{(5 - x)^2 + 16}.$$ **[1]**

By differentiation and without using a graphic calculator, find the value of x that minimizes $f(x)$. Hence find the minimum length of the path APB. **[5]**

8 In an experiment, glucose is infused into the bloodstream at a constant rate of B mg/min. Glucose is also converted and removed from the bloodstream at a rate proportional to the amount of glucose, x mg, present in the bloodstream at time t minutes after the start of the experiment. If $x = 150$, the amount of glucose in the bloodstream remains constant.

Show that

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{B}{150}(150 - x). \qquad [2]$$

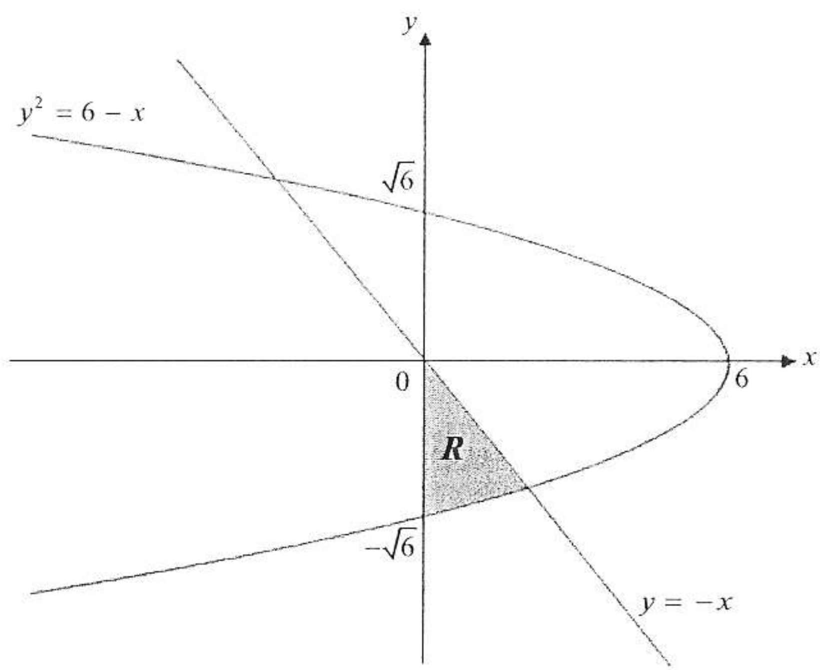
Find x in terms of B and t given that the bloodstream contains 70mg of glucose when $t = 0$. **[4]**

9 The diagram shows the graphs of $y^2 = 6 - x$ and $y = -x$.

(i) State the coordinates of their points of intersection. **[1]**

(ii) Find the area of the finite region bounded by the 2 graphs. **[3]**

(iii) The region R is bounded by the line $y = -x$, a part of the curve $y^2 = 6 - x$ and the y-axis, as shown in the diagram. Find the volume of the solid formed when R is rotated through 2π radians about the y-axis, giving your answer correct to two decimal places. **[3]**



10 **(i)** Without using a calculator, solve the inequality $7+\frac{4}{x}>\frac{22}{x+1}$, leaving your answer in exact form. **[4]**

(ii) Hence, solve the inequality $7+4\mathrm{e}^{x}>\frac{22}{\mathrm{e}^{-x}+1}$, leaving your answer in exact form. **[3]**

11 The n[th] term of a geometric progression is $x^{n-1}(x+1)^{n}$, where $x\neq -1$, $x\neq 0$.

(i) Determine the range of values of x for which S, the sum to infinity exists. **[3]**

(ii) Given $x=\frac{1}{5}$, find

(a) the exact value of S, **[2]**

(b) the least value of n for which $\left|S_n-S\right|$ is less than 10^{-6}, where S_n is the sum of the first n terms of the progression. **[2]**

12 Given that $y=\mathrm{e}^{-\sqrt{1+x}}$, $x\geq -1$, show that

$$4(x+1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2\frac{\mathrm{d}y}{\mathrm{d}x}-y=0.$$

Find the Maclaurin's series for y up to and including the term in x^3. **[6]**

Hence find the expansion for $\mathrm{e}^{2x-\sqrt{1+x}}$ up to and including the term in x^3. **[2]**

13 **(i)** Express $\frac{1}{(2r+1)(2r+3)}$ in the form $\frac{A}{2r+1}+\frac{B}{2r+3}$ where A and B are constants to be determined. **[2]**

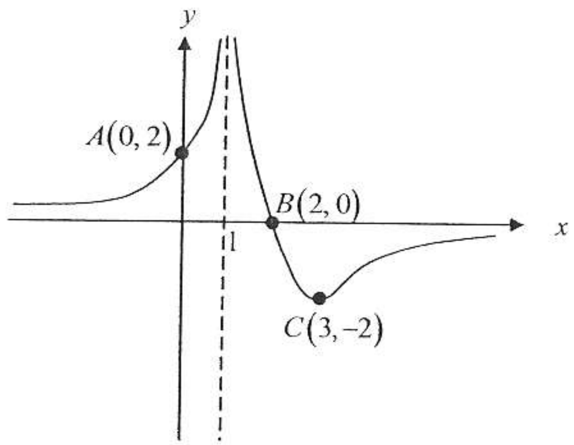
(ii) Hence find the sum of the series

$$\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\frac{1}{9.11}+\frac{1}{11.13}+\ldots+\frac{1}{(2n+1).(2n+3)},$$

giving your answer in the form $k-\mathrm{f}(n)$, where k is a constant. **[4]**

(iii) Hence find the sum of $\frac{1}{2.7.9}+\frac{1}{2.9.11}+\frac{1}{2.11.13}+\ldots$ **[2]**

14



The diagram shows the graph of $y = \mathrm{f}(x)$. The graph passes through the points $A\ (0, 2)$, $B\ (2, 0)$ and $C\ (3, -2)$. The equations of the asymptotes are $y = 0$ and $x = 1$.

(i) Sketch, on separate clearly labeled diagrams, the graphs of

(a) $y = \mathrm{f}'(x)$, **[3]**

(b) $y = \dfrac{1}{\mathrm{f}(x)}$, **[3]**

(c) $y = -\sqrt{\mathrm{f}(x)}$. **[3]**

(ii) Copy the above sketch of $y = \mathrm{f}(x)$ and by drawing a sketch of another suitable curve on the same diagram, state the number of real roots of the equation $\left[\mathrm{f}(x)\right]^2 - x + 1 = 0$. **[2]**

15 A sequence of real numbers $v_1, v_2, v_3, \ldots$ is defined by the following equation

$$v_n = \frac{1}{n}\sum_{r=1}^{n}(-1)^{r+1}r^2, \quad n \geq 1.$$

(i) Calculate the values of v_n for $n = 1$, 2, 3 and 4. **[2]**

(ii) Conjecture a formula for v_n in terms of n and prove your conjecture by mathematical induction. **[5]**

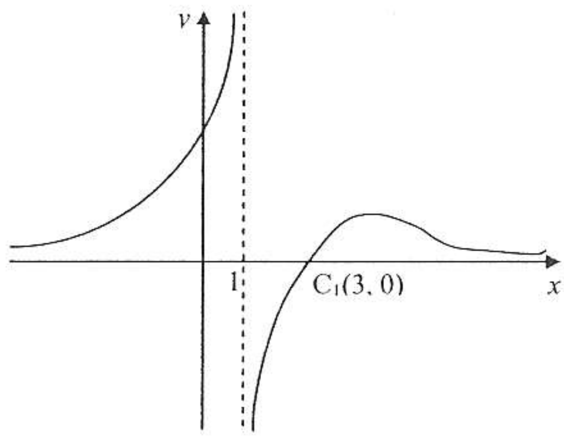
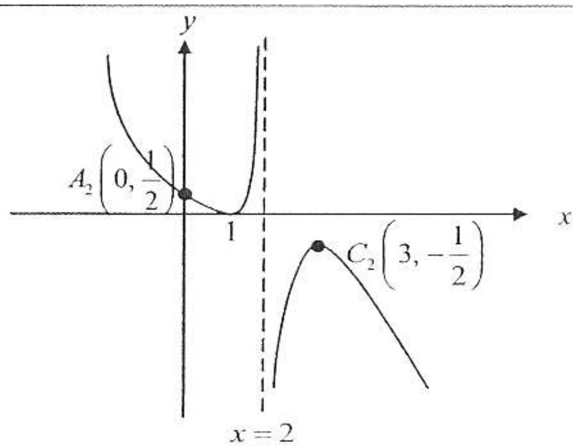
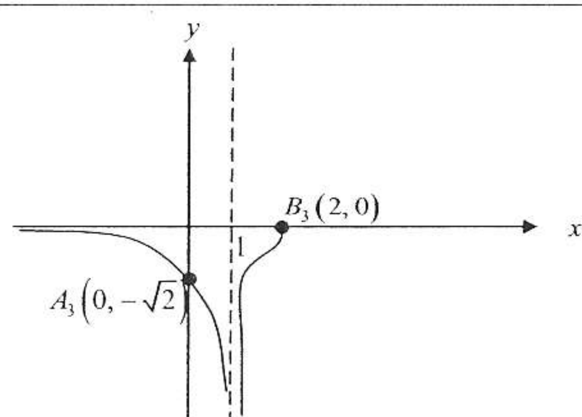
END OF PAPER

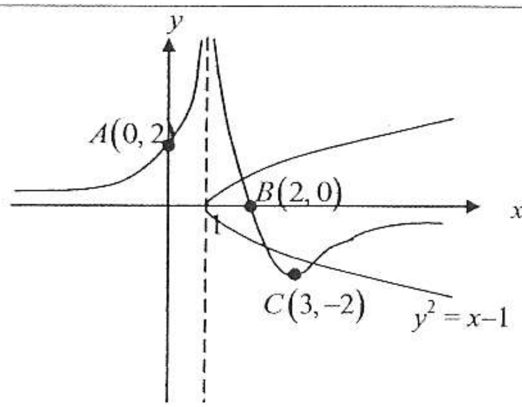
Raffles Junior College
H2 Mathematics
Year 5 (2008)

2008 JC1 H2 Mathematics Promotion Examination Solutions

1(i)	$3^x \ln 3$
(ii)	$\frac{1}{\ln 3}(3^x)+c$
(iii)	$\frac{x}{\ln 3}3^{x+1}-\frac{1}{(\ln 3)^2}3^{x+1}+c$
2(i)	Note that the range of f $=\mathbb{R}\setminus\{1\}$
(ii)	$\mathrm{f}^{-1}: x \to \frac{2(1+x)}{1-x},\ x\in\mathbb{R},\ x\neq 1$ or $\mathrm{f}^{-1}: x \to \frac{4}{1-x}-,\ x\in\mathbb{R}, x\neq 1$
(iii)	$a=-\sqrt{2}$.
3(i)	Hence, required rate $=10\pi\ \text{cm}^2/\text{min}$. □
(ii)	required rate $=\sqrt{\frac{\pi}{\pi-1}}$ cm/min.
4	**<u>EITHER</u>** I. Translate the graph 4 units in the negative x-direction, then II. Reflect the graph about/in the y-axis **<u>OR</u>** I. Reflect the graph about/in the y-axis, then II. Translate the graph 4 units in the positive x-direction.
	<u>Graph of $y = g(x)$</u>: Any graph that **1.** has asymptotes $x = 2$, $y = 4 - x$ and a maximum point $(3, y_0)$, $y_0 < 1$. **2.** is symmetric about the line $x = 2$. **3.** has asymptotes $y = x$, (with all 3 asymptotes intersecting at (2, 2)) **4.** has a maximum point $(1, y_0)$, $y_0 < 1$.
5(a)	1.13 (to 3 s.f) (from GC)
(b)	$\frac{1}{2}\sin x^2+c$

(c)	$\ln\frac{3}{2}$
6(i)	$l = 1$ or 2.
(ii)	The range of the function is[1,2].
7	Hence f(x) is minimum when $x = 15/7$, and on substituting x gives $f(x) = \sqrt{74}$ or 8.60 (to 3.s.f)
8	$x = 150 - 80e^{-\frac{B}{150}t}$
9(i)	Coordinates of the points of intersection are (−3, 3) and (2, −2).
(ii)	$\frac{125}{6}$ units^2 or 20.8 units^2 (to 3 s.f.)
(ii)	Volume of R= 10.36 units^3 (to 2 d.p.)
10(i)	$x < -1$ or $0 < x < \frac{4}{7}$ or $x > 1$
(ii)	$x < 0$ or $x > -\ln\left(\frac{4}{7}\right)$
11(i)	$r = x(x+1)$; $-1.62 < x < 0.618$, $x \neq -1, 0$. (rounded off to 3 s.f.)
(ii)(a)	$r = \frac{6}{25}$ or 0.24; $\frac{30}{19}$
(ii)(b)	Least n = 11
12	$e^{2x-\sqrt{1+x}} \approx \left(e^{-1}\right)\left(1 + \frac{3}{2}x + \frac{5}{4}x^2 + \frac{11}{16}x^3\right)$
13(i)	$A = \frac{1}{2}$; $B = -\frac{1}{2}$
(ii)	$\frac{1}{6} - \frac{1}{2}\left(\frac{1}{2n+3}\right)$
(iii)	$\frac{1}{28}$

14(a)(i)	y 1 C1(3, 0) x
(ii)	y $A_2\left(0, \frac{1}{2}\right)$ 1 $C_2\left(3, -\frac{1}{2}\right)$ x $x = 2$
(iii)	y $B_3\left(2, 0\right)$ 1 $A_3\left(0, -\sqrt{2}\right)$ x

(b)	 $[f(x)]^2 - x + 1 = 0 \Rightarrow f(x) = \pm\sqrt{x-1}$ There are 3 real roots to the equation. (Since there are 3 points of intersection between graphs of $y = f(x)$ and $y = \pm\sqrt{x-1}$)
15(i)	We have $v_1 = 1$, $v_2 = -\frac{3}{2}$, $v_3 = \frac{6}{3} = 2$, $v_4 = -\frac{10}{4} = -\frac{5}{2}$.
(ii)	Conjecture that $v_n = (-1)^{n+1}\frac{n+1}{2}$.