2012 RI H2 Mathematics Preliminary Examination Paper 1

Qn	Solution
1 **[3]**	$x+y+z=21$ **------ (1)** $3x-z=0$ **------- (2)** $0.05y+0.8z=3(0.9x) \Rightarrow 2.7x-0.05y-0.8z=0$ **----- (3)** Using GC, $x=2.1$, $y=12.6$ and $z=6.3$
2 **[2]** **[3]**	$\frac{1}{(1+2x^2)^2}=(1+2x^2)^{-2}$ $=1+(-2)(2x^2)+\frac{(-2)(-3)}{2!}(2x^2)^2+\frac{(-2)(-3)(-4)}{3!}(2x^2)^3+...$ $=1-4x^2+12x^4-32x^6+...$ Coefficient of x^{2r} is $\frac{(-2)(-3)...(-r-1)}{r!}2^r=\frac{(-1)^r(r+1)!}{r!}2^r=(-1)^r(r+1)2^r$ Expansion is valid for $\left\|2x^2\right\|<1$ $\Rightarrow x^2<\frac{1}{2}$ $\Rightarrow -\frac{1}{\sqrt{2}}<x<\frac{1}{\sqrt{2}}$ $\therefore$ Range of validity is $\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$.
3 **[6]**	$w^2=a^2-b^2+2ab\mathrm{i}=5+12\mathrm{i}$ $a^2-b^2=5$ (1) $2ab=12$ (2) Sub $a=\frac{6}{b}$ in (1) $\left(\frac{6}{b}\right)^2-b^2=5 \Rightarrow \left(b^2\right)^2+5\left(b^2\right)-36=0$ $\Rightarrow \left(b^2+9\right)\left(b^2-4\right)=0$ $\Rightarrow b=2$ or -2 $b=2, a=3$ or $b=-2, a=-3$ $w=3+2\mathrm{i}$ or $-\left(3+2\mathrm{i}\right)$
	$z^2-z-\left(1+3\mathrm{i}\right)=0$ $z=\frac{1\pm\sqrt{1+4\left(1+3\mathrm{i}\right)}}{2}=\frac{1\pm\sqrt{5+12\mathrm{i}}}{2}$ $=\frac{1\pm\left(3+2\mathrm{i}\right)}{2}$ $z=2+\mathrm{i}$ or $-1-\mathrm{i}$

<table>
<tr><td></td><td colspan="2">(complete the square is acceptable)</td></tr>
<tr><td>4(i)
[6]

(ii)</td><td colspan="2">$u_1 = 812$ and $u_n = 2012 = 812 + (n-1)2$
$n = 601$

$v_1 = 812$ and $v_m = 2002 = 812 + (m-1)14$
$m = 86$

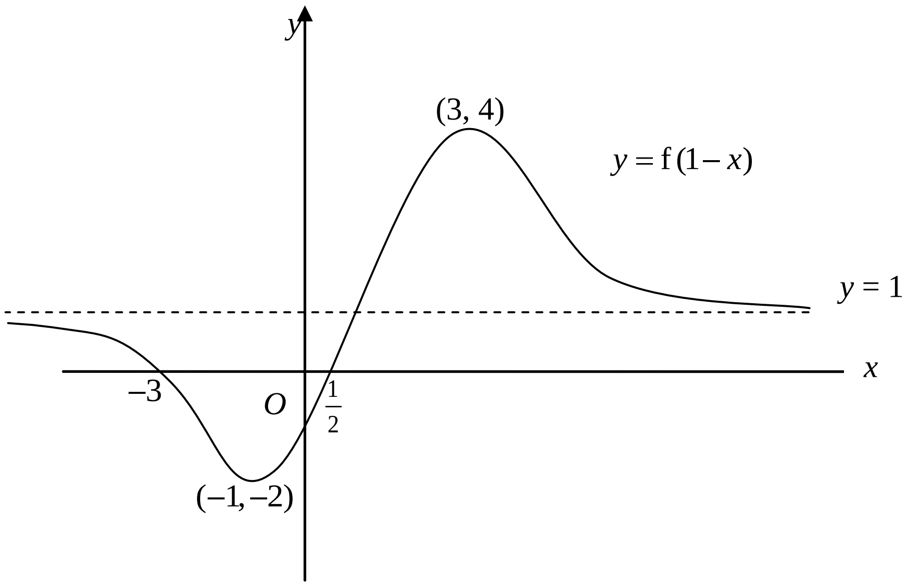
Required sum $= \frac{601}{2}(812+2012) - \frac{86}{2}(812+2002) = 727610$</td></tr>
<tr><td>5(a)
[2]</td><td colspan="2">$\int \ln(2e^{\frac{1}{\sqrt{1-4x^2}}})\,dx = \int \ln 2 + \frac{1}{\sqrt{1-4x^2}}\,dx$
$= \int \ln 2 + \frac{1}{2} \times \frac{2}{\sqrt{1-(2x)^2}}\,dx$
$= x\ln 2 + \frac{1}{2}\sin^{-1}(2x) + c$</td></tr>
<tr><td>5(b)
[5]</td><td>$\int_1^4 \frac{1}{1+2\sqrt{x}+x}\,dx$
$= \int_1^2 \frac{1}{1+2u+u^2}\,2u\,du$
$= \int_1^2 \frac{2u+2}{1+2u+u^2} - \frac{2}{(1+u)^2}\,du$
$= \left[\ln(1+2u+u^2) + \frac{2}{1+u}\right]_1^2$
$= \left(\ln 9 + \frac{2}{3}\right) - (\ln 4 + 1)$
$= \ln\frac{9}{4} - \frac{1}{3} = 2\ln\frac{3}{2} - \frac{1}{3}$</td><td>$u = \sqrt{x}$
$\Rightarrow \frac{du}{dx} = \frac{1}{2\sqrt{x}} = \frac{1}{2u}$
$\Rightarrow \frac{dx}{du} = 2u$

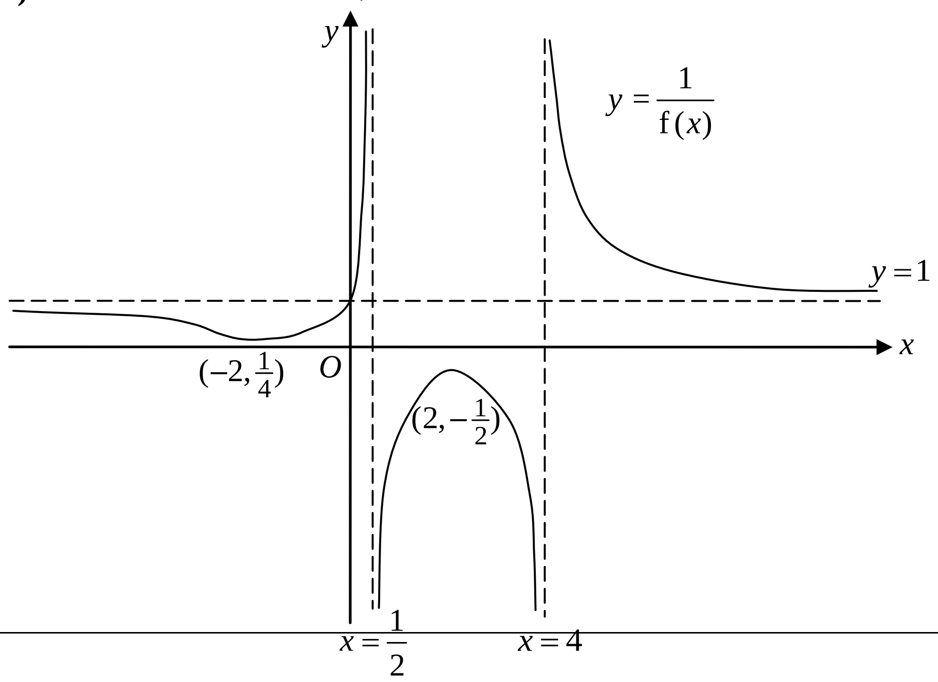
When $x = 1,\ u = \sqrt{1} = 1$
When $x = 4,\ u = \sqrt{4} = 2$

Let $\frac{2u}{(1+u)^2} = \frac{A}{1+u} + \frac{B}{(1+u)^2}$
$\Rightarrow 2u = A(1+u) + B$

Subst $u = -1,\ B = -2$
Subst $u = 0,\ A = -B = 2$</td></tr>
</table>

<table>
<tr><td></td><td>Alternatively,
$\int_1^4 \frac{1}{1+2\sqrt{x}+x}\ \mathrm{d}x$
$=\int_1^2 \frac{1}{1+2u+u^2}2u\ \mathrm{d}u$
$=\int_1^2 \frac{2u}{(1+u)^2}\ \mathrm{d}u$
$=2\int_1^2 \frac{1}{1+u}-\frac{1}{(1+u)^2}\ \mathrm{d}u$
$=2\left[\ln(1+u)+\frac{1}{1+u}\right]_1^2$
$=2\left(\ln 3+\frac{1}{3}\right)-2\left(\ln 2+\frac{1}{2}\right)$
$=2\ln\frac{3}{2}-\frac{1}{3}$</td><td></td></tr>
<tr><td>**6**
[3]

[3]</td><td colspan="2">**(i)**


(ii)

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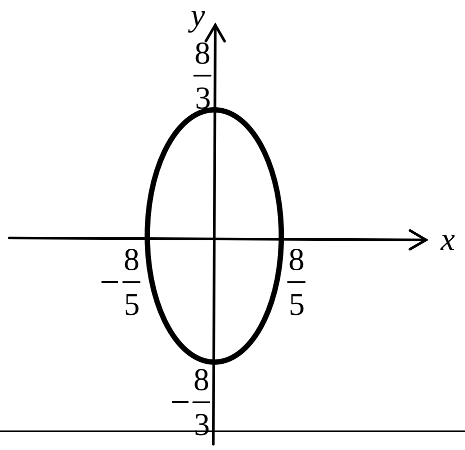
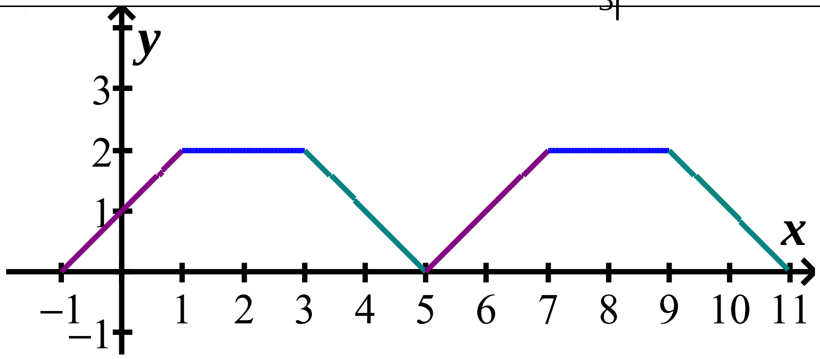
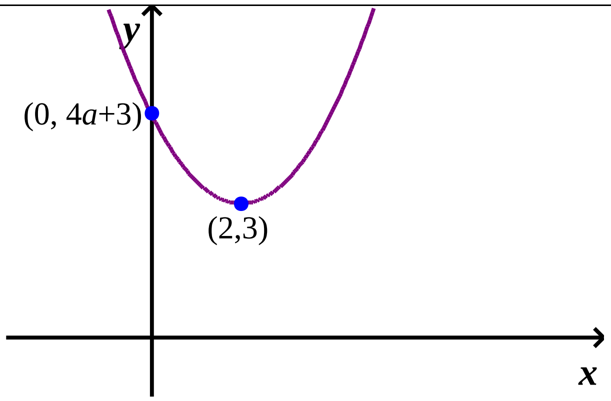
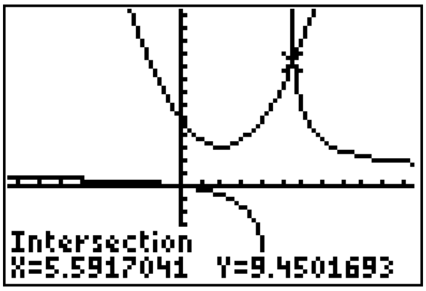
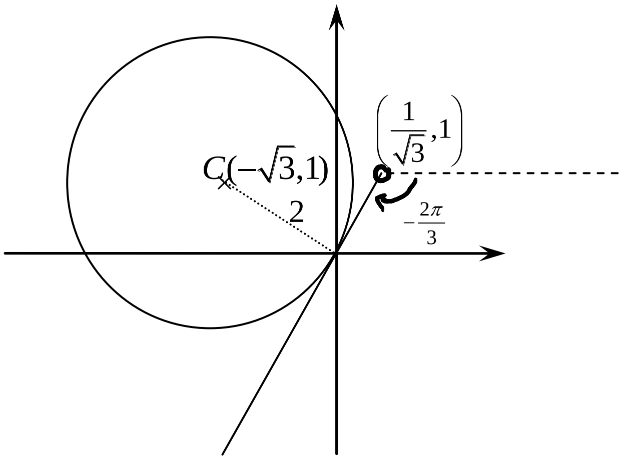
7(a) **[3]**	$\frac{PR}{RQ}=\frac{1}{3}, \quad \overrightarrow{OR}=\frac{1}{4}\left(3\overrightarrow{OP}+\overrightarrow{OQ}\right)$ $=\frac{3}{4}(\underset{\sim}{a}+\underset{\sim}{b})+\frac{1}{4}(3\underset{\sim}{a}-3\underset{\sim}{b})$ $=\frac{3}{2}\underset{\sim}{a}=\frac{3}{2}\overrightarrow{OA}$ $\because \overrightarrow{OR}//\overrightarrow{OA}$ and O is a common point O, A, R are collinear
7bi **[2]**	From GC $x=z$ $y=\frac{3}{2}z$ $z=z$ Cartesian equation is $\frac{x}{2}=\frac{y}{3}=\frac{z}{2}$
7bii **[2]**	Any point (x,y,z) on l has $x=\lambda$, $y=\frac{3\lambda}{2}$ and $z=\lambda$ for some real λ. So $(x-2y+2z)+c(2x-2y+z)$ $=\left(\lambda-2\left(\frac{3\lambda}{2}\right)+2\lambda\right)+c\left(2\lambda-2\left(\frac{3\lambda}{2}\right)+\lambda\right)$ $=0$ for all c Hence l lies in p_3 for all c. OR All points (x,y,z) on l satisfy the equations $x-2y+2z=0$ and $2x-2y+z=0$. So $(x-2y+2z)+c(2x-2y+z)=0+c(0)=0$. Hence l lies in p_3 for all c.
biii **[1]**	$d\neq 0$
8(i) **[5]**	$y=\tan^{-1}(x)$ At $x=-1, y=\tan^{-1}(-1)=-\frac{\pi}{4}$ and $y=-2(-1)-2-\frac{\pi}{4}=-\frac{\pi}{4}$

	Area of region $R=\frac{1}{2}\left(\frac{\pi}{8}\right)\left(\frac{\pi}{4}\right)+\int_{-1}^{0}\left(0-\tan^{-1}(x)\right)\,\mathrm{d}x$ $=\frac{\pi^2}{64}-\left\{\left[x\tan^{-1}(x)\right]_{-1}^{0}-\int_{-1}^{0}\frac{x}{1+x^2}\,\mathrm{d}x\right\}$ $=\frac{\pi^2}{64}-\left\{\left[x\tan^{-1}(x)\right]_{-1}^{0}-\left[\frac{1}{2}\ln\left\|1+x^2\right\|\right]_{-1}^{0}\right\}$ $=\frac{\pi^2}{64}+\frac{\pi}{4}-\frac{1}{2}\ln 2$ OR Area of region $R=\frac{1}{2}\left(1+\frac{\pi}{8}+1\right)\left(\frac{\pi}{4}\right)-\int_{-\frac{\pi}{4}}^{0}\left(0-\tan(y)\right)\,\mathrm{d}y$ $=\frac{\pi^2}{64}+\frac{\pi}{4}+\left[\ln\left\|\sec y\right\|\right]_{-\frac{\pi}{4}}^{0}$ $=\frac{\pi^2}{64}+\frac{\pi}{4}+\left(0-\ln\sqrt{2}\right)$ $=\frac{\pi^2}{64}+\frac{\pi}{4}-\frac{1}{2}\ln 2$
8(ii) **[2]** **[2]**	After transformation, $y=-2(x+\frac{\pi}{8})-2-\frac{\pi}{4}$ $=-2x-2-\frac{\pi}{2}$ After transformation, $y=\tan^{-1}(x+\frac{\pi}{8})$ Volume $=\pi\int_{-\frac{\pi}{4}}^{0}\left(-\frac{y}{2}-1-\frac{\pi}{4}\right)^2-\left(\tan y-\frac{\pi}{8}\right)^2\,\mathrm{d}y$ $=4.35$
9 **[4]**	$\frac{\mathrm{d}y}{\mathrm{d}x}+2y=(x+1)\mathrm{e}^{-2x}$ --- (1) Let $z=y\mathrm{e}^{2x}\Rightarrow\frac{\mathrm{d}z}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{e}^{2x}+2\mathrm{e}^{2x}y$ Sub. into (1), $\frac{\mathrm{d}y}{\mathrm{d}x}+2y=(x+1)\mathrm{e}^{-2x}$ $\Rightarrow\mathrm{e}^{2x}\frac{\mathrm{d}y}{\mathrm{d}x}+2y\mathrm{e}^{2x}=x+1$ $\Rightarrow\frac{\mathrm{d}z}{\mathrm{d}x}=x+1$ $\Rightarrow z=\int(x+1)\,\mathrm{d}x$ $\Rightarrow z=\frac{x^2}{2}+x+c$ $\Rightarrow y=\mathrm{e}^{-2x}\left(\frac{x^2}{2}+x+c\right)$

9(i) **[1]**	When $x=0, y=1, \therefore c=1$. The particular solution is $y=\mathrm{e}^{-2x}\left(\dfrac{x^2}{2}+x+1\right)$.
9(ii) **[4]**	$\dfrac{\mathrm{d}y}{\mathrm{d}x}+2y=(x+1)\mathrm{e}^{-2x}$ $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+2\dfrac{\mathrm{d}y}{\mathrm{d}x}=\mathrm{e}^{-2x}-2(x+1)\mathrm{e}^{-2x}=-(2x+1)\mathrm{e}^{-2x}$ $\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}+2\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=-2\mathrm{e}^{-2x}+2(2x+1)\mathrm{e}^{-2x}=4x\mathrm{e}^{-2x}$ When $x=0, y=1, \dfrac{\mathrm{d}y}{\mathrm{d}x}=-1, \dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=1, \dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}=-2$ Using Maclaurin's expansion, $y=1-x+\dfrac{1}{2!}x^2-\dfrac{2}{3!}x^3+\ldots$ $=1-x+\dfrac{1}{2}x^2-\dfrac{1}{3}x^3+\ldots$
9iii **[2]**	Replace x by $-2x$ in the standard series expansion of e^x and perform an expansion for $\mathrm{e}^{-2x}\left(\dfrac{x^2}{2}+x+1\right)$ up to and including the term in x^3. Compare the coefficients of this series with that of the expansion in **(ii)** to verify the correctness.
10a **[7]**	$V=\dfrac{1}{3}\pi r^2h=120 \Rightarrow h=\dfrac{360}{\pi r^2}$ From diagram, $r^2+h^2=l^2$ $S=\pi rl$ $S^2=\pi^2r^2l^2$ $=\pi^2r^2(r^2+h^2)$ $=\pi^2r^2\left(r^2+\dfrac{360^2}{\pi^2r^4}\right)$ $=\pi^2r^4+\dfrac{129600}{r^2}$ Differentiate w.r.t. r, we get $2S\dfrac{\mathrm{d}S}{\mathrm{d}r}=4\pi^2r^3-\dfrac{259200}{r^3}$ For stationary values of S, set $\dfrac{\mathrm{d}S}{\mathrm{d}r}=0$ $4\pi^2r^3=\dfrac{259200}{r^3} \Rightarrow r^6=\dfrac{64800}{\pi^2}$ $\Rightarrow r=\dfrac{\sqrt[6]{64800}}{\sqrt[3]{\pi}}=4.33$ (3 s.f.) and $h=\dfrac{360}{\pi r^2}=6.11$ (3 s.f.)
10b **i** **[3]**	$\dfrac{\mathrm{d}y}{\mathrm{d}x}=-\dfrac{3\cos t}{5\sin t}$ Equation of l

	$y-3\sin t=\dfrac{5\sin t}{3\cos t}(x-5\cos t)$ $y=\dfrac{5}{3}x\tan t-\dfrac{25}{3}\sin t+3\sin t$ or $y=\dfrac{5}{3}x\tan t-\dfrac{16}{3}\sin t$
10b ii [4]	At A, $y=0.\therefore x=\dfrac{16}{5}\cos t$ At B, $x=0.\therefore y=-\dfrac{16}{3}\sin t$ Mid-point of AB, M has coordinates $\left(\dfrac{8}{5}\cos t,-\dfrac{8}{3}\sin t\right)$ Let $x=\dfrac{8}{5}\cos t$ and $y=-\dfrac{8}{3}\sin t$ Then $\dfrac{x^2}{\left(\frac{8}{5}\right)^2}+\dfrac{y^2}{\left(\frac{8}{3}\right)^2}=1$ $\Leftrightarrow\ 25x^2+9y^2=64$ 
11a i [2]	
11a ii [2]	$\int_o^a \mathrm{h}(x)\,\mathrm{d}x=\dfrac{63}{2}$ $\int_{-1}^{a}\mathrm{h}(x)\mathrm{d}x=\dfrac{63}{2}+\dfrac{1}{2}=32$ $\int_{-1}^{a}\mathrm{h}(x)\mathrm{d}x=4(8)$ $a=-1+4(6)=23$

11b i [2]	 $y=a(x-2)^2+3$
11b ii [3]	In order for gk to exist, $R_k \subseteq D_g$ Domain of g is $(-\infty,5)\cup(5,\infty,)$ $k(x)=a(x-2)^2+3,\ x\geq 7$ Range of k is $[25a+3,\infty)$ So, $25a+3>5$ $a>\frac{2}{25}$ or 0.08
11b iii [3]	 When $f(x)>g(x)$, $x<5$ or $x>5.59$ (3 s.f.) Replace x with $-x$, $x>-5$ or $x<-5.59$ (3 s.f.)
12(a) [4]	$zz^{*}=\left\|\left(\sqrt{3}+i\right)^{\frac{2}{7}}\right\|^2=\left\|\sqrt{3}+i\right\|^{\frac{4}{7}}$ $=2^{\frac{4}{7}}$ $\arg z=\arg\left(\left(\sqrt{3}+i\right)^{\frac{2}{7}}\right)$ $=\frac{2}{7}\arg\left(\sqrt{3}+i\right)$ $=\frac{2}{7}\left(\frac{\pi}{6}\right)=\frac{\pi}{21}$

(b) **[5]**	$\arg\left(\mathrm{i}w-\frac{\mathrm{i}}{\sqrt{3}}+1\right)=-\frac{\pi}{6}$ $\arg\left(\mathrm{i}\left(w-\frac{1}{\sqrt{3}}-i\right)\right)=-\frac{\pi}{6}$ $\arg\left(w-\left(\frac{\sqrt{3}}{3}+i\right)\right)=-\frac{\pi}{2}-\frac{\pi}{6}=-\frac{2\pi}{3}$ 
12b i **[2]**	Least value of $\left\|z-\frac{1}{\sqrt{3}}-\mathrm{i}\right\|$ $=\frac{1}{\sqrt{3}}-\left(-\sqrt{3}\right)-2$ $=\frac{4\sqrt{3}}{3}-2$
12b ii **[2]**	$\frac{2\pi}{3}\le\arg\left(z-\frac{1}{\sqrt{3}}-\mathrm{i}\right)\le\pi$ or $-\pi<\arg\left(z-\frac{1}{\sqrt{3}}-\mathrm{i}\right)\le-\frac{2\pi}{3}$