VICTORIA JUNIOR COLLEGE

2023 PROMO PRACTICE PAPER C

(Modified from 2020 VJC H2 Math PROMO)

- Exam conditions (one sitting, 3 hours)
- Manage your time well
- Check against solutions and learn from your mistakes before the next practice

1 RVHS Promo 9758/2020/Q2(i)

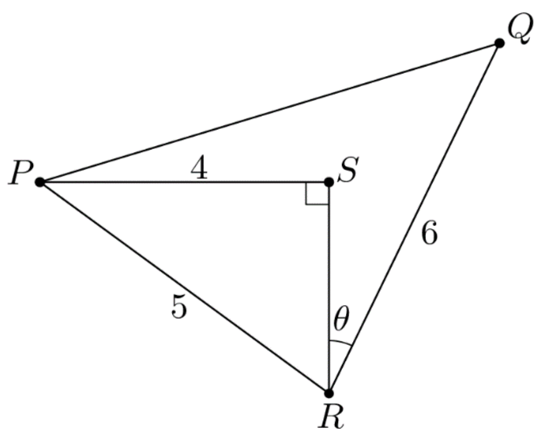
Show that $x^2+x+3>0$ for all real values of x.
Without the use of a calculator, find the range of values of x that satisfy

$$\frac{x^2+2x+6}{x^2-x-12} \geq \frac{1}{x-4}.$$ [4]

2 YIJC Prelim 9758/2020/02/Q2(a)

Given that $z=\dfrac{\lambda-4\mathrm{i}}{1-\lambda\mathrm{i}}$, $\lambda \in \mathbb{R}$ and $\arg(z)=\pi$, find the value of z. [4]

3 NJC Prelim 9758/2018/01/Q7(b)



In the diagram above, $QR=6$, $PS=4$, $PR=5$, $\angle PSR=\dfrac{\pi}{2}$ and $\angle QRS=\theta$ radians.

(i) Show that $PQ=\left(61-36\cos\theta+48\sin\theta\right)^{\frac{1}{2}}$. [2]

(ii) Given that θ is a sufficiently small angle, show that

$$PQ \approx 5+p\theta+q\theta^2$$

for some rational constants p and q to be determined exactly. [4]

4 It is given that $\dfrac{\mathrm{d}y}{\mathrm{d}x}+xy=\mathrm{e}^{-x}$ and $y=1$ when $x=0$.

(i) Find the Maclaurin series for y, up to and including the term in x^3. [5]

(ii) Hence, or otherwise, state the first three non-zero terms of the Maclaurin series for $\dfrac{\mathrm{d}y}{\mathrm{d}x}$. [1]

5 (a) Find

(i) $\int \frac{1}{x^2+2x+5}\,dx$, [2]

(ii) $\int \frac{(\ln x)^3}{x}\,dx$. [2]

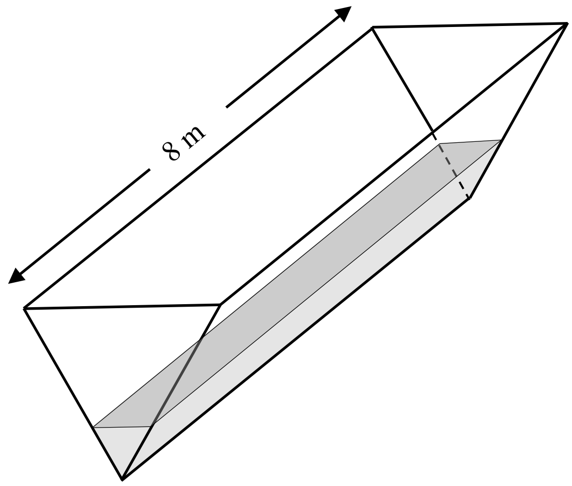
(b) Find the exact value of $\int_0^{\frac{\pi}{2}} x\cos x\,dx$. [3]

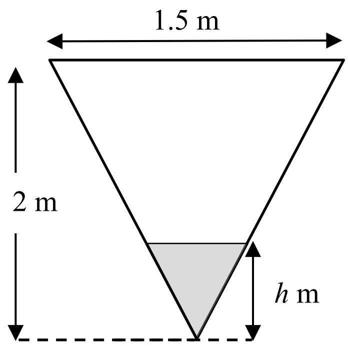
6 The curve C has equation $x^2+ay^2+bx+cy=0$ where a, b and c are constants.

(a) Given that C passes through the point $(5,-3)$ and that the tangent to C at the point $\left(\frac{1}{2},-\frac{3}{2}\right)$ is parallel to the y-axis, find the values of a, b and c. [4]

(b) Given **instead** that $a=-1$, $b=-10$ and $c=-6$, sketch C. [4]

7 The diagram below shows a drain in the form of a triangular prism 8 m long and 2 m deep. The opening of the drain is 1.5 m wide.





Cross section of drain

During a heavy downpour, rainwater flows into the drain at a constant rate of 0.03 m^3 per second. At the same time, the rainwater is drained away at a constant rate of 0.02 m^3 per second. At time t seconds after the start, the depth of the rainwater in the drain is h m.

(i) Show that the volume of rainwater in the drain, V m^3, is given by $V=3h^2$. [3]

(ii) Find the rate at which h is increasing when $V=7.2$. [4]

8 An infinite geometric series has first term a and common ratio r, where $r>0$. The third term is 36 and the sum to infinity is 243.

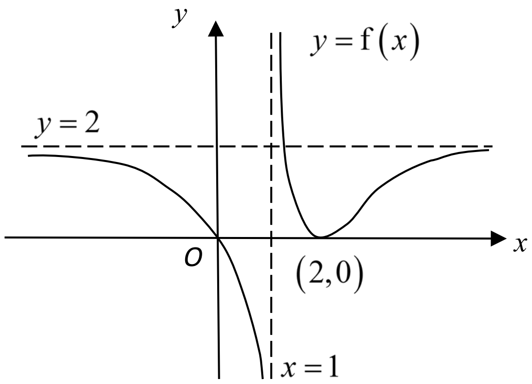
(i) Find the value of a and r. [3]

An arithmetic series has first term 1 and common difference d. The sum of the first 6 terms of the arithmetic series is equal to the sum of the first 3 terms of the geometric series.

(ii) Find the value of d. [3]

(iii) Find the least value of n for which the nth term of the arithmetic series is more than the sum of the first $2n$ terms of the geometric series. [3]

9 **(a)** The graph of $y = \mathrm{f}(x)$ is shown in the diagram. It passes through the origin and has a minimum point at (2, 0). The lines $x = 1$ and $y = 2$ are the asymptotes of the graph.



Sketch the graph of $y = 2 - \mathrm{f}(x)$, stating the coordinates of the point where the graph crosses the y-axis, the coordinates of any stationary point(s) and the equations of any asymptotes. [2]

(b) The curve C has equation $y = \dfrac{2x^2 + kx - 2}{x+2}$, where k is a constant.

(i) Find the set of values of k for which C has 2 stationary points. [3]

For the rest of the question, take $k = -3$.

(ii) By expressing the equation $y = \dfrac{2x^2 - 3x - 2}{x+2}$ in the form $y = ax + b + \dfrac{c}{x+2}$, where a, b and c are constants, write down the equations of the asymptotes of C. [3]

(iii) Hence sketch C, giving the equations of any asymptotes, the coordinates of any stationary points and of the points where C crosses the axes. [2]

(iv) Describe a pair of transformations which transform the graph of $y = x + \dfrac{6}{x+2}$ to C. [2]

10 It is given that the complex number $w = -\left(\sqrt{3}\right) - \mathrm{i}$.

(a) Find the value of $|w|$. [1]

(b) Given that a is a real number, find $|w + a\mathrm{i}|$ in terms of a. [2]

(c) Given that b is a real number, find the imaginary part of $w + \dfrac{b\mathrm{i}}{w}$ in terms of b. [3]

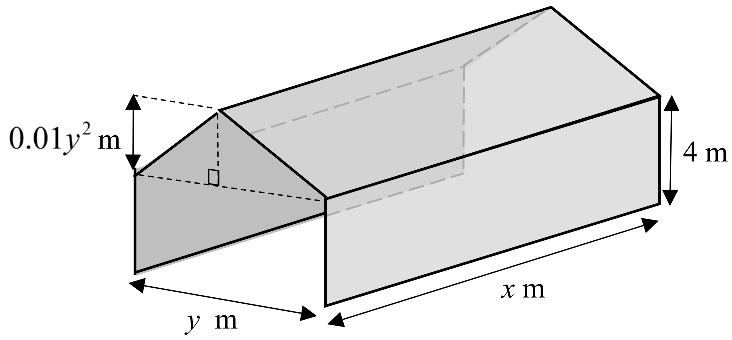
(d) Find a cubic equation where all its coefficients are real numbers and where two of its roots are 4 and w, giving the exact values of the coefficients. [3]

(e) **(i)** Find the exact value of $\arg(w)$. [1]

(ii) Without using a calculator, find the three smallest positive whole number values of n such that $w^n w^*$ is a real number. [3]

11 To combat a particular infectious disease, the government planned to build a temporary structure to house patients. This structure comprises two identical rectangular vertical walls and a roof, which is formed by two identical rectangular metal sheets as shown in the diagram below. The structure has a length of x m and width y m, and a total floor area of 2000 m^2. The height of the vertical walls is 4 m, and the roof adds another $0.01y^2$ m to the overall height of this structure.

You may assume that the vertical walls and metal sheets of the roof are of negligible thickness.



(i) Express y in terms of x and show that the total external surface area S m^2 of the vertical walls and the roof is given by

$$S = 8x + 2000\sqrt{\frac{1600}{x^2} + 1}\,. \qquad [4]$$

It is desired that S should be as small as possible.

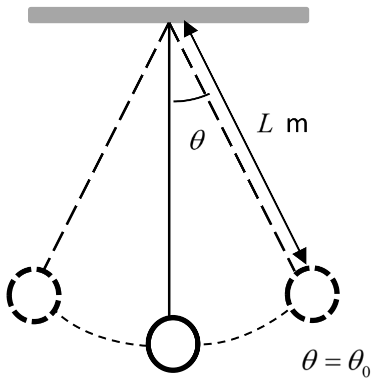
(ii) Given that $x = x_1$ is the value of x which gives a stationary value of S, show that x_1 satisfies the equation $x^6 + 1600x^4 - 1.6 \times 10^{11} = 0$. [3]

(iii) Hence, find the minimum value of S, showing that this value is a minimum. [3]

Due to space constraint on the site where the structure is to be built, an additional requirement is imposed such that the length of the structure is at most two times its width.

(iv) Find the smallest possible value of S under this additional requirement. [2]

12 A pendulum is a weight suspended from a fixed point so that it can swing freely. The period of a pendulum is the time it takes for the pendulum to make a complete back-and-forth swing and θ radians is the angle of the pendulum measured from the vertical. The diagram below shows a pendulum of length L m whose initial position is at θ_0 radians from the vertical.



The period T seconds is given by

$$T = 4\sqrt{\frac{L}{2g}}\int_0^{\theta_0} \frac{1}{\sqrt{\cos\theta - \cos\theta_0}}\,\mathrm{d}\theta,$$

where $0 \leqslant \theta \leqslant \theta_0$ and g is the acceleration due to gravity.

(i) Show that if $\sin x = \dfrac{\sin\frac{\theta}{2}}{\sin\frac{\theta_0}{2}}$, where $0 \leqslant x \leqslant \dfrac{\pi}{2}$, then

$$\cos x = \frac{\sqrt{\cos\theta - \cos\theta_0}}{\left(\sqrt{2}\right)\sin\frac{\theta_0}{2}} \quad \text{and} \quad \frac{\mathrm{d}x}{\mathrm{d}\theta} = \frac{\cos\frac{\theta}{2}}{\left(\sqrt{2}\right)\sqrt{\cos\theta - \cos\theta_0}}.$$ [4]

(ii) Hence use the substitution $\sin x = \dfrac{\sin\frac{\theta}{2}}{\sin\frac{\theta_0}{2}}$ to show that the period T can be expressed in the form

$$T = 4\sqrt{\frac{L}{g}}\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - k^2\sin^2 x}}\,\mathrm{d}x,$$

where $k = \sin\left(\dfrac{\theta_0}{2}\right)$. [3]

(iii) Find the first two non-zero terms of the series expansion of $\dfrac{1}{\sqrt{1-y}}$. [1]

(iv) Given that k is sufficiently small, use your answer in part **(iii)** to show that

$$T \approx 4\sqrt{\frac{L}{g}}\int_0^{\frac{\pi}{2}} \left(1 + \frac{1}{a}k^2\sin^2 x\right)\mathrm{d}x,$$

where a is a positive constant to be determined. [1]

(v) Hence show that T is approximately $2\pi\sqrt{\dfrac{L}{g}}\left(1 + \dfrac{k^2}{4}\right)$. [3]