CANDIDATE NAME: ……………………………………………………………………

CLASS: ……………………… INDEX NUMBER: …….........

ADDITIONAL MATHEMATICS **4049/01**

Paper 1 **November 2024**

Secondary 4 Express **2 hours 15 minutes**

Setter: itzpipey :)))

Candidates answer on the question paper.
No Additional Materials are required.

READ THESE INSTRUCTIONS FIRST

Write your name, class, and index number in the spaces at the top of this page.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved scientific calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 90.

90

This document consists of **19** printed pages and **1** blank page.

MATHEMATICAL FORMULAE

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Binomial expansion

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\ldots(n-r+1)}{r!}$

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$
$$\sec^2 A = 1 + \tan^2 A$$
$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$
$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$
$$\sin 2A = 2 \sin A \cos A$$
$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$
$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$
$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$\Delta = \frac{1}{2} bc \sin A$$

1 **(a)** Find the equation of the normal to the curve $y = tan(\frac{x}{2} - \pi)$ at the origin. [3]

$$\frac{dy}{dx} = \frac{1}{2}\sec^2\left(\frac{x}{2} - \pi\right)$$

$$= \frac{1}{2\cos^2\left(\frac{x}{2} - \pi\right)} \quad M1$$

at the origin, $x = 0$, $\frac{dy}{dx} = \frac{1}{2\cos^2(0 - \pi)}$

$$= \frac{1}{2\cos^2(-\pi)}$$

$$= \frac{1}{2(-1)^2}$$

$$= \frac{1}{2}$$

grad of normal $= -\frac{1}{\frac{1}{2}} = -2$ M1

eqn of normal: $y - 0 = -2(x - 0)$

$y = -2x$ // A1

(b) A particle moves along the curve. When $x = \frac{\pi}{3}$, y is decreasing at a rate of π $units/s$. Find the rate of change of x at this instant. [3]

$$\frac{dy}{dt} = -\pi \quad M1$$

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

$$-\pi = \frac{1}{2\cos^2\left(\frac{\pi}{6} - \pi\right)} \cdot \frac{dx}{dt} \quad M1$$

$$\frac{dx}{dt} = -2\pi\cos^2\left(-\frac{5\pi}{6}\right)$$

$$= -2\pi\left(-\frac{\sqrt{3}}{2}\right)^2$$

$$= -\frac{3}{2}\pi$$

$$= -4.71238$$

$= -4.71$ units/s // A1

2 Show that the curve $(m-1)x^2 - 2y^2 = mxy + 1$ and the line $y = mx + 1$ will never intersect for all values of m. [5]

sub 2nd eqn into 1st:

$(m-1)x^2 - 2(mx+1)^2 = mx(mx+1)+1$

$(m-1)x^2 - 2(m^2x^2+2mx+1) = m^2x^2+mx+1$

$(m-1-2m^2-m^2)x^2 - 3mx - 1 = 0$

$(-3m^2+m-1)x^2 - 3mx - 1 = 0$ M1 for quadratic format

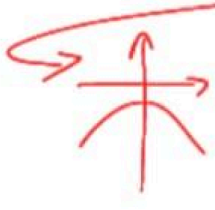
$b^2-4ac = (-3m)^2 - 4(-3m^2+m-1)(-1)$

$= 9m^2 - 12m^2 + 4m - 4$

$= -3m^2+4m-4$ M1

$\hookrightarrow b^2-4ac = 4^2 - 4(-3)(-4)$

$= -32 < 0$ M1



frown shaped graph with no roots hence $-3m^2+4m-4$ is $\boxed{\text{ALWAYS}}$ → M1 $\boxed{\text{NEGATIVE}}$ hence b^2-4ac (the first one) < 0, $\therefore$ there are no values of m such that the curve and line intersect // (shown) A1

3 **(a)** Given that $log_3 p = q$ and $log_3 q = r$, express $log_{27}(pq^6)$ in terms of r in the **simplest form.** [4]

$\log_3 p = q \qquad \log_3 q = r$

$q = 3^r$ M1

$$\log_{27}(pq^6) = \frac{\log_3(pq^6)}{\log_3 27} \quad \text{M1}$$

$$= \frac{\log_3 p + \log_3(q^6)}{\log_3(3^3)} \quad \text{M1 for simplifying and substituting}$$

$$= \frac{q + 6\log_3 q}{3}$$

$$= \frac{3^r + 6r}{3}$$

$$= 3^{r-1} + 2r \text{ // A1}$$

(b) Solve $lg(x^2 + 1) + ln(x^2 + 1) = e$. [5]

$$\lg(x^2+1) + \frac{\lg(x^2+1)}{\lg e} = e \quad \text{M1}$$

accept change all bases to e too! :)

$$\lg(x^2+1)\left(1 + \frac{1}{\lg e}\right) = e$$

$$\lg(x^2+1) = \frac{e}{1+\frac{1}{\lg e}} \quad \text{M1}$$

$$x^2+1 = 10^{\frac{e}{1+\frac{1}{\lg e}}} \quad \text{M1}$$

$$x = \pm\sqrt{10^{\frac{e}{1+\frac{1}{\lg e}}} - 1} \quad \text{M1}$$

$x = 2.37779$ or $x = -2.37779$

$= 2.38$ // (3sf) $\quad = -2.38$ // (3sf) } A1

4 **(a)** Explain why all the powers of x in the binomial expansion of $(2x - \frac{1}{ax})^9$ are odd. [3]

$T_{r+1} = \binom{9}{r}(2x)^{9-r}\left(-\frac{1}{ax}\right)^r$ M1

$= \binom{9}{r}(2)^{9-r}(x)^{9-r}\left(-\frac{1}{a}\right)^r(x^{-1})^r$

$= \binom{9}{r}(2)^{9-r}\left(-\frac{1}{a}\right)^r(x)^{9-2r}$ M1

the powers of x are given by $9-2r$. since r is a positive integer, $2r$ is even. Since odd − even is always odd, $9-2r$ is always odd, ∴ all powers of x in expansion are odd //

A1 for justification

(b) Find the range of values of a such that the term independent of x in the expansion of $(\frac{a}{x^7} + \frac{1}{x^5})(2x - \frac{1}{ax})^9$ has a positive coefficient. [5]

for x^7 term, $9-2r=7$

$2r=2$

$r=1$

coeff. of x^7 term $= \binom{9}{1}(2)^{9-1}\left(-\frac{1}{a}\right)^1$

$= -\frac{2304}{a}$

for x^5 term, $9-2r=5$

$2r=4$

$r=2$

coeff. of x^5 term $= \binom{9}{2}(2)^{9-2}\left(-\frac{1}{a}\right)^2$

$= \frac{4608}{a^2}$

M1

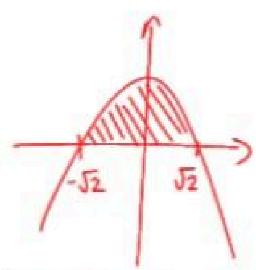
term indep. of x in $\left(\frac{a}{x^7}+\frac{1}{x^5}\right)\left(2x-\frac{1}{ax}\right)^9 = \left(\frac{-2304}{a}\right)(a) + \left(\frac{4608}{a^2}\right)(1)$

$= \frac{4608}{a^2} - 2304$ M1

for term indep. of x to have +ve coeff, $\frac{4608}{a^2} - 2304 > 0$

$2 - a^2 > 0$

$(\sqrt{2}+a)(\sqrt{2}-a) > 0$ M1



(frowney as coeff. of a^2 is −ve)

$\underline{-\sqrt{2} < a < \sqrt{2}}$, AND $a \neq 0$ (else binomial undefined)

M1

$\boxed{\therefore -\sqrt{2} < a < \sqrt{2},\ a \neq 0 //}$ A1

(alt: $-\sqrt{2} < a < 0$, $0 < a < \sqrt{2}$)

5 For the function $f(x) = px^3 + p^2x^2 + 18x - 54$, where p is a constant, the remainder when $f(x)$ is divided by $x - 1$ is the same as when it is divided by $x + 2$.

(a) Find the possible values of p. [3]

$$f(1) = f(-2)$$

$$p(1^3) + p^2(1^2) + 18(1) - 54 = p(-2)^3 + p^2(-2)^2 + 18(-2) - 54$$

$$3p^2 - 9p - 54 = 0$$

$$p^2 - 3p - 18 = 0 \quad M1$$

$$(p-6)(p+3) = 0$$

$$p - 6 = 0 \quad \text{or} \quad p + 3 = 0$$

$$p = 6 \;/\!/\; A1 \qquad p = -3 \;/\!/\; A1$$

(b) Given further that $f(x)$ is divisible by $x + \sqrt{6}$, find $f(x)$. [3]

if $f(x)$ divisible by $x+\sqrt{6}$, $f(-\sqrt{6})=0$

if $p=6$, $f(-\sqrt{6}) = (6)(-\sqrt{6})^3 + (6^2)(-\sqrt{6})^2 + 18(-\sqrt{6}) - 54 = 29.72755 \neq 0$

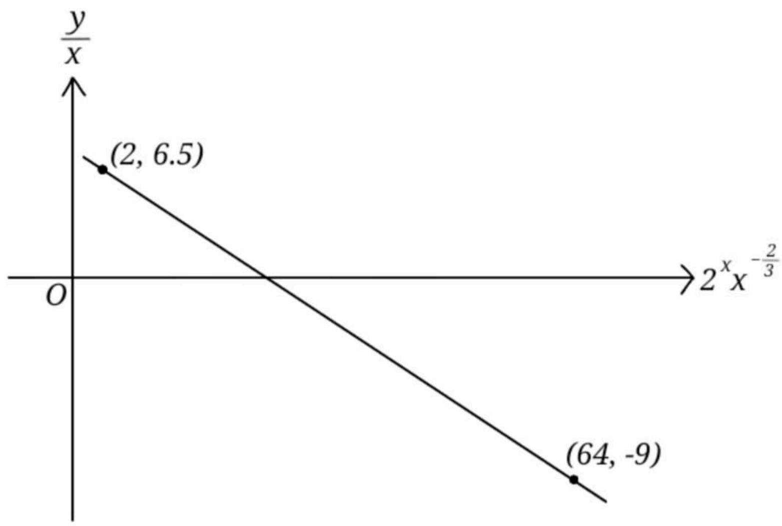
hence $p \neq 6$ M1

if $p=-3$, $f(-\sqrt{6}) = (-3)(-\sqrt{6})^3 + (-3)^2(-\sqrt{6})^2 + 18(-\sqrt{6}) - 54 = 0$

hence $p=-3$ M1

$\therefore f(x) = -3x^3 + (-3)^2x^2 + 18x - 54 = -3x^3 + 9x^2 + 18x - 54$ // A1

6



(a) The variables x and y are connected by the equation $2^{3x-m}x = (nx - y)^3$ where m and n are constants. The diagram shows the straight line graph obtained by plotting $\frac{y}{x}$ against $2^x x^{-\frac{2}{3}}$, of which $(2,\ 6.5)$ and $(64, -\ 9)$ are points on. Find the values of m and n and hence express y in terms of x. [6]

$$2^{3x-m}x = (nx-y)^3$$

$$2^{x-\frac{m}{3}}x^{\frac{1}{3}} = nx - y$$

$$(2^x)(2^{-\frac{m}{3}})(x^{-\frac{2}{3}}) = n - \frac{y}{x}$$

$$\frac{y}{x} = -2^{-\frac{m}{3}}(2^x x^{-\frac{2}{3}}) + n \quad M1$$

$$\text{Let } -2^{-\frac{m}{3}} = k$$

$$6.5 = 2k + n \text{ — } ①$$
$$-9 = 64k + n \text{ — } ② \quad M1$$

$$② - ①: 62k = -\frac{31}{2}$$

$$k = -\frac{1}{4} \text{ — } ③ \quad M1$$

$$-2^{-\frac{m}{3}} = -\frac{1}{4}$$

$$2^{-\frac{m}{3}} = 2^{-2}$$

$$-\frac{m}{3} = -2$$

$$m = 6 \quad A1$$

$$\text{Sub } ③ \text{ in } ①: 6.5 = 2(-\tfrac{1}{4}) + n$$

$$n = 7 \quad A1$$

$$\text{So, } 2^{x-\frac{6}{3}}x^{\frac{1}{3}} = 7x - y$$

$$y = 7x - 2^{x-2}x^{\frac{1}{3}} \quad A1$$

(b) V and t are connected by the equation $V = Ae^{kt+1}$. Explain how a straight line graph may be drawn using given values of V and t and how the values of A and k may be found using the straight line graph. [4]

$\ln V = \ln(Ae^{kt+1})$

$\ln V = \ln A + \ln e^{kt+1}$

$\ln V = \ln A + kt + 1$

$\ln V = kt + \ln A + 1$ M1 A1

$\ln V$ can be plotted against t. k can be calculated by measuring the graph's gradient. A1 A can be found like this: $\ln A + 1 =$ vertical axis intercept (call this VAI)

$\ln A = VAI - 1$

$A = e^{VAI - 1}$ A1

7 A particle moves in a straight line such that its velocity, $v\ m/s$, at time t seconds for a few seconds is given by the equation $v = -\frac{2}{pt+3}$, where p is a negative constant.

(a) Express the particle's acceleration in terms of p and t. Explain what the expression says about the particle's movement. Show all workings. [3]

$$a = \frac{dv}{dt} = -\left(-\frac{2p}{(pt+3)^2}\right)$$

$$= \frac{2p}{(pt+3)^2} \text{ A1}$$

$$(pt+3)^2 > 0$$

$$\frac{2}{(pt+3)^2} > 0$$

$$\frac{2p}{(pt+3)^2} < 0$$

$$a < 0 \text{ M1}$$

∴ the particle is decelerating / slowing down // A1

(b) **(i)** Express the particle's displacement from a fixed point O in terms of p and t, with c as an arbitrary constant of integration. [1]

$$s = \int -\frac{2}{pt+3}\, dt$$

$$= -\frac{2}{p}\ln(pt+3) + c \text{ // A1}$$

(ii) The particle starts with a displacement of ln 243 metres from O. Given now that $c = 0$, find p. [1]

when $t = 0$, $s = \ln 243$

$$-\frac{2}{p}\ln(0+3) = \ln(3^5)$$

$$-\frac{2}{p}\ln 3 = 5\ln 3$$

$$-\frac{2}{p} = 5$$

$$p = -\frac{2}{5} \text{ A1}$$

(c) Using your value of p obtained in part **b(ii)**,

(i) find the time at which the particle crosses O, [2]

$$s=0$$
$$-\frac{2}{p}\ln(pt+3)=0$$
$$\ln\left(-\frac{2}{5}t+3\right)=0 \quad \text{M1}$$
$$-\frac{2}{5}t+3=e^{0}$$
$$-\frac{2}{5}t=-2$$
$$t=5 \,/\!/ \quad \text{A1}$$

(ii) show that the particle does not change directions from $t=0$ to $t=7$, [2]

if $v=0$

$$-\frac{2}{pt+3}=0$$
$$-2=0 \quad \text{M1}$$

↳ false!

hence $v\neq 0$, the particle does not come to instantaneous rest, hence it does not change directions // A1

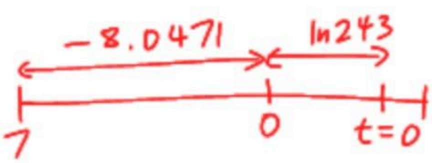
alt. mtd.

$$0\le t\le 7$$
$$0\ge -\frac{2}{5}t\ge -\frac{14}{5}$$
$$3\ge -\frac{2}{5}t+3\ge \frac{1}{5}$$
$$\frac{1}{3}\le \frac{1}{-\frac{2}{5}t+3}\le 5$$
$$-\frac{2}{3}\ge -\frac{2}{-\frac{2}{5}t+3}\ge -10$$
$$-\frac{2}{3}\ge v\ge -10 \text{ when } 0\le t\le 7 \quad \text{M1}$$

since v remains $-$ve, it does not change direction // A1

(iii) find the distance travelled by the particle in the first 7 seconds. [2]

$$\text{when } t=7,\ s=-\frac{2}{-\frac{2}{5}}\ln\left(-\frac{2}{5}(7)+3\right)$$
$$=-8.0471 \quad \text{M1}$$



$$\text{dist. travelled} = \ln 243 + 8.0471$$
$$=13.540$$
$$=13.5\text{ m} \,/\!/ \ (3\text{sf}) \quad \text{A1}$$

8 **(a)** Show that $cosec^2(67.5°) = 4 - 2\sqrt{2}$. [5]

$$\cos 45° = 2\cos^2 22.5° - 1 \quad \text{M1}$$

$$2\cos^2 22.5° = \frac{\sqrt{2}}{2} + 1$$

$$= \frac{\sqrt{2}+2}{2} \quad \text{M1}$$

$$\cos^2 22.5° = \frac{\sqrt{2}+2}{4}$$

$$\cos^2 22.5° = \sin^2(90° - 22.5°) = \sin^2(67.5°)$$

$$\text{so } \sin^2 67.5° = \frac{\sqrt{2}+2}{4} \quad \text{M1}$$

$$\operatorname{cosec}^2 67.5° = \frac{4}{\sqrt{2}+2} \times \frac{\sqrt{2}-2}{\sqrt{2}-2}$$

$$= \frac{4\sqrt{2}-8}{2-4} \quad \text{M1}$$

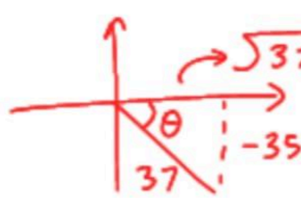
$$= \frac{4\sqrt{2}-8}{-2}$$

$$= 4 - 2\sqrt{2} \text{ // (shown)} \quad \text{A1}$$

(b) Showing your workings fully, find the **exact** value of $7cot[sin^{-1}(-\frac{35}{37})]$. [2]

$$-\frac{\pi}{2} \le \sin^{-1}\left(\frac{-35}{37}\right) \le \frac{\pi}{2}$$

since $-\frac{35}{37} < 0$, 4th quad



$$\sqrt{37^2 - (-35)^2} = 12 \quad \text{M1}$$

$$7\cot\left[\sin^{-1}\left(-\frac{35}{37}\right)\right] = 7\left(\frac{12}{-35}\right)$$

$$= -\frac{12}{5} \text{ //} \quad \text{A1}$$

(c) Solve $5sin^2A + cosA = 2sin(90° - 2A)$, where $-\frac{\pi}{2} \le A \le \frac{\pi}{2}$. [5]

$5\sin^2 A + \cos A = 2\cos 2A$

$5(1-\cos^2 A) + \cos A = 2(2\cos^2 A - 1)$

$9\cos^2 A - \cos A - 7 = 0$ M1

$\cos A = \dfrac{-(-1) \pm \sqrt{(-1)^2 - 4(9)(-7)}}{2(9)}$ M1

$\cos A = 0.93922$ M1 or $\cos A = -0.82811$

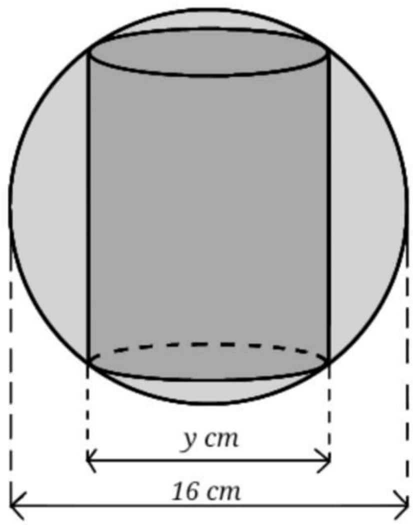
$\alpha = \cos^{-1}(0.93922)$ ↳ rej, $\cos A > 0$

$= 0.350445$ M1

$\therefore A = -0.350445, 0.350445$

$= -0.350, 0.350$ // (3sf) A1

9

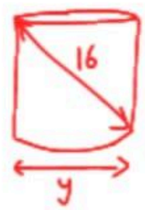


A cylinder of diameter $y\ cm$ is inscribed in and is to be cut out of a plastic sphere of diameter 16 cm.

(a) After the cylinder is cut out, there will be some waste remains from the sphere. Show that the volume, V, of these remains can be expressed as

$$V = \pi\left(\frac{2048}{3} - \frac{\sqrt{256y^4 - y^6}}{4}\right)$$

[4]



by pyth. thm., height of cylinder $= \sqrt{16^2 - y^2}$

$= \sqrt{256 - y^2}$ M1

$V = \underline{\frac{4}{3}\pi\left(\frac{16}{2}\right)^3}$ M1 $\underline{- \pi\left(\frac{y}{2}\right)^2\left(\sqrt{256 - y^2}\right)}$ M1

$= \pi\left(\frac{2048}{3} - \frac{y^2}{4}\sqrt{256 - y^2}\right)$

$= \pi\left(\frac{2048}{3} - \frac{\sqrt{256y^4 - y^6}}{4}\right)$ // (shown) A1

(b) Given that y can vary, find the optimal volume of the cylinder to be cut out such that minimal waste is produced. Explain why the value you obtain is optimal. [7]

$$\frac{dV}{dy} = -\pi\left(\frac{1024y^3 - 6y^5}{8\sqrt{256y^4 - y^6}}\right) \quad \text{M1}$$

for stationary values of V, $-\pi\left(\frac{1024y^3-6y^5}{8\sqrt{256y^4-y^6}}\right) = 0$

$$1024y^3 - 6y^5 = 0$$

$$y^3(1024 - 6y^2) = 0 \quad \text{M1}$$

$y^3 = 0$ or $1024 - 6y^2 = 0$

$y = 0$ (rej, $y>0$) M1

$y^2 = \frac{512}{3}$

$y = \sqrt{\frac{512}{3}}$ M1 or $y = -\sqrt{\frac{512}{3}}$ (rej, $y>0$) M1

y	$\sqrt{\frac{512}{3}}-0.1$	$\sqrt{\frac{512}{3}}$	$\sqrt{\frac{512}{3}}+0.1$
$\frac{dV}{dy}$	-8.47	0	8.944
sketch of V against y	\	—	/

$\Rightarrow V$ is minimal when $y = \sqrt{\frac{512}{3}}$ M1

when $y = \sqrt{\frac{512}{3}}$, vol. of cylinder $= \pi\left(\frac{\sqrt{\frac{512}{3}}}{2}\right)^2\left[\sqrt{256 - \left(\sqrt{\frac{512}{3}}\right)^2}\right]$

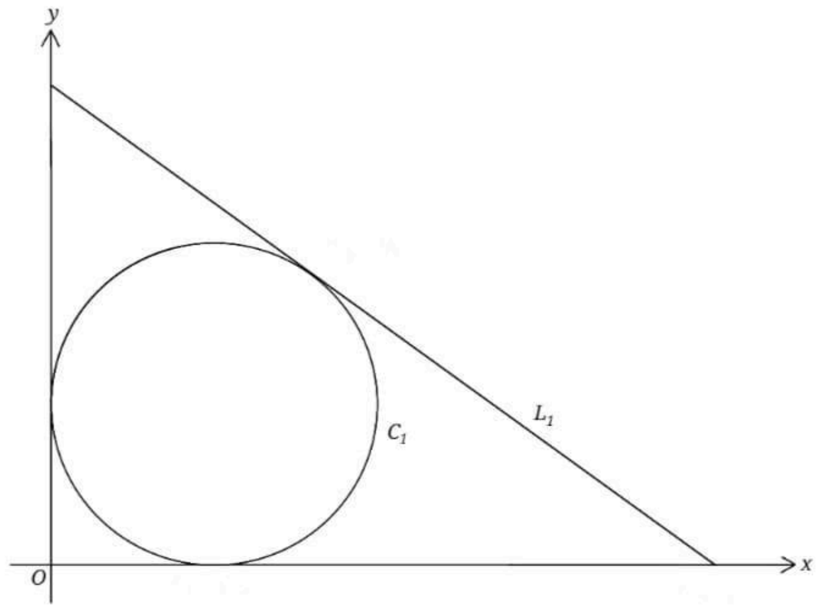
$= 1238.220$

$= 1240\text{ cm}^3$ (3sf)

with this value of y, V has a minimum point, hence waste produced is minimal and optimal

A1

10



A cylindrical barrel rests on its side on the floor against a wall. These are represented in the diagram above by the circle C_1, the x-axis and the y-axis respectively. A straight rod is situated on the side of the cylinder such that it lies in the same plane as C_1 and can be represented by the line L_1. The barrel has a cross-sectional area of $100\pi\ cm^2$.

(a) Find the equation of C_1. [2]

let radius be r

$100\pi = \pi r^2$

$r = \pm 10$

$r = 10$ (rej $-$ve as $r > 0$) M1

since y-axis and x-axis tangent to C_1,

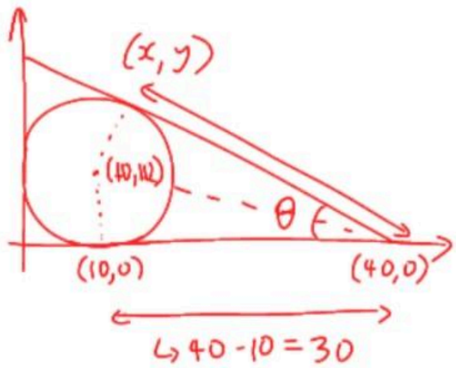
C_1 centre is $(10, 10)$

C_1 eqn: $(x-10)^2 + (y-10)^2 = 10^2$

$(x-10)^2 + (y-10)^2 = 100$ A1

(also accept $x^2 + y^2 - 20x - 20y + 100 = 0$)

(b) Find the angle at which the rod is lifted off the floor at. [4]



(x,y), $(10,10)$, $(10,0)$ and $(40,0)$ form a kite M1 for mention of this or angle being bisected

$\text{dist}_{(10,10) \text{ to } (10,0)} = \text{dist}_{(x,y) \text{ to } (10,10)}$ (radii of circle)

$\text{dist}_{(x,y) \text{ to } (40,0)} = \text{dist}_{(10,0 \text{ to } 40,0)}$ (tan. from ext. pt.) M1

let angle rod lifted off floor from be θ,

$\frac{1}{2}\theta = \tan^{-1}\left(\frac{10}{30}\right)$ M1

$\theta = 2\tan^{-1}\left(\frac{1}{3}\right)$

$\theta = 0.64350$

$= 0.644\text{rad}$ A1

(or $36.9°$)

(c) Find the equation of L_1. [2]

gradient of $L_1 = -\tan\theta$

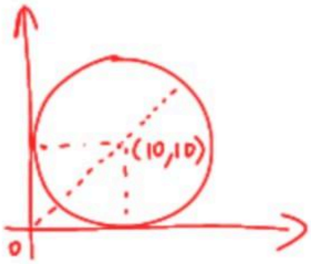
$= -\tan\left(2\tan^{-1}\left(\frac{1}{3}\right)\right)$

$= -\frac{3}{4}$ M1

eqn of L_1 : $y - 0 = -\frac{3}{4}(x - 40)$

$y = -\frac{3}{4}x + 30$ A1

(d) Find the **exact** coordinates of the point on C_1 that is the furthest away from the origin O. [4]



let this point be (x, y)
this pt. should lie on the circle's diameter that crosses O when extended

eq'n : $y - 0 = \frac{10-0}{10-0}(x-0)$

$y = x$ M1

Sub $y = x$ into C_1 eq'n: $(x-10)^2 + (x-10)^2 = 100$

$2(x-10)^2 = 100$

$(x-10)^2 = 50$

$x - 10 = \pm\sqrt{50}$

$x = 10 \pm 5\sqrt{2}$ M1

(cannot be $10 - 5\sqrt{2}$ as $x > 10$, so rej) M1

$x = 10 + 5\sqrt{2}$

since $y = x$, $y = 10 + 5\sqrt{2}$

$\therefore$ the point on C_1 furthest from O is $(10+5\sqrt{2}, 10+5\sqrt{2})$ A1

END OF PAPER

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