Chapter 6: Techniques of Differentiation

Content Outline

- Differentiation of simple functions
- Differentiation of simple functions defined implicitly or parametrically
- Finding the approximate value of a derivative at a given point using a graphing calculator

Derivatives listed in MF26

$f(x)$	$f'(x)$
$\sin^{-1} x$	$\dfrac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-\dfrac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\dfrac{1}{1+x^2}$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\sec x$	$\sec x \tan x$

References

- http://www.h2maths.site
  [Demonstration on various differentiation techniques through keying in different expressions.]
- http://www.calculus-help.com/tutorials/
  [Animated demonstration on various differentiation techniques.]
- http://www.mathbits.com/MathBits/TISection/Openpage.htm
  [Using TI Graphing Calculator in differentiation.]
- AS: Use of Maths – Calculus (Publisher: Nelson Thornes)
- Calculus DeMystified – by Steven G. Krantz
- Calculus – The Easy Way – by Douglas Downing

Prerequisite

Secondary school knowledge of calculus (differentiation), algebra and coordinate geometry.

Introduction: What is Calculus?

Calculus is the mathematics of motion and change, which is why calculus is a prerequisite for many courses. Whenever we move from the static to the dynamic, we would consider using calculus. In the 17th century, calculus was developed and researched in attempt to answer some fundamental questions about the world and the way things work. These investigations led to two fundamental concepts of calculus – derivative and integral. The breakthrough in the development of these concepts was the formulation of a mathematical tool called a limit.

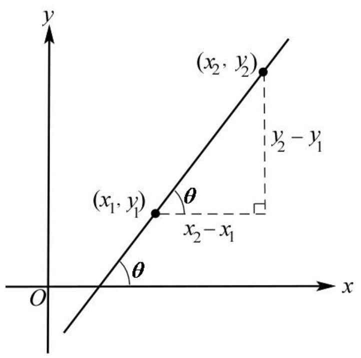
1. Definitions

(i) Gradient

The gradient of a function $f(x)$ defines the direction of the graph of f(x) and shows how the function f(x) changes with x.

(a) Linear function: $y = mx + c$

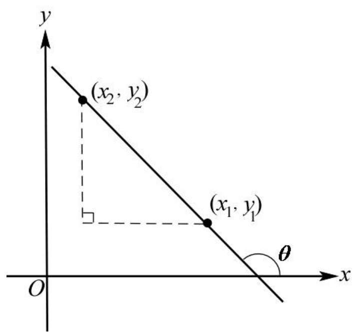
The gradient of a linear function $y = mx + c$ is the constant m where m is the tangent of the angle that the line makes with the **positive direction of x-axis**.



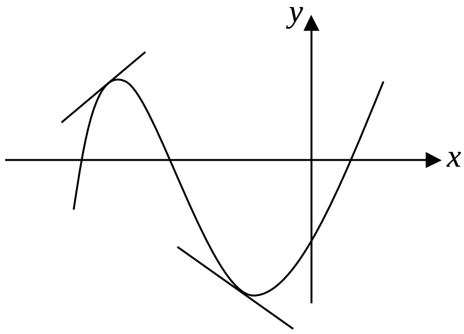
Gradient of the line l = $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \tan\theta$

Note: $m \geq 0$ in the above case since $0 \leq \theta < \dfrac{\pi}{2}$.

For the case below, since $\dfrac{\pi}{2} < \theta < \pi$, $m < 0$.



(b) Non-linear function:
The gradient of a curve at any point is defined to be the gradient of the tangent drawn at that point. Thus, the gradient of a curve is not a constant but has different values at different points on the curve.



(ii) Differentiation
The process of determining the rate of change of a function with respect to one of its variables, e.g. the rate of change of y with respect to x, is known as **differentiation.** The general expression for the gradient is called the **derivative** or the **gradient function** and is denoted by the symbol $\frac{\mathrm{d}y}{\mathrm{d}x}$. The derivative of a function $\mathrm{f}(x)$ with respect to x is denoted by $\mathrm{f}'(x)$.

2. Differentiation of Basic Functions (from 'O' Level Mathematics)

	Differentiation with respect to x	**Results**
1	$\frac{\mathrm{d}}{\mathrm{d}x}\left(ax^n\right)$ (note that a and n are ***constants***)	nax^{n-1}
(i)	$\frac{\mathrm{d}}{\mathrm{d}x}\left(ax\right)$ (note that a is a ***constant***)	a
(ii)	$\frac{\mathrm{d}}{\mathrm{d}x}\left(a\right)$ (note that a is a ***constant***)	0
2	$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin x\right)$	$\cos x$
3	$\frac{\mathrm{d}}{\mathrm{d}x}\left(\cos x\right)$	$-\sin x$
4	$\frac{\mathrm{d}}{\mathrm{d}x}\left(\tan x\right)$	$\sec^2 x$
5	$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^x\right)$	e^x
6	$\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln x\right)$	$\frac{1}{x}$

Note: You are required to remember the above differentiation results.

3. Basic Rules of Differentiation

1	**Sum/Difference Rule** $$\frac{\mathrm{d}}{\mathrm{d}x}(u \pm v) = \frac{\mathrm{d}u}{\mathrm{d}x} \pm \frac{\mathrm{d}v}{\mathrm{d}x}$$ (Note: u and v are functions of x.)
2	**Product Rule** $$\frac{\mathrm{d}}{\mathrm{d}x}(uv) = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$$ (Note: u and v are functions of x.)
3	**Quotient Rule** $$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{v}\right) = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$$ (Note: u and v are functions of x.)

4. Chain Rule

Chain rule is a process that allows us to differentiate composite functions e.g. $\left(2x^3+1\right)^{-5}$, $\sin 3x$, $\ln(\ln x)$ etc.

To find the derivative of a composite function, we use:

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}}$$

Students' note:
Composite functions can be seen as combination of basic functions. (i.e. basic function within another basic function)

How chain rule works:

Find $\frac{\mathrm{d}}{\mathrm{d}x}(2x+1)^4$.

Let $u = 2x+1$ and $y = u^4$

Then $\frac{\mathrm{d}u}{\mathrm{d}x} = 2$, $\frac{\mathrm{d}y}{\mathrm{d}u} = 4u^3$

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} \\ &= 4u^3(2) \\ &= 4(2x+1)^3(2) = 8(2x+1)^3\end{aligned}$$

Example 1

Find $\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{x^2}\right)$.

Solution:

$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{x^2}\right) =$ $\qquad = 2x\mathrm{e}^{x^2}$

Example 2

Find $\frac{\mathrm{d}}{\mathrm{d}x}(\sin 3x)$.

Solution:

$\frac{\mathrm{d}}{\mathrm{d}x}(\sin 3x) =$

$= 3\cos 3x$

Example 3

Differentiate the following functions with respect to x:

(a) $\left(1+\frac{1}{x}\right)^{\frac{1}{2}}$; (b) $\cos^3 2x$.

Solution:

(a) $\frac{\mathrm{d}}{\mathrm{d}x}\left(1+\frac{1}{x}\right)^{\frac{1}{2}} =$

(b) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\cos^3 2x\right) = \frac{\mathrm{d}}{\mathrm{d}x}[\cos 2x]^3$

$=$

$=$

Example 4

Differentiate the following exponential and logarithmic functions with respect to x:

(a) $e^{\sqrt{x}}$; (b) $\ln(\sin x)$; (c) $\ln(px)$; (d) $\log_3 x$; (e) $\lg\left(x^2+1\right)$.

Note: "lg" means "$\log_{10}$".
Question:
Is there an equivalent differentiation formula for $\log_a$?

Solution:

(a) $\frac{d}{dx}\left(e^{\sqrt{x}}\right) = \qquad = \frac{1}{2\sqrt{x}}e^{\sqrt{x}}$

(b) $\frac{d}{dx}\left[\ln(\sin x)\right] = \qquad = \cot x$

(c) $\frac{d}{dx}\left[\ln(px)\right] = \qquad = \frac{1}{x}$ **Alternatively** $\frac{d}{dx}\left[\ln(px)\right] = \frac{d}{dx}\left[\qquad \right] = 0 + \frac{1}{x} = \frac{1}{x}$

(d) $\frac{d}{dx}\left(\log_3 x\right) = \qquad = \left(\frac{1}{\ln 3}\right)\left(\frac{d}{dx}(\ln x)\right) = \left(\frac{1}{\ln 3}\right)\left(\frac{1}{x}\right) = \frac{1}{x\ln 3}$

(e) $\frac{d}{dx}\lg\left(x^2+1\right) = \frac{d}{dx}\left[\qquad \right] = \frac{1}{\ln 10}\frac{d}{dx}\left[\ln(x^2+1)\right]$

$$= \frac{1}{\ln 10}\left(\frac{1}{x^2+1}\right)(2x) = \frac{2x}{(\ln 10)\left(x^2+1\right)}$$

Recall:

Before differentiating a logarithmic function, simplify the function first using the following Laws of Logarithms:

Laws of Logarithms:

For all $m>0, n>0$ and $a>0, a\neq 1$,

(i) $\log_a m^k = k\log_a m$ **(Power Law)**

(ii) $\log_a(mn) = \log_a m + \log_a n$ **(Product Law)**

(iii) $\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$ **(Quotient Law)**

Change of Base of Logarithms:

For all $m>0, a>0, a\neq 1$ and $b>0, b\neq 1$,

$$\log_a m = \frac{\log_b m}{\log_b a}$$

In particular, $\log_a m = \frac{\ln m}{\ln a}$

5. Differentiation of Other Basic Functions (Trigonometric)

	Differentiation with respect to x	Results
1	$\frac{\mathrm{d}}{\mathrm{d}x}(\sec x)$	$\sec x \tan x$
2	$\frac{\mathrm{d}}{\mathrm{d}x}(\operatorname{cosec} x)$	$-\operatorname{cosec} x \cot x$
3	$\frac{\mathrm{d}}{\mathrm{d}x}(\cot x)$	$-\operatorname{cosec}^2 x$
4	$\frac{\mathrm{d}}{\mathrm{d}x}(\sin^{-1} x)$	$\frac{1}{\sqrt{1-x^2}}$
5	$\frac{\mathrm{d}}{\mathrm{d}x}(\cos^{-1} x)$	$-\frac{1}{\sqrt{1-x^2}}$
6	$\frac{\mathrm{d}}{\mathrm{d}x}(\tan^{-1} x)$	$\frac{1}{1+x^2}$

Note: Formulae 1, 2, 4, 5 and 6 are in MF26 ('A' level formulae list) while formula 3 is NOT. Hence, you need to remember formula 3.

Example 5

Prove that $\frac{\mathrm{d}}{\mathrm{d}x}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$.

Solution:

$\frac{\mathrm{d}}{\mathrm{d}x}(\operatorname{cosec} x) = \frac{\mathrm{d}}{\mathrm{d}x}\Big(\qquad\Big)$

$=$

$= -(\sin x)^{-2}(\cos x)$

$= -\frac{\cos x}{(\sin x)^2}$

$= -\left(\frac{1}{\sin x}\right)\left(\frac{\cos x}{\sin x}\right)$

$= -\operatorname{cosec} x \cot x$

Note : $(\sin x)^{-1} \neq \sin^{-1} x$

Similarly,

$\sec x = \frac{1}{\cos x} = (\cos x)^{-1} \neq \cos^{-1} x$

$\cot x = \frac{1}{\tan x} = (\tan x)^{-1} \neq \tan^{-1} x$

Note: Formulae 1 and 3 can also be similarly proven. Try it yourself.
Formulae 4, 5 and 6 will be proven using implicit differentiation which will be covered in a later part of the chapter.

Example 6

Find (a) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\cot\frac{1}{x}\right)$; (b) $\frac{\mathrm{d}}{\mathrm{d}x}\left[\sin^{-1}(2x-3)\right]$; (c) $\frac{\mathrm{d}}{\mathrm{d}x}\left[\cos^{-1}(\sqrt{x})\right]$.

Solution:

(a) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\cot\frac{1}{x}\right)=$ $=\frac{1}{x^2}\operatorname{cosec}^2\frac{1}{x}$

(b) $\frac{\mathrm{d}}{\mathrm{d}x}\left[\sin^{-1}(2x-3)\right]=$ $=\frac{2}{\sqrt{1-(2x-3)^2}}$

(c) $\frac{\mathrm{d}}{\mathrm{d}x}\left[\cos^{-1}\left(\sqrt{x}\right)\right]=$ $=-\frac{1}{2\sqrt{x(1-x)}}$

Example 7

Find (a) $\frac{\mathrm{d}}{\mathrm{d}x}\left(x\tan^{-1}x^2\right)$; (b) $\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{\sec u}{1+u}\right)$; (c) $\frac{\mathrm{d}}{\mathrm{d}t}\left[\ln\left(\sin t^3\right)\right]$.

Solution:

(a) $\frac{\mathrm{d}}{\mathrm{d}x}\left(x\tan^{-1}x^2\right)=$

$$=\tan^{-1}x^2+\frac{2x^2}{1+x^4}$$

(b)

$\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{\sec u}{1+u}\right)=$

$$=\frac{(\sec u)\left[(\tan u)(1+u)-1\right]}{(1+u)^2}$$

(c) $\frac{\mathrm{d}}{\mathrm{d}t}\ln\left(\sin t^3\right)=$

$$=3t^2\cot\left(t^3\right)$$

6. Higher Order Derivatives

Given $y = \mathrm{f}(x)$, we have the following:

(i) $\frac{\mathrm{d}y}{\mathrm{d}x}$ is called the **first derivative,** obtained by differentiating *y* with respect to *x*.

In other words, $\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}'(x).}$

(ii) $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ is called the **second derivative,** obtained by differentiating $\frac{\mathrm{d}y}{\mathrm{d}x}$ with respect to *x*, In other words, $\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \mathrm{f}''(x).}$

(iii) $\frac{\mathrm{d}^3 y}{\mathrm{d}x^3}$ is called the **third derivative,** obtained by differentiating $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ with respect to *x*. In other words, $\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right) = \frac{\mathrm{d}^3 y}{\mathrm{d}x^3} = \mathrm{f}'''(x).}$

(iv) $\frac{\mathrm{d}^n y}{\mathrm{d}x^n}$ is called the ***n*th derivative,** obtained by differentiating $\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}}$ with respect to *x*. In other words, $\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}}\right) = \frac{\mathrm{d}^n y}{\mathrm{d}x^n} = \mathrm{f}^{(n)}(x).}$

Note that the "*n*" is written with a bracket.

Note: (i) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}y}}$ (only true for 1st order derivative)

(ii) $\frac{\mathrm{d}^n y}{\mathrm{d}x^n} \neq \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^n$, e.g. $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} \neq \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2$.

Example 8

If $y = a\sin x + b\cos x$, where *a* and *b* are constants, show that $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + y = 0$

Solution:

Let $y = a\sin x + b\cos x$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = a\cos x - b\sin x$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -a\sin x - b\cos x$$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -y \quad \text{(Why?}$$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + y = 0 \quad \text{(shown)}$$

7. <u>Implicit Differentiation</u>

For many curves, it is rather difficult (or impossible) to express y explicitly in terms of x (e.g. $2x+y^2-3xy=0$). In this case, to find $\frac{\mathrm{d}y}{\mathrm{d}x}$, we have to differentiate the functions of y **implicitly** with respect to x.

In general, the idea of implicit differentiation is to differentiate every term in the equation/expression with respect to x.

The example below illustrates the idea of implicit differentiation when y is a function of x:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(y^2\right)=2y\left(\frac{\mathrm{d}}{\mathrm{d}x}(y)\right) \quad \text{(By Chain Rule)}$$

$$=2y\frac{\mathrm{d}y}{\mathrm{d}x}$$

Similarly,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\right)$$

$$=$$

<u>Note</u>:
Chain rule is used in implicit differentiation, for example:
$\frac{\mathrm{d}}{\mathrm{d}x}(3x+1)^2=2(3x+1)\left(\frac{\mathrm{d}}{\mathrm{d}x}(3x+1)\right)$

<u>Example 9</u>
Find:

(a) $\frac{\mathrm{d}}{\mathrm{d}x}\left(4y^3\right)$; (b) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)^4$.

Solution:

(a) $\frac{\mathrm{d}}{\mathrm{d}x}\left(4y^3\right)=4\frac{\mathrm{d}}{\mathrm{d}x}\left(y^3\right)$

$=$

$=$

(b) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)^4=$

$=$

When differentiating the product of two different variables, we apply product rule.

For instance, differentiating xy , with respect to x gives $x\frac{\mathrm{d}y}{\mathrm{d}x}+y$.

Example 10

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y if $x^3+y^3=3xy^2$.

Solution:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^3\right)+\frac{\mathrm{d}}{\mathrm{d}x}\left(y^3\right)=3\frac{\mathrm{d}}{\mathrm{d}x}\left(xy^2\right)$$

$\Rightarrow$ (product rule)

$$\Rightarrow \quad x^2+y^2\frac{\mathrm{d}y}{\mathrm{d}x}=\left[x\left(2y\frac{\mathrm{d}y}{\mathrm{d}x}\right)+y^2(1)\right]$$

$$\Rightarrow \quad y^2\frac{\mathrm{d}y}{\mathrm{d}x}-2xy\frac{\mathrm{d}y}{\mathrm{d}x}=y^2-x^2$$

$$\Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y^2-x^2}{y^2-2xy}$$

Example 11

Differentiate the following implicitly with respect to x: (you need not make $\frac{\mathrm{d}y}{\mathrm{d}x}$ the subject)

(a) $x^2-2xy+2y^2=4$,

(b) $\ln y=y\ln x$, (Note that it is implied that $x>0, y>0$.)

(c) $x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^3$.

Solution:

(a)

(b)

(c)

<u>Example 12</u> [An application of Implicit Differentiation]

Prove that $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\tan^{-1} x\right)=\dfrac{1}{1+x^2}$.

(Recall the above formula was introduced in the earlier section and provided in MF26 also.)

Solution:

Let $y=\tan^{-1} x$ (Note : Principal values : $-\dfrac{1}{2}\pi<\tan^{-1} x<\dfrac{1}{2}\pi$ obtained from MF 26)

$\Rightarrow$ $\tan y = x$

Differentiate implicitly with respect to *x*,

Note: $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ has to be in terms of *x* only as requested by question.

Note : Results for $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\cos^{-1} x\right)$ and $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\sin^{-1} x\right)$ (also in MF26) can be similarly derived.

8. <u>Differentiation of Parametric Equations</u>

Suppose a curve *C* is defined by the pair of parametric equations

$$x=\mathrm{f}(t) \text{ and } y=\mathrm{g}(t),$$

where *t* is a parameter.

The gradient of the curve *C* at a point (x, y) can be found using the Chain Rule,

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}t}\times\frac{\mathrm{d}t}{\mathrm{d}x} \quad \text{(Since } y \text{ is a function of } t \text{ and } t \text{ is a function of } x\text{)}$$

$$=\frac{\mathrm{d}y}{\mathrm{d}t}\times\frac{1}{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)}$$

Thus, $$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}}.$$

Note that $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ will be in terms of the parameter *t*.

Example 13

The parametric equations of a curve are $x = 1 - \cos t$ and $y = t + \sin t$.

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t.

Solution:

$$x = 1 - \cos t \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} =$$

$$y = t + \sin t \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t} =$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} =$$

Example 14

Given that $x = 2t - \ln(2t)$ and $y = t^2 - \ln(t^2)$, where $t > 0$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t.

Solution:

$$\frac{\mathrm{d}x}{\mathrm{d}t} =$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} =$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} =$$

$$=$$

Example 15

A curve C has parametric equations $x = 1 - \cos\theta$, $y = \theta - \sin\theta$, where $0 \le \theta \le 2\pi$.

Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \tan\left(\frac{1}{2}\theta\right)$.

Solution:

$$x = 1 - \cos\theta \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}\theta} = \qquad y = \theta - \sin\theta \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}\theta} =$$

$$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}\theta}}{\frac{\mathrm{d}x}{\mathrm{d}\theta}} =$$

$$= \frac{\sin\left(\frac{1}{2}\theta\right)}{\cos\left(\frac{1}{2}\theta\right)} = \tan\left(\frac{1}{2}\theta\right) \quad \text{[Shown]}$$

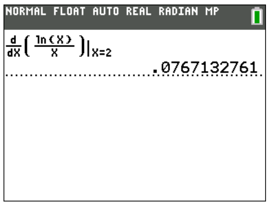
9. Finding the Approximate Value of a Derivative at a Given Point Using a GC

Example 16

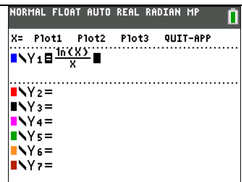
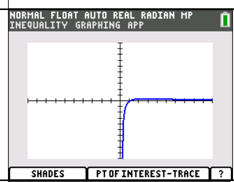
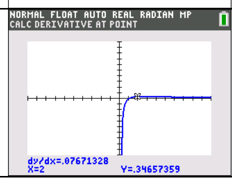
Find the numerical value of the derivative of $\dfrac{\ln x}{x}$ at $x = 2$.

Solution: There are 2 ways to calculate the numerical value of the derivative at $x = 2$ using the GC.

Method 1 Using the Home Screen:

Step	Keystrokes	Screenshots
1	Press [ALPHA] then [WINDOW] and press [3] to select **"3:nDeriv("**.	
2	Complete the expression and then press [ENTER]	

Method 2 Using the Graph Method:

Step	Keystrokes	Screenshots
1	Press [Y=] and enter the function: $Y_1 = \dfrac{\ln x}{x}$	
2	Press [GRAPH] to graph the function.	
3	Press [2nd] and then [TRACE] to select **"6:dy/dx"**.	
4	Enter 2 and press [ENTER] to calculate the value of the derivative at $x = 2$.	

From GC, the numerical value of the derivative of $\dfrac{\ln x}{x}$ at $x = 2$ is <u>0.0767</u> (3sf)

Example 17

Use your calculator to find the gradient of the curve $y = 2^{\cos x}$ at the points where $x = 0$ and $x = \frac{1}{2}\pi$.

Solution: