Section A: Pure Mathematics

1

Solution

Finding table values in SGD,

	Cheese/kg	Chocolate/kg	Candy/kg
Price/SGD	4	6	6
Price/SGD	8	10	4
Price/SGD	8	5	7

Let x, y, z be the number of three kg packs bought from Denmark, England and Russia respectively.

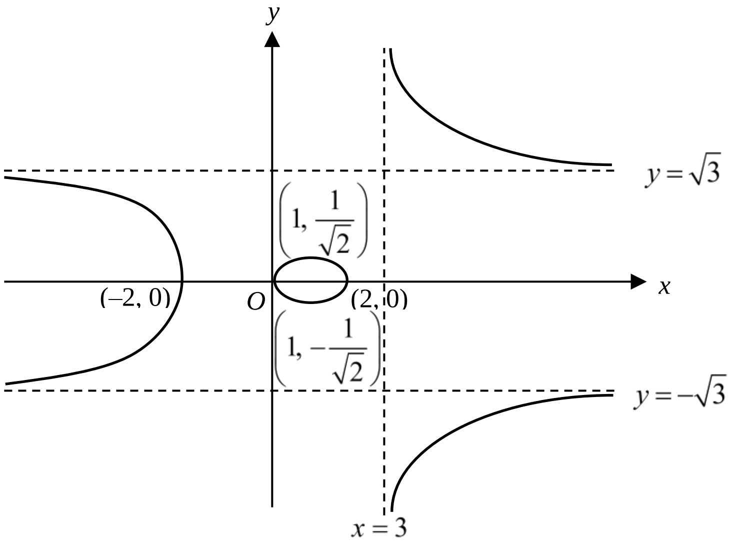
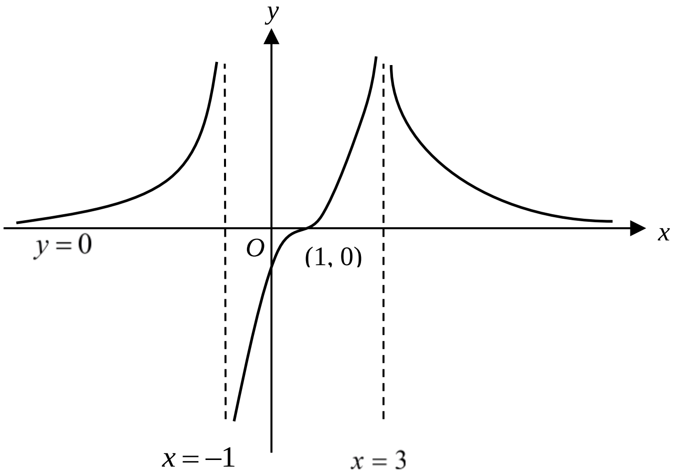
$$4x+8y+8z=84$$

$$6x+10y+5z=85$$

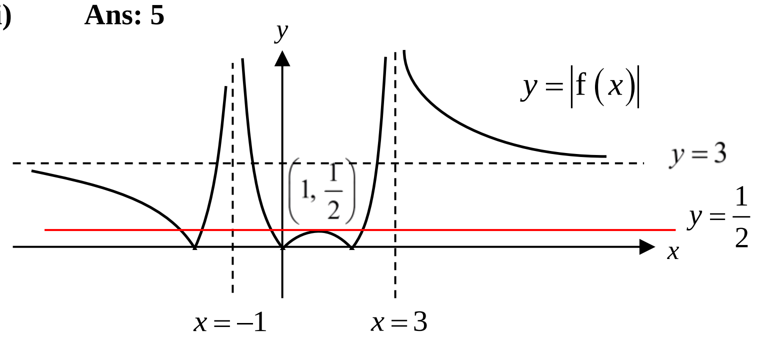
$$6x+4y+7z=77$$

$$x=5,\quad y=3,\quad z=5$$

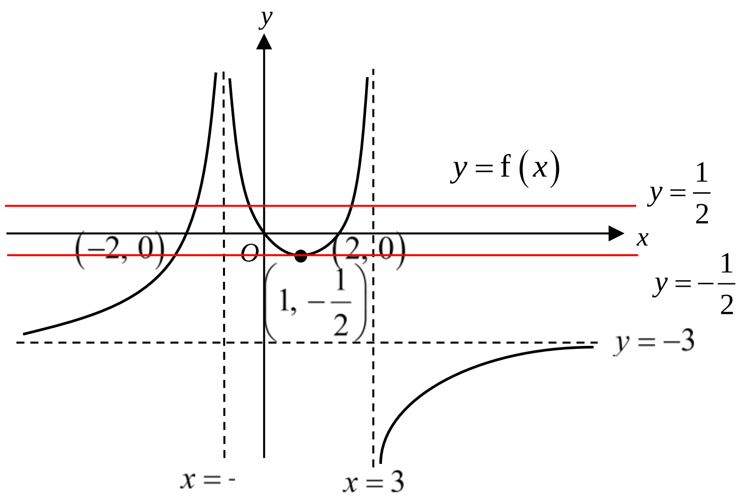
She should buy 13 packs in total.

2	
	Solution
	(i)(a) $y = \mathrm{f}(x) \to y = -\mathrm{f}(x) \to y^2 = -\mathrm{f}(x)$ 
	(b) $y = \mathrm{f}'(x)$ 

(ii) **Ans: 5**



Alternative



(iii) **Method ①:**

$y=f(x)$

$\downarrow$ I : translation of 1 units in the negative x-direction

$y=f(x+1)$

$\downarrow$ II : scaling parallel to x-axis by a scale factor of $\frac{1}{2}$

$y=f[2(x)+1]$

$\downarrow$ III : translation of 3 units in the negative y-direction

$y=f(2x+1)-3$

Method ②:

	$y=\mathrm{f}(x)$ $\downarrow$ I : scaling parallel to x-axis by a scale factor of $\frac{1}{2}$ $y=\mathrm{f}(2x)$ $\downarrow$ II : translation of $\frac{1}{2}$ units in the negative x-direction $y=\mathrm{f}\left[2\left(x+\frac{1}{2}\right)\right]$ $\downarrow$ III : translation of 3 units in the negative y-direction $y=\mathrm{f}(2x+1)-3$

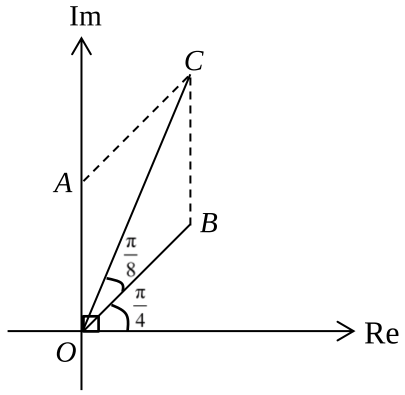
3	
	Solution

(i) $|a|=\left|\frac{1+\mathrm{i}}{1-\mathrm{i}}\right|=\frac{|1+\mathrm{i}|}{|1-\mathrm{i}|}=\frac{\sqrt{2}}{\sqrt{2}}=1$ $|b|=\left|\frac{\sqrt{2}}{1-\mathrm{i}}\right|=\frac{\sqrt{2}}{|1-\mathrm{i}|}=\frac{\sqrt{2}}{\sqrt{2}}=1$

$$\begin{aligned}\arg(a)&=\arg\left(\frac{1+\mathrm{i}}{1-\mathrm{i}}\right)\\&=\arg(1+\mathrm{i})-\arg(1-\mathrm{i})\\&=\frac{\pi}{4}-\left(-\frac{\pi}{4}\right)\\&=\frac{\pi}{2}\end{aligned}$$

$$\begin{aligned}\arg(b)&=\arg\left(\frac{\sqrt{2}}{1-\mathrm{i}}\right)\\&=\arg\left(\sqrt{2}\right)-\arg(1-\mathrm{i})\\&=0-\left(-\frac{\pi}{4}\right)\\&=\frac{\pi}{4}\end{aligned}$$

(ii)



$$\angle BOC = \frac{\frac{\pi}{2} - \frac{\pi}{4}}{2} = \frac{\pi}{8}$$

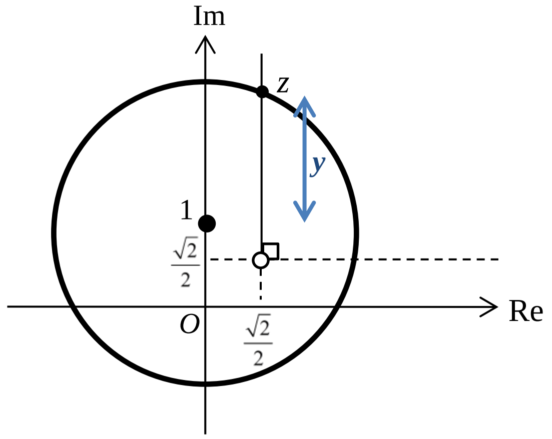
$$\arg(a+b) = \frac{\pi}{8} + \frac{\pi}{4} = \frac{3\pi}{8}$$

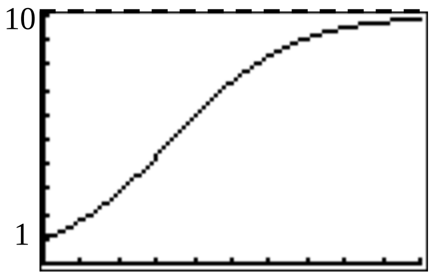
$$a = \frac{1+\mathrm{i}}{1-\mathrm{i}} \cdot \frac{1+\mathrm{i}}{1+\mathrm{i}} = \frac{1+2\mathrm{i}+\mathrm{i}^2}{1-\mathrm{i}^2} = \frac{2\mathrm{i}}{2} = \mathrm{i}$$

$$b = \frac{\sqrt{2}}{1-\mathrm{i}} \cdot \frac{1+\mathrm{i}}{1+\mathrm{i}} = \frac{\sqrt{2}(1+\mathrm{i})}{1-\mathrm{i}^2} = \frac{\sqrt{2}}{2}(1+\mathrm{i})$$

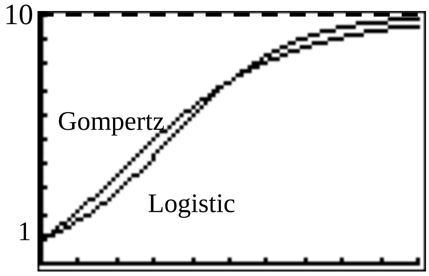
$$\therefore a+b = \mathrm{i} + \frac{\sqrt{2}}{2}(1+\mathrm{i}) = \frac{\sqrt{2}}{2} + \left(\frac{\sqrt{2}}{2}+1\right)\mathrm{i}$$

$$\tan\left(\frac{3\pi}{8}\right) = \frac{\frac{\sqrt{2}}{2}+1}{\frac{\sqrt{2}}{2}} = 1 + \frac{2}{\sqrt{2}} = 1+\sqrt{2} \quad \text{(shown)}$$

	(iii)	**(a)** $\lvert z-\mathrm{i}\rvert = 2 \Rightarrow \lvert z-(0+\mathrm{i})\rvert = 2$ Locus is a circle, centre (0, 1) and radius 2 **(b)** $\arg(z-b)=\frac{\pi}{2} \Rightarrow \arg\left(z-\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\mathrm{i}\right)\right)=\frac{\pi}{2}$ Locus is a half-line from the point $\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$ (excluding the angle itself) making an angle of $\frac{\pi}{2}$ to the positive Re-axis direction.  Equation of circle: $x^2+(y-1)^2=2^2$ When $x=\frac{\sqrt{2}}{2}$, $\left(\frac{\sqrt{2}}{2}\right)^2+(y-1)^2=2^2$ $\frac{1}{2}+(y-1)^2=4$ $(y-1)^2=\frac{7}{2}$ $y-1=\pm\sqrt{\frac{7}{2}}$ $y=1+\sqrt{\frac{7}{2}}$ or $1-\sqrt{\frac{7}{2}}$ (rej. $\because$ y is positive) $\therefore z=\frac{\sqrt{2}}{2}+\left(1+\sqrt{\frac{7}{2}}\right)\mathrm{i}$ **<u>Alternative for finding *y*</u>** From diagram, using Pythagoras' theorem, $y=1+\sqrt{2^2-\left(\frac{\sqrt{2}}{2}\right)^2}=1+\sqrt{\frac{7}{2}}$

	Solution
	(i) $\frac{\mathrm{d}P}{\mathrm{d}t}=0.64P\left(1-\frac{P}{10}\right)$ $\frac{\mathrm{d}P}{\mathrm{d}t}=0.064P(10-P)$ $\frac{1}{P(10-P)}\frac{\mathrm{d}P}{\mathrm{d}t}=0.064$ $\int\frac{1}{P(10-P)}\mathrm{d}P=\int 0.064\mathrm{d}t$ $\frac{1}{10}\int\frac{1}{P}+\frac{1}{10-P}\mathrm{d}P=\int 0.064\mathrm{d}t$ $\int\frac{1}{P}+\frac{1}{10-P}\mathrm{d}P=\int 0.64\mathrm{d}t$ $\ln\lvert P\rvert-\ln\lvert 10-P\rvert=0.64t+C$ $\ln\left\lvert\frac{P}{10-P}\right\rvert=0.64t+C$ $\frac{P}{10-P}=\pm e^{0.64t+C}$ $\frac{P}{10-P}=Ae^{0.64t}$, where $A=\pm e^{c}$ $\frac{10-P}{P}=Be^{-0.64t}$, where $B=\frac{1}{A}$ $\frac{10}{P}=1+Be^{-0.64t}$ $P=\frac{10}{1+Be^{-0.64t}}$ When $t=0$, $P=1$, $1=\frac{10}{1+Be^{0}}$ $B=9$ $\therefore P=\frac{10}{1+9e^{-0.64t}}$
	 The population will approach 10 asymptotically in the long run.
	(ii) $\frac{\mathrm{d}P}{\mathrm{d}t}=0.4P\ln\left(\frac{10}{P}\right)$ $\frac{1}{P(\ln 10-\ln P)}\frac{\mathrm{d}P}{\mathrm{d}t}=0.4$ $\int\frac{1}{P(\ln 10-\ln P)}dP=\int 0.4\mathrm{d}t$

	$\ln\lvert\ln 10-\ln P\rvert=-0.4t+C$ $\ln 10-\ln P=\pm e^{-0.4t+C}$ $\ln 10-\ln P=Ae^{-0.4t}$, where $A=\pm e^{C}$ $\ln P=\ln 10-Ae^{-0.4t}$ $P=10e^{-Ae^{-0.4t}}$

		When $t=0$, $P=1$, $1=10e^{-Ae^{0}}$ $A=\ln 10$ $\therefore P=10e^{-(\ln 10)e^{-0.4t}}$  (Similarity) The population approaches 10 asymptotically in the long run in both models. (Difference) • The population in the second model would increase faster first then slower compared to the first model. OR • The Gompertz function will take a longer time to approach 10 as compared to the Logistic function.

Section B: Statistics

5

Solution

(i) Simple random sampling method.
Possible answers for disadvantages:
not representative of the gender make-up of the class.

(ii) **<u>Method ①:</u>**

$$\mathrm{P}\left(\text{3 females and 2 males are selected}\right)$$
$$=\mathrm{P}\left(\text{female,female,female,male,male}\right)\times\frac{5!}{3!2!}$$
$$=\frac{15}{25}\times\frac{14}{24}\times\frac{13}{23}\times\frac{10}{22}\times\frac{9}{21}\times\frac{5!}{3!2!}$$
$$=\frac{195}{506}$$
$$\approx 0.385\left(3\text{ s.f}\right)$$

<u>Method ②:</u>

$$\mathrm{n}\left(S\right)={}^{25}\mathrm{C}_5$$

To select 3 females and 2 males

Stage 1: Select 3 females out of 15, no, of ways $={}^{15}\mathrm{C}_3$

Stage 2: Select 2 males out of 10, no. of ways $={}^{10}\mathrm{C}_2$

Total number of ways $={}^{15}\mathrm{C}_3\times{}^{10}\mathrm{C}_2$

$$\mathrm{P}\left(\text{3 females and 2 males are selected}\right)$$
$$=\frac{{}^{15}\mathrm{C}_3\times{}^{10}\mathrm{C}_2}{{}^{25}\mathrm{C}_5}$$
$$=\frac{195}{506}$$
$$\approx 0.385\left(3\text{ s.f}\right)$$

	(iii) **<u>Method ①:</u>** $\text{P}\left(\text{Amy and Bertrand are selected}\right)$ $= \text{P}\left(\text{Amy,Bertrand,any student,any student,any student}\right) \times \frac{5!}{3!}$ $= \frac{1}{25} \times \frac{1}{24} \times \frac{23}{23} \times \frac{22}{22} \times \frac{21}{21} \times \frac{5!}{3!}$ $= \frac{1}{30}$ $\approx 0.0333\left(3 \text{ s.f}\right)$ **<u>Method ②:</u>** To select Amy, Bertrand and 3 other students *Stage 1*: Select Amy $= 1$ *Stage 2*: Select Bertrand $= 1$ *Stage 3*: Select any 3 students $= {}^{23}\text{C}_3$ Total number of ways $= {}^{23}\text{C}_3$ $\text{P}\left(\text{Amy and Bertrand are selected}\right)$ $= \frac{{}^{23}\text{C}_3}{{}^{25}\text{C}_5}$ $= \frac{1}{30}$ $\approx 0.0333\left(3 \text{ s.f}\right)$

(iv) **Method ①:**

$$\mathrm{P}\left(\text{Amy and Bertrand are selected} \mid 3 \text{ females and } 2 \text{ males are selected}\right)$$

$$= \frac{\mathrm{P}\left(\text{Amy and Bertrand are selected} \cap 3 \text{ females and } 2 \text{ males are selected}\right)}{\mathrm{P}\left(3 \text{ females and } 2 \text{ males are selected}\right)}$$

$$= \frac{\mathrm{P}\left(\text{Amy,Bertrand,female,female,male}\right) \times \frac{5!}{2!}}{\frac{195}{506}}$$

$$= \frac{\frac{1}{25} \times \frac{1}{24} \times \frac{14}{23} \times \frac{13}{22} \times \frac{9}{21} \times \frac{5!}{2!}}{\frac{195}{506}}$$

$$= \frac{\frac{39}{2530}}{\frac{195}{506}}$$

$$= \frac{1}{25} \text{ or } 0.04$$

	Method ②: $\mathrm{P}\left(\text{Amy and Bertrand are selected} \mid \text{3 females and 2 males are selected}\right)$ $=\dfrac{\mathrm{P}\left(\text{Amy and Bertrand are selected} \cap \text{3 females and 2 males are selected}\right)}{\mathrm{P}\left(\text{3 females and 2 males are selected}\right)}$ $=\dfrac{\dfrac{1\times1\times{}^{14}\mathrm{C}_2\times{}^{9}\mathrm{C}_1}{{}^{25}\mathrm{C}_5}}{\dfrac{195}{506}}$ $=\dfrac{\dfrac{39}{2530}}{\dfrac{195}{506}}$ $=\dfrac{1}{25}$ or 0.04 **Note:** $1\times1\times{}^{14}\mathrm{C}_2\times{}^{9}\mathrm{C}_1$ is select Amy, Select Bertrand, Select 2 females out of 14 and select 1 male out of 9 **Method ③: Reduced sample space** $\mathrm{n}\left(S'\right)={}^{15}\mathrm{C}_3\times{}^{10}\mathrm{C}_2=20475$ Number of ways Amy, Bertrand, 2 females and 1 male are selected $=1\times1\times{}^{14}\mathrm{C}_2\times{}^{9}\mathrm{C}_1=819$ $\mathrm{P}\left(\text{Amy and Bertrand are selected} \mid \text{3 females and 2 males are selected}\right)$ $=\dfrac{\mathrm{P}\left(\text{Amy and Bertrand are selected} \cap \text{3 females and 2 males are selected}\right)}{\mathrm{P}\left(\text{3 females and 2 males are selected}\right)}$ $=\dfrac{819}{20475}$ $=\dfrac{1}{25}$ or 0.04

	(v)	The teacher can write each student's name (all the students' names are distinct) on a small piece paper of the same size, folded it into half. He can then place all the females' students name into a large bowl. He then shook the bowl and took out 3 pieces of paper one by one without replacement. Next, he placed all the male students' name into a large bowl. He then shook the bowl and took out 2 pieces of paper one by one without replacement.
		<u>Method ①:</u> $\frac{1}{25}$ or 0.04 same answers in **(iv)** **<u>Method ②:</u>** $\mathrm{P}\left(\text{Amy and Bertrand are selected}\right)$ $= \frac{1 \times 1 \times {}^{14}C_2 \times {}^{9}C_1}{{}^{15}C_3 \times {}^{10}C_2}$ $= \frac{1}{25}$ or 0.04

6

Solution

(i) Assume that the event of obtaining a rotten cherry is independent of another.
Assume that the probability of obtaining a rotten cherry is constant.

(ii) Let X be the random variable denoting the number of rotten cherries out of 26.

$X \sim \mathrm{B}(26, 0.08)$

X	$\mathrm{P}(X = x)$
1	0.25868
2	0.28117
3	0.19560

The most likely number of rotten cherries is 2.

(iii) Let Y be the random variable denoting the number of rotten cherries out of n.

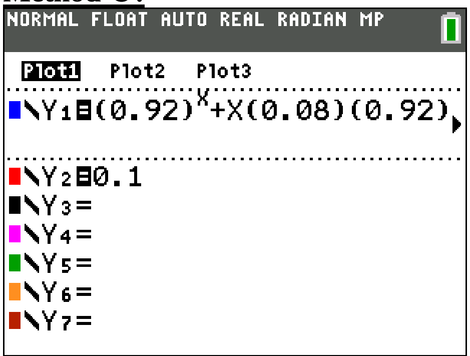
$Y \sim \mathrm{B}(n, 0.08)$

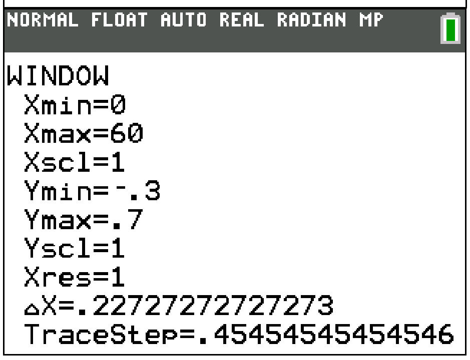
$$\mathrm{P}(Y = 0) + \mathrm{P}(Y = 1) < 0.1$$

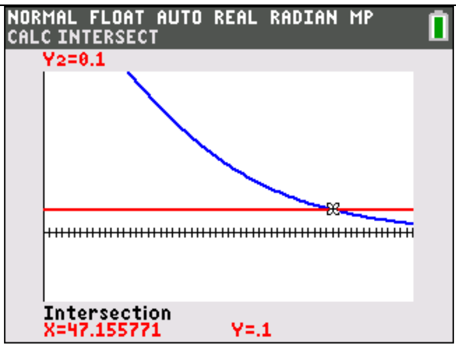
$${}^{n}\mathrm{C}_{0}(0.08)^{0}(1-0.08)^{n} + {}^{n}\mathrm{C}_{1}(0.08)^{1}(1-0.08)^{n-1} < 0.1$$

$$(0.92)^{n} + n(0.08)(0.92)^{n-1} < 0.1$$

<u>Method ①:</u>

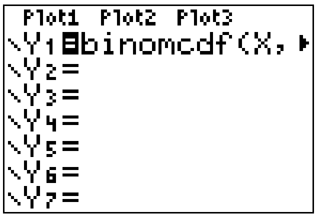


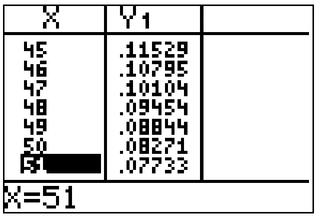




The least number of n is 48.

Method ②:





n	$\mathrm{P}(Y \le 1)$
47	0.10104 (> 0.1)
48	0.09454 (< 0.1)
49	0.08844 (< 0.1)

The least number of n is 48.

(iv) Let W be the random variable denoting the number of rotten cherries out of 60.

$W \sim \mathrm{B}(60, 0.08)$

Since $n = 60$, n is large.

$np = 60 \times 0.08 = 4.8 < 5$

$W \sim \mathrm{Po}(4.8)$ approximately

$$\mathrm{P}(W \le 1) = 0.0477325329$$
$$\approx 0.0477 \text{ (to 3 s.f.)}$$

	(v)	Let C be the random variable denoting the number of boxes of cherries thrown away in a year. $C \sim \text{Po}(12)$ Since $\lambda = 12 > 10$, λ is large. $C \sim \text{N}(12,12)$ approximately $\text{P}(2 \le C \le 5) \xrightarrow{c.c.} \text{P}(1.5 < C < 5.5)$ $\text{P}(1.5 < C < 5.5) = 0.0291$ (to 3 s.f.)

7	
	Solution
	(i) Let C and O be the random variable denoting the masses of a randomly selected carrot and onion, in grams, respectively.

$2C \sim \mathrm{N}\left(c, 5^2\right)$ and $\mathrm{O} \sim \mathrm{N}\left(75, 3^2\right)$

$2C - \left(O_1 + O_2 + O_3 + O_4 + O_5\right)$
$\sim \mathrm{N}\left(2c - 5(75), 4\left(5^2\right) + 5\left(3^2\right)\right)$

$2C - \left(O_1 + O_2 + O_3 + O_4 + O_5\right) \sim \mathrm{N}\left(2c - 375, 145\right)$

$\mathrm{P}\left(2C > O_1 + O_2 + O_3 + O_4 + O_5\right) > 0.9$

$\mathrm{P}\left(2C - \left(O_1 + O_2 + O_3 + O_4 + O_5\right) > 0\right) > 0.9$

$$\mathrm{P}\left(Z > \frac{0 - 2c + 375}{\sqrt{145}}\right) > 0.9$$

$$\mathrm{P}\left(Z < \frac{-2c + 375}{\sqrt{145}}\right) < 0.1$$

$$\frac{-2c + 375}{\sqrt{145}} < -1.281551567$$

$c > 195$ (to 3 s.f.)

The distributions of the masses of all carrots and onions are independent of one another.

	(ii)	$\dfrac{C_1+C_2+O_1+O_2+O_3}{5} \sim \mathrm{N}(125,\ 3.08)$ $\mathrm{P}\left(\dfrac{C_1+C_2+O_1+O_2+O_3}{5} > 130\right)$ $= 0.00219\ (3\text{ s.f.})$
	(ii)	$0.0018(C_1+C_2+C_3)-0.0015(O_1+O_2+O_3+O_4)$ $\sim \mathrm{N}\begin{pmatrix}(0.0018\times 3\times 200)-(0.0015\times 4\times 75)\\ ,(0.0018^2\times 3\times 5^2)+(0.0015^2\times 4\times 3^2)\end{pmatrix}$ $0.0018(C_1+C_2+C_3)-0.0015(O_1+O_2+O_3+O_4)$ $\sim \mathrm{N}(0.63, 0.000324)$ $\mathrm{P}\left(\left\lvert\begin{array}{l}0.0018(C_1+C_2+C_3)\\ -0.0015(O_1+O_2+O_3+O_4)\end{array}\right\rvert < 0.6\right)$ $= \mathrm{P}\begin{pmatrix}-0.6 < 0.0018(C_1+C_2+C_3)\\ -0.0015(O_1+O_2+O_3+O_4) < 0.6\end{pmatrix}$ $= 0.0478\ (\text{to } 3\text{ s.f.})$

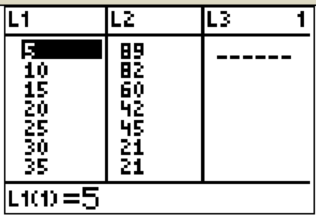
8	
	Solution
	(i) Let X be the random variable denoting the mass of peanut butter in a random jar. Sample mean = $\frac{\Sigma x}{n} = \frac{2695}{10} = 269.5$ (exact) $s^2 = \frac{1}{n-1}[\Sigma x^2 - \frac{(\Sigma x)^2}{n}] = \frac{1}{9}[726313 - \frac{(2695)^2}{10}]$ $= \frac{7}{6} = 1.166666667$ $H_0 : \mu = 270$ $H_1 : \mu < 270$ At 10% level of significance, reject H_0 if $p < 0.10$
	Since $n = 10$ is small and population variance is unknown, assume X is normally distributed and use t-test. Under H_0, test-statistic, $T \sim t(9)$ From GC, $t = -1.463850109$ $p = 0.0886338513 < 0.10$ Hence reject H_0 and conclude at 10% level of significance there is sufficient evidence that the manufacturer has overstated the mean mass of peanut butter in each jar.
	Since n is too small for CLT to be used, we need to assume that X is normally distributed and then t-test can be used.
	In this case, $H_0 : \mu = 270$ $H_1 : \mu \neq 270$ So p-value = 2×0.0886338513 $= 0.1772677026 > 0.10$ So the conclusion would change to do not reject H_0 and that there is insufficient evidence at 10% level that the mean mass differs from 270 g.
	(ii) In this case, $H_0 : \mu = \mu_o$ $H_1 : \mu \neq \mu_o$ Under H_0, since X is normally distributed and population variance is given, $\sim N(\mu_o , \frac{1.1^2}{10})$ Since H_o is not rejected at 5% level of significance, $-1.959963986 < \frac{269.5 - \mu_o}{\frac{1.1}{\sqrt{10}}} < 1.959963986$ $-1.959963986 \frac{1.1}{\sqrt{10}} < 269.5 - \mu_o < 1.959963986 \frac{1.1}{\sqrt{10}}$ $-269.5 - 1.959963986 \frac{1.1}{\sqrt{10}} < -\mu_o < -269.5 + 1.959963986 \frac{1.1}{\sqrt{10}}$ $-270.1817745 < -\mu_o < -268.8182255$ $268.8182255 < \mu_o < 270.1817745$

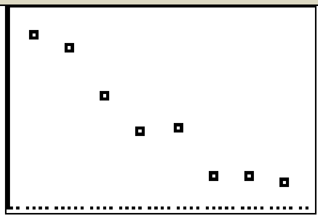
	$268.8 < \mu_o < 270.2$ (to 1 d.p.) **or** $268.9 \leq \mu_o \leq 270.1$ (to 1 d.p.)

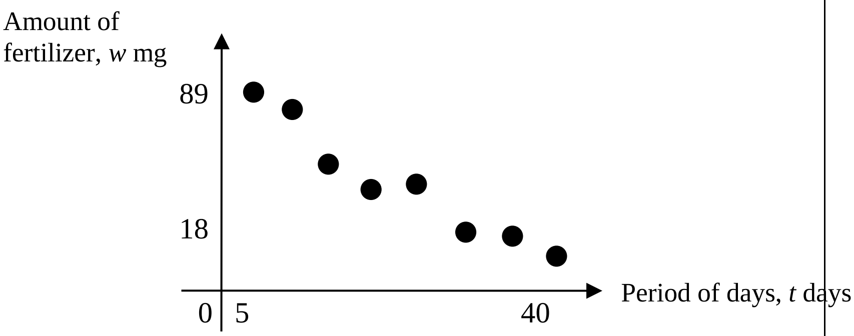
9

Solution

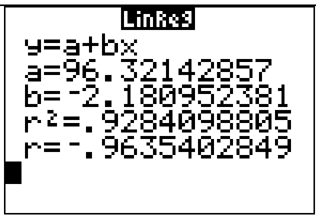
(i)





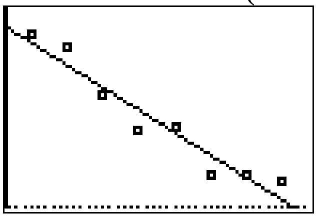
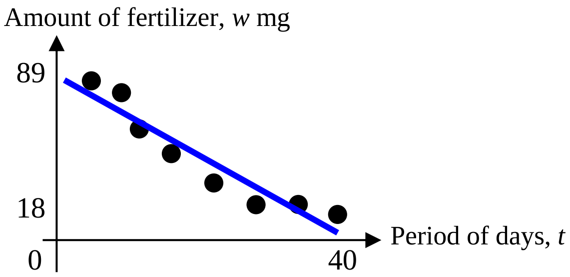


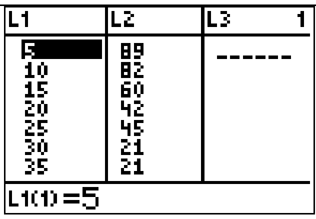
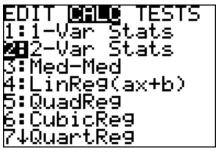
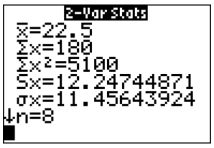
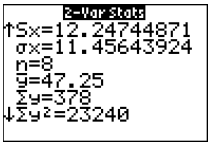
(ii)



For $r \approx$ - 0.964 (to 3 s.f.)

From the product moment correlation coefficient value, r is close to – 1 indicating that there is a strong negative linear relationship between *w* and *t*, i.e. as the number of days increases, the amount of fertilizer retained decreases.

	(iii)	Regression line w on in the form $w = a + bt$: $w = 96.32142857 - 2.180952381t$ $\Rightarrow w = 96.3 - 2.18t$ (to 3 s.f.)  
	(iv)	b: This value gives the estimated rate at which the fertilizer is lost from the soil, i.e. for every additional day, the amount of fertiliser lost from the soil is approximately 2.18 mg.
	(v)	When $w = 30.7$ mg, Using $w = 96.32142857 - 2.180952381t$ $30.7 = 96.32142857 - 2.180952381t$ $\Rightarrow \quad t = 30.08842795$ ≈ 30.1 days Since $w = 30.7$ mg is within the data range $18 \leq w \leq 89$ and $r = -0.964$ is close to -1 showing a strong negative correlation between t and w, the estimate is reliable.

	(vi)	     Since $(\bar{t}, \bar{w})$ is on the line of regression, so the additional point is (22.5, 47.25).