1 To celebrate Senior Citizens' Week, Sports Hub is giving a 30% discount for every 5 Senior Citizen tickets purchased and a 10% discount for every 2 Senior Citizen tickets purchased. There is no discount for the Adult and Child tickets in that week. During that week, Anand, Ben and Charlie purchased tickets with the maximal discount on the Senior Citizen tickets. The number of tickets purchased and the total amount spent are shown in the table below.

	Number of tickets purchased			
	Senior Citizen	Adult	Child	Total amount spent ($)
Anand	16	20	15	840
Ben	27	30	10	1093
Charlie	19	25	12	946

Find the original price of a Senior Citizen ticket. [4]

2 Given that $2y=\cos^{-1}\left(x^2\right)$, show that $(\sin 2y)\frac{\mathrm{d}y}{\mathrm{d}x}=-x$. [1]

By further differentiation of this result, show that $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=0$ when $x=0$. [3]

Hence find the Maclaurin series of y in ascending powers of x, up to and including the term in x^2. [2]

3 **(a)** Show algebraically that x^2-4x+5 is positive for all real values of x. [1]

It is given that k is a constant where $0<k<\frac{1}{2}$.

(b) Solve the inequality $\frac{x^2-4x+5}{(kx-1)(x-2)}<0$, giving your answer in terms of k. [3]

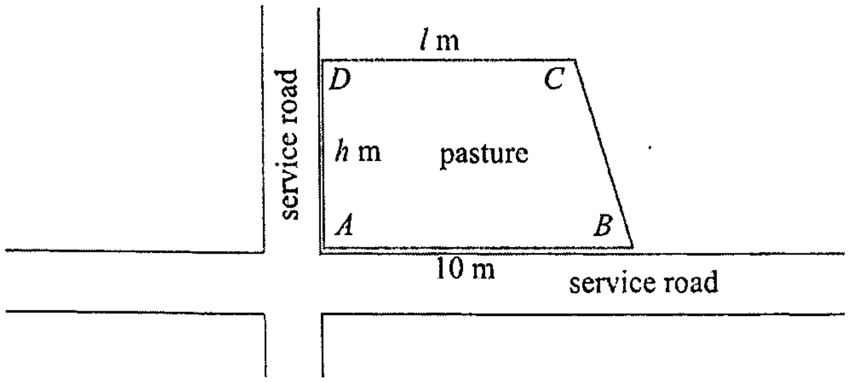
(c) Hence deduce the solution to the inequality $\frac{x^2-4|x|+5}{(k|x|-1)(|x|-2)}<0$. [3]

4 **(a)** Using integration by parts, find the exact value of $\int_1^2 x^2\ln x\,\mathrm{d}x$. [3]

(b) Use the substitution $x=\sec\theta$ to find $\int\frac{x-1}{\sqrt{x^2-1}}\,\mathrm{d}x$. [5]

5 [It is given that the area of a trapezium is $\frac{1}{2}(a+b)h$ where a and b are the lengths of the parallel sides and h is the perpendicular distance between the 2 parallel sides.]

At the junction of two perpendicular service roads, a farmer wishes to create a pasture in the shape of a trapezium $ABCD$ for his herd of goats. It is given that AB is perpendicular to AD, $AB = 10$ metres, $DC = l$ metres and $AD = h$ metres.



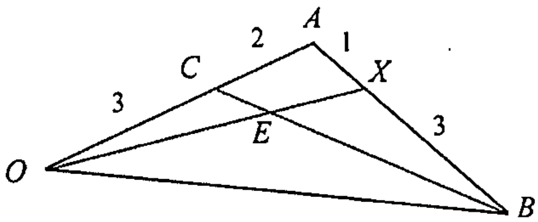
The farmer wishes to use exactly 30 metres of fencing to create the pasture.

(a) Show that $h = \dfrac{10l-150}{l-20}$. [3]

(b) Express the area S metre squares of the pasture in the form $5\left(l+c+\dfrac{d}{l-20}\right)$ where c and d are constants to be determined. [2]

(c) Use differentiation to find the exact value of l that maximises S. [3]

6 Referred to the origin O, points A and B have position vectors **a** and **b** respectively where **a** and **b** are non-zero and non-parallel vectors. Point C lies between O and A such that $OC : CA = 3 : 2$. Point X lies between A and B such that $AX : XB = 1 : 3$. The lines BC and OX meet at the point E.



(a) Show that the vector equation of the line BC can be written as $\mathbf{r} = \frac{3}{5}\lambda\mathbf{a} + (1-\lambda)\mathbf{b}$ where λ is a parameter. [1]

(b) By finding a vector equation of the line OX and using the result in **(a)**, show that the position vector of E is $\frac{1}{2}\underset{\sim}{a} + \frac{1}{6}\underset{\sim}{b}$. [4]

(c) It is given that angle AOB is 30° and **b** is a unit vector. Find the exact perpendicular distance from E to the line OA. [3]

7 The curve C has equation $y=\dfrac{x^2+2x+a^2}{(x+a)^2}$, where a is a real constant and $a \neq 0,1$.

(a) Show that the x-coordinate of the stationary point on C is a. [3]

(b) By completing the square for x^2+2x+a^2, or otherwise, find the exact range of values of a for which C always lies above the x-axis. [3]

It is given that $a > 1$.

(c) Find the equations of the asymptotes of C. [2]

(d) Sketch C, giving the coordinates of the axial intercept and the stationary point. [2]

8 Functions f and g are defined by

$$\mathrm{f}: x \mapsto (x-2)^2-1, \quad x \in \ , x \leq 1,$$

$$\mathrm{g}: x \mapsto \frac{1}{1-x}, \quad x \in \ , x > 1.$$

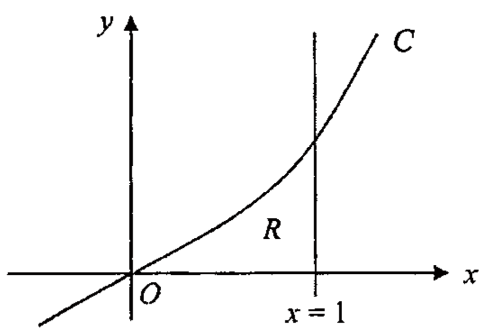
(a) Find $\mathrm{f}^{-1}(x)$ and state the domain of f^{-1}. [3]

(b) On the same diagram, sketch the graphs of $y=\mathrm{f}(x)$ and $y=\mathrm{f}^{-1}(x)$, showing clearly the relationship between the graphs and giving the coordinates of the axial intercepts. Hence find the exact solution of the equation $\mathrm{f}(x)=\mathrm{f}^{-1}(x)$. [5]

(c) Determine whether the composite function fg exists, justifying your answer. [2]

9 **(a)** State a sequence of transformations that will transform the circle with equation $x^2+y^2=1$ onto the curve with equation $\dfrac{(x-2)^2}{9}+y^2=1$. [2]

(b) A curve C with cartesian equation $y=\dfrac{x}{\sqrt{5-x^2}}$ is shown below. R is the finite region enclosed by C, the x-axis and the line $x=1$.



(i) Find the exact area of R. [3]

(ii) Find the exact volume of the solid formed when R is rotated through 2π radians about the y-axis. [6]

10 The line l has equation $\mathbf{r}=\begin{pmatrix}2\\-1\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\0\\3\end{pmatrix}$ where λ is a parameter. The point A has coordinates $(2,-2,1)$.

(a) Find the position vector of the foot of perpendicular from A to l. [3]

The plane p_1 contains A and l.

(b) Find, in scalar product form, a vector equation of p_1. [3]

It is given that l also lies on a second plane p_2 with equation $\mathbf{r}\cdot\begin{pmatrix}1\\a\\b\end{pmatrix}=-1$ where a and b are constants.

(c) Show that $b=-\frac{1}{3}$ and find the value of a. [3]

(d) Hence find the acute angle between p_1 and p_2. [2]

11 A curve C has the parametric equations

$$x=\cos 2\theta,\quad y=1+\sin\theta,\quad \text{for } \frac{\pi}{2}\le\theta\le\frac{3\pi}{2}.$$

(a) Find the exact values of θ at the points where C meets the y-axis. [2]

(b) Sketch C, giving the coordinates of the end-points. [2]

(c) Find $\frac{dy}{dx}$ in terms of θ. Hence find the equation of the tangent to C at the point where $\theta=\frac{3\pi}{4}$, giving your answer in the form $y=px+q$ where p and q are exact constants. [4]

(d) Write down the area of the region bounded by C and the y-axis in the form

$$\int_{\theta_1}^{\theta_2} f(\theta)\,d\theta,$$

where θ_1, θ_2 and $f(\theta)$ should be stated. Hence find the exact value of this area. [5]

(e) Show that if θ is sufficiently small, $\frac{x}{y^2}\approx(1-\theta)^2$. [4]

Q1	Solution
	Let \$$x$, \$$y$ and \$$z$ be the price of a Senior Citizen, an Adult and a Child ticket respectively. $15(0.7)x+1x+20y+15z=840$ $11.5x+20y+15z=840$ ------(1) $25(0.7)x+2(0.9)x+30y+10z=1093$ $19.3x+30y+10z=1093$ ------(2) $15(0.7)x+4(0.9)x+25y+12z=946$ $14.1x+25y+12z=946$ ------(3) Using GC, $x=10$, $y=25$, $z=15$ The original price of a Senior Citizen ticket is \$10.

Q2	Solution
	$2y = \cos^{-1}\left(x^2\right)$ --- (1)

$\cos 2y = x^2$

Differentiating wrt x,

$-2(\sin 2y)\dfrac{dy}{dx} = 2x$

$(\sin 2y)\dfrac{dy}{dx} = -x$ (shown) --- (2)

Differentiating wrt x,

$(\sin 2y)\dfrac{d^2y}{dx^2} + 2(\cos 2y)\left(\dfrac{dy}{dx}\right)^2 = -1$ --- (3)

Differentiating wrt x,

$$(\sin 2y)\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2}\left(2(\cos 2y)\frac{dy}{dx}\right)$$
$$+2(\cos 2y)\left(2\frac{dy}{dx}\right)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\left(-2(\sin 2y)\frac{dy}{dx}\right) = 0$$

i.e. $(\sin 2y)\dfrac{d^3y}{dx^3} + 4(\cos 2y)\dfrac{d^2y}{dx^2}\left(\dfrac{dy}{dx}\right) - 2(\sin 2y)\left(\dfrac{dy}{dx}\right)^3 = 0$ --- (4)

When $x = 0$, $y = \dfrac{1}{2}\cos^{-1} 0 = \dfrac{\pi}{4}$

$$\sin\frac{\pi}{2}\frac{dy}{dx} = 0 \quad \Rightarrow \frac{dy}{dx} = 0$$

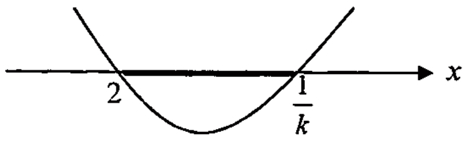
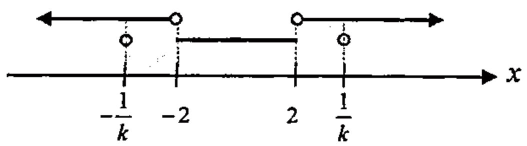
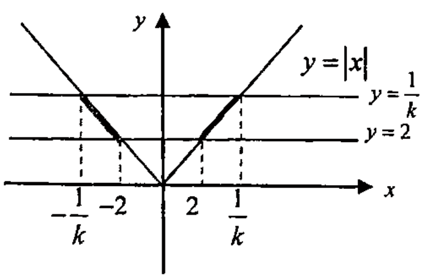
$$\sin\frac{\pi}{2}\frac{d^2y}{dx^2} + 2\cos\frac{\pi}{2}(0)^2 = -1 \quad \Rightarrow \frac{d^2y}{dx^2} = -1$$

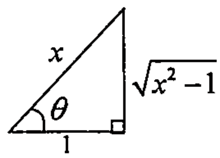
$$\sin\frac{\pi}{2}\frac{d^3y}{dx^3} + 4\cos\frac{\pi}{2}(-1)(0) - 2\sin\frac{\pi}{2}(0)^3 = 0$$

$$\Rightarrow \quad \frac{d^3y}{dx^3} = 0 \text{ (shown)}$$

$y = \dfrac{\pi}{4} + \dfrac{(-1)}{2}x^2 + \ldots$

$= \dfrac{\pi}{4} - \dfrac{1}{2}x^2 + \ldots$

Q3	Solution
(a)	$x^2 - 4x + 5 = (x-2)^2 + 1 > 0$ since $(x-2)^2 \geq 0$ <u>for all real values of x.</u> OR Since discriminant = $(-4)^2 - 4(1)(5) = -4 < 0$, **<u>and coefficient of $x^2 > 0$</u>**, the curve lies entirely above the x-axis. Thus $x^2 - 4x + 5 > 0$ <u>for all real values of x.</u>
(b)	$\dfrac{x^2 - 4x + 5}{(kx-1)(x-2)} < 0$ From (a), $x^2 - 4x + 5 > 0$ <u>for all</u> $x \in \mathbb{R}$, thus $(kx-1)(x-2) < 0$ Given $0 < k < \dfrac{1}{2} \Rightarrow \dfrac{1}{k} > 2$  The solution is $2 < x < \dfrac{1}{k}$
(c)	Replace x by $\|x\|$, $2 < \|x\| < \dfrac{1}{k}$ $\|x\| > 2$ **and** $\|x\| < \dfrac{1}{k}$ $(x > 2$ **or** $x < -2)$ **and** $-\dfrac{1}{k} < x < \dfrac{1}{k}$  The solution is $-\dfrac{1}{k} < x < -2$ <u>or</u> $2 < x < \dfrac{1}{k}$ **<u>Alternative Method</u>**:  From graph, $-\dfrac{1}{k} < x < -2$ <u>or</u> $2 < x < \dfrac{1}{k}$

Q4	Solution
(a)	$\int_1^2 x^2 \ln x \, dx$ $= \left[\left(\frac{x^3}{3}\right)\ln x\right]_1^2 - \int_1^2 \left(\frac{x^3}{3}\right)\left(\frac{1}{x}\right) dx$ $= \frac{8}{3}\ln 2 \; - \; \frac{1}{3}\int_1^2 x^2 \, dx$ $= \frac{8}{3}\ln 2 \; - \; \frac{1}{9}\left[x^3\right]_1^2$ $= \frac{8}{3}\ln 2 \; - \; \frac{7}{9}$
(b)	$\int \frac{x-1}{\sqrt{x^2-1}} \, dx$ $= \int \frac{\sec\theta - 1}{\sqrt{\sec^2\theta - 1}}(\sec\theta\tan\theta) \, d\theta$ $= \int \frac{\sec\theta - 1}{\tan\theta}(\sec\theta\tan\theta) \, d\theta$ $= \int \sec^2\theta - \sec\theta \, d\theta$ $= \tan\theta - \ln\lvert\sec\theta + \tan\theta\rvert + c$ $= \sqrt{x^2-1} - \ln\left\lvert x + \sqrt{x^2-1}\right\rvert + C$ $x = \sec\theta$ $\frac{dx}{d\theta} = \sec\theta\tan\theta$ $x = \sec\theta \Rightarrow \cos\theta = \frac{1}{x}$  $\therefore \tan\theta = \frac{\sqrt{x^2-1}}{1}$ OR $\tan\theta = \sec^2\theta - 1 = \sqrt{x^2-1}$

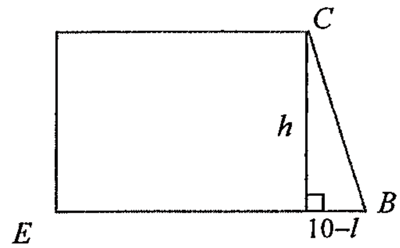
Q5	Solution

(a)

Given the perimeter of pasture = 30,

$10+h+l+BC=30$

$BC = 20-(h+l)$



Using Pythagoras' Theorem,

$h^2+(10-l)^2 = BC^2$

$h^2+(10-l)^2 = \left(20-(h+l)\right)^2$

$h^2+100-20l+l^2 = 400-40(h+l)+(h+l)^2$

$h^2+100-20l+l^2 = 400-40h-40l+h^2+2hl+l^2$

$h(40-2l) = 300-20l$

Hence $h = \dfrac{300-20l}{40-2l} = \dfrac{10l-150}{l-20}$ (shown)

(b)

$S = \dfrac{1}{2}(10+l)\left(\dfrac{10l-150}{l-20}\right)$

$= 5\left(\dfrac{(10+l)(l-15)}{l-20}\right)$

$= 5\left(\dfrac{l^2-5l-150}{l-20}\right)$

$= 5\left(l+15+\dfrac{150}{l-20}\right)$ by long division

(c)

$\dfrac{dS}{dl} = 5\left(1-\dfrac{150}{(l-20)^2}\right)$

When $\dfrac{dS}{dl}=0$, $1-\dfrac{150}{(l-20)^2}=0$

$(l-20)^2 = 150 \Rightarrow l = 20\pm\sqrt{150}$

Since $l<10$, $l = 20-\sqrt{150}$

<u>2nd derivative test</u>

$\dfrac{d^2S}{dl^2} = -5(-2)\left(\dfrac{150}{(l-20)^3}\right) = \dfrac{1500}{(l-20)^3}$

When $l = 20-\sqrt{150}$, $\dfrac{d^2S}{dl^2} = -\dfrac{1500}{\left(\sqrt{150}\right)^3} < 0$

Thus S is a maximum.

1st derivative test

Value of l	$\left(20-\sqrt{150}\right)^-$ e.g. 7.75	$20-\sqrt{150}$	$\left(20-\sqrt{150}\right)^+$ 7.76
$\dfrac{dS}{dl}$	0.00208	0	–0.00608
Sign	+	0	-

Q6	Solution
(a)	$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \frac{3}{5}\underset{\sim}{a} - \underset{\sim}{b}$ Line BC : $\underset{\sim}{r} = \overrightarrow{OB} + \lambda\overrightarrow{BC} = \underset{\sim}{b} + \lambda\left(-\underset{\sim}{b} + \frac{3}{5}\underset{\sim}{a}\right)$ $\underset{\sim}{r} = \frac{3}{5}\lambda\underset{\sim}{a} + (1-\lambda)\underset{\sim}{b}$ where $\lambda \in \square$ (shown)
(b)	Using ratio theorem, $\overrightarrow{OX} = \frac{3\underset{\sim}{a} + \underset{\sim}{b}}{4}$ Line OX: $\underset{\sim}{r} = \frac{\beta}{4}(3\underset{\sim}{a} + \underset{\sim}{b})$ or $\mu(3\underset{\sim}{a} + \underset{\sim}{b})$ At point of intersection E, $\frac{3}{5}\lambda\underset{\sim}{a} + (1-\lambda)\underset{\sim}{b} = \frac{\beta}{4}(3\underset{\sim}{a} + \underset{\sim}{b})$ Since $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are non-zero and non-parallel vectors, $\frac{3}{5}\lambda = \frac{3}{4}\beta$ and $1-\lambda = \frac{\beta}{4}$ $\frac{3}{5}\left(1 - \frac{\beta}{4}\right) = \frac{3}{4}\beta \Rightarrow \beta = \frac{2}{3}$ $\overrightarrow{OE} = \frac{1}{2}\underset{\sim}{a} + \frac{1}{6}\underset{\sim}{b}$ (shown)
(c)	Perpendicular distance of E from line OA $= \frac{\left\lvert\overrightarrow{OE} \times \underset{\sim}{a}\right\rvert}{\lvert\underset{\sim}{a}\rvert}$ $= \frac{\left\lvert\frac{1}{6}(3\underset{\sim}{a} + \underset{\sim}{b}) \times \underset{\sim}{a}\right\rvert}{\lvert\underset{\sim}{a}\rvert}$ $= \frac{1}{6}\frac{\lvert\underset{\sim}{b} \times \underset{\sim}{a}\rvert}{\lvert\underset{\sim}{a}\rvert}$ $(\because \underset{\sim}{a} \times \underset{\sim}{a} = \underset{\sim}{0})$ $= \frac{1}{6}\frac{\lvert\underset{\sim}{a}\rvert\lvert\underset{\sim}{b}\rvert\sin 30^\circ}{\lvert\underset{\sim}{a}\rvert}$ $= \frac{1}{6}\lvert\underset{\sim}{b}\rvert\sin 30^\circ$ $= \frac{1}{12}$

Q7	Solution
(a)	

$$y=\frac{x^2+2x+a^2}{(x+a)^2} \quad (x\neq -a)$$

$$\frac{dy}{dx}=\frac{(x+a)^2(2x+2)-(x^2+2x+a^2)2(x+a)}{(x+a)^4}$$

$$=\frac{2(x+a)\left[(x+a)(x+1)-(x^2+2x+a^2)\right]}{(x+a)^4}$$

$$=\frac{2\left[(a-1)x+a-a^2\right]}{(x+a)^3}$$

$$=\frac{2(a-1)(x-a)}{(x+a)^3}$$

When $\frac{dy}{dx}=0$ and $a\neq 1$, $x=a$ (shown)

Alternative method 1:

$$y=\frac{x^2+2x+a^2}{(x+a)^2}$$

$$y=1+\frac{2(1-a)x}{(x+a)^2}$$

$$\frac{dy}{dx}=\frac{2(1-a)\left[(x+a)^2-x.2(x+a)\right]}{(x+a)^4}$$

$$=\frac{2(1-a)(x+a)\left[(x+a)-2x\right]}{(x+a)^4}$$

$$=\frac{2(1-a)\left[a-x\right]}{(x+a)^3}$$

When $\frac{dy}{dx}=0$ and $a\neq 1$, $x=a$ (shown)

Alternative method 2:

$$y=\frac{x^2+2x+a^2}{(x+a)^2} \quad \text{--- (1)}$$

$$(x+a)^2y=x^2+2x+a^2$$

Differentiate w.r.t. x:

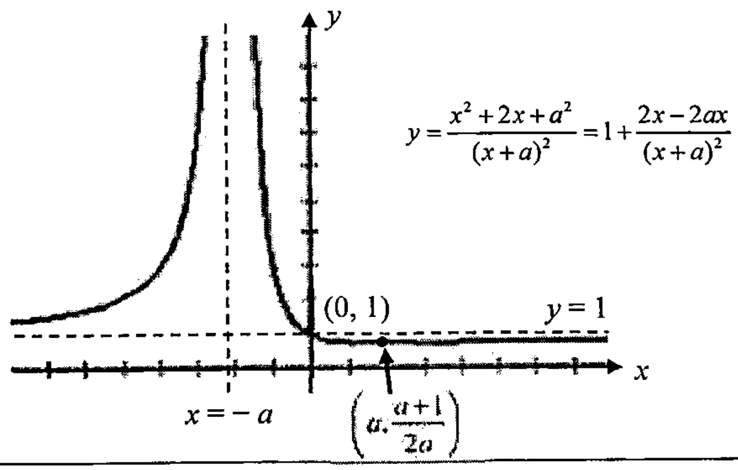
$$(x+a)^2\frac{dy}{dx}+2(x+a)y=2x+2$$

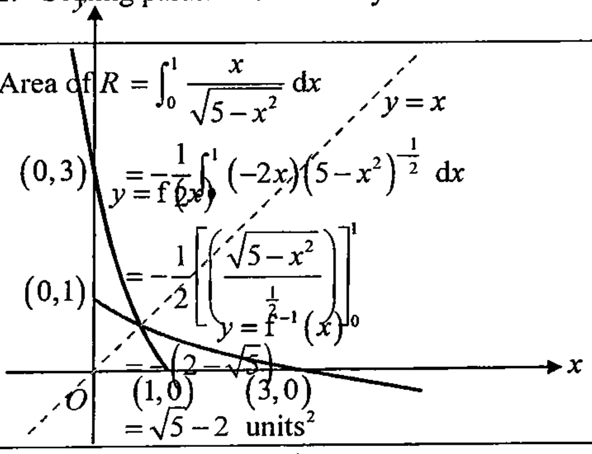
When $\frac{dy}{dx}=0$,

$$(x+a)y=x+1$$

Substitute (1):

$$(x+a)\left(\frac{x^2+2x+a^2}{(x+a)^2}\right)=x+1$$

	Since $x+a\neq 0$, $x^2+2x+a^2=(x+a)(x+1)$ $x^2+2x+a^2=x^2+ax+x+a$ $(1-a)x=a-a^2$ $\therefore\ x=\dfrac{a(1-a)}{1-a}=a$ (shown) since $a\neq 1$
(b)	$y=\dfrac{x^2+2x+a^2}{(x+a)^2}=\dfrac{(x+1)^2+a^2-1}{(x+a)^2},\quad x\neq -a$ For $y>0$ for all real values of x, and since $(x+a)^2>0$ and $(x+1)^2\geq 0$, $a^2-1>0$ $(a+1)(a-1)>0$ $a<-1$ or $a>1$ **<u>Alternative method</u>:** For $y>0$ for all real values of x, and since $(x+a)^2>0$, $x^2+2x+a^2>0$ for all real values of x. Discriminant $=2^2-4a^2<0$ $a^2-1>0$ $(a+1)(a-1)>0$ $a<-1$ or $a>1$
(c)	Equations of asymptotes: $x=-a$ and $y=1$
(d)	

Q8	Solution
(a)	Let $y = f(x) = (x-2)^2 - 1$, $x \le 1$ $(x-2)^2 = y+1$ $x - 2 = \pm\sqrt{y+1}$ Since $x \le 1$, $x = 2 - \sqrt{y+1}$ $\therefore \; f^{-1}(x) = 2 - \sqrt{x+1}$ $D_{f^{-1}} = [0, \infty)$
(b)	
(b)(ii)	At the intersection point, $f(x) = f^{-1}(x) = x$ $(x-2)^2 - 1 = x$ $x^2 - 5x + 3 = 0$ $x = \dfrac{5 \pm \sqrt{13}}{2}$ Since $x \le 1$, $x = \dfrac{5 - \sqrt{13}}{2}$
(c)	$R_g = (-\infty, 0)$, $D_f = (-\infty, 1]$ Since $R_g \subseteq D_f$, fg exists.

Q9	Solution
(a)	$x^2 + y^2 = 1 \xrightarrow{\text{replace } x \text{ by } \frac{x}{3}} \left(\dfrac{x}{3}\right)^2 + y^2 = 1 \xrightarrow{\text{replace } x \text{ by } x-2} \dfrac{(x-2)^2}{9} + y^2 = 1$ The transformations are 1. Scaling parallel to x-axis by a factor 3. 2. Translation of 2 units in the positive direction of x-axis. OR $x^2 + y^2 = 1 \xrightarrow{\text{replace } x \text{ by } x-\frac{2}{3}} \left(x - \dfrac{2}{3}\right)^2 + y^2 = 1 \xrightarrow{\text{replace } x \text{ by } \frac{x}{3}} \left(\dfrac{x}{3} - \dfrac{2}{3}\right)^2 + y^2 = 1$ 1. Translation of $\frac{2}{3}$ units in the positive direction of x-axis.
(b)	2. Scaling parallel to x-axis by a factor 3.
(b)(i)	Area of $R = \int_0^1 \dfrac{x}{\sqrt{5-x^2}}\, dx$ $= -\dfrac{1}{2}\int_0^1 (-2x)\left(5-x^2\right)^{-\frac{1}{2}}\, dx$ $= -\dfrac{1}{2}\left[\left(\dfrac{\sqrt{5-x^2}}{\frac{1}{2}}\right)\right]_0^1$ $= -\left(2 - \sqrt{5}\right)$ $= \sqrt{5} - 2$ units^2
(b)(ii)	$y^2 = \dfrac{x^2}{5-x^2} \Rightarrow 5y^2 - y^2x^2 = x^2 \Rightarrow \left(1+y^2\right)x^2 = 5y^2$ $\Rightarrow x^2 = \dfrac{5y^2}{1+y^2}$ When $x = 1$, $y = \dfrac{1}{\sqrt{4}} = \dfrac{1}{2}$ Volume of solid formed $= \pi(1)^2\left(\dfrac{1}{2}\right) - \pi\int_0^{\frac{1}{2}} x^2\, dy$ $= \dfrac{1}{2}\pi - \pi\int_0^{\frac{1}{2}} \dfrac{5y^2}{1+y^2}\, dy$ $= \dfrac{1}{2}\pi - 5\pi\int_0^{\frac{1}{2}} 1 - \dfrac{1}{1^2+y^2}\, dy$ $= \dfrac{1}{2}\pi - 5\pi\left[y - \tan^{-1} y\right]_0^{\frac{1}{2}}$ $= \dfrac{1}{2}\pi - 5\pi\left[\dfrac{1}{2} - \tan^{-1}\left(\dfrac{1}{2}\right)\right]$ $= \pi\left[5\tan^{-1}\left(\dfrac{1}{2}\right) - 2\right]$ unit^3

Q10	Solution
(a)	Line l: $\underset{\sim}{r} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$ where $\lambda \in \mathbb{R}$ Let N be the foot of perpendicular of A from l. $\overrightarrow{ON} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$ for some $\lambda \in \mathbb{R}$ $\overrightarrow{AN} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$ Since $\overrightarrow{AN}$ perpendicular to l, $\left[\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}\right] \cdot \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = 0$ $-3 + 10\lambda = 0 \Rightarrow \lambda = \frac{3}{10}$ $\therefore \overrightarrow{ON} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \frac{3}{10}\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 23/10 \\ -1 \\ 9/10 \end{pmatrix}$
(b)	A normal of p_1 is $\underset{\sim}{n_1} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}$ Equation of p_1 is $\underset{\sim}{r} \cdot \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} = 7$
(c)	Since l lies on p_2, the direction vector of l and the normal of p_2 are perpendicular. $\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ a \\ b \end{pmatrix} = 0$ $1 + 3b = 0 \Rightarrow b = -\frac{1}{3}$ (shown) The point (2, – 1, 0) lies on l and so it also lies on p_2. $\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ a \\ b \end{pmatrix} = -1$ $2 - a = -1 \Rightarrow a = 3$

Alternative method 1 (if $a = 3$ is already found)

The point on l with position vector $\underset{\sim}{r} = \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix}$ (when $\lambda = 1$) lies on p_2.

$$\begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ a \\ b \end{pmatrix} = -1$$

$$3 - a + 3b = -1$$

$$3b = a - 4 \Rightarrow b = -\frac{1}{3}$$

Alternative method 2

Since l lies on p_2, $\begin{pmatrix} 2+\lambda \\ -1 \\ 3\lambda \end{pmatrix} \square \begin{pmatrix} 1 \\ a \\ b \end{pmatrix} = -1$ for all $\lambda \in \square$.

$$2 + \lambda - a + 3\lambda b = -1$$

$$3 - a + (1 + 3b)\lambda = 0$$

Hence by comparing coefficient ($\because$ equation is true for all $\lambda \in \square$),

$$a = 3$$

$$1 + 3b = 0 \Rightarrow b = -\frac{1}{3}$$

Alternative method 3 (tedious method and can only be used to find b, not a)

Note that since l lies on p_1 and p_2, so l is the line of intersection of the two planes, hence

$\begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ a \\ b \end{pmatrix}$ is parallel to $\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$ i.e $\begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ a \\ b \end{pmatrix} = k \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$ for some $k \in \square$

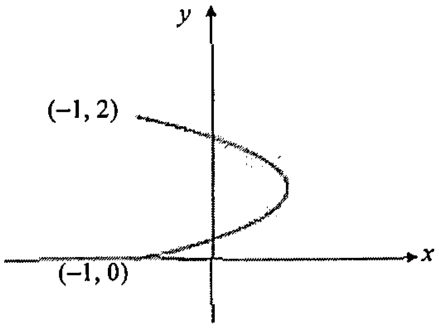
$\begin{pmatrix} -b + a \\ -(3b+1) \\ 3a + 1 \end{pmatrix} = k \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$ for some $k \in \square$

$$\therefore 3b + 1 = 0 \Rightarrow b = -\frac{1}{3}$$

(d) Let the acute angle between the two planes be θ

$$\cos\theta = \frac{\left| \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ -\frac{1}{3} \end{pmatrix} \right|}{\sqrt{11}\sqrt{\frac{91}{9}}} = \frac{1}{\sqrt{1001}}$$

$\theta = 88.2^\circ$ or 1.54 radians

Q11	Solution
(a)	Let $x=\cos 2\theta = 0$ $2\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$ $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \ldots$ $\theta = \frac{3\pi}{4}$ or $\frac{5\pi}{4}$ for $\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$
(b)	When $\theta = \frac{\pi}{2}$, $x=-1, y=2$ (end-point) When $\theta = \frac{3\pi}{2}$, $x=-1, y=0$ (end-point) 
(c)	$x=\cos 2\theta \Rightarrow \frac{dx}{d\theta} = -2\sin 2\theta$ $y = 1+\sin\theta \Rightarrow \frac{dy}{d\theta} = \cos\theta$ $\frac{dy}{dx} = \frac{\cos\theta}{-2\sin 2\theta}$ When $\theta = \frac{3\pi}{4}$, $x = \cos\frac{3\pi}{2} = 0$ $y = 1+\sin\frac{3\pi}{4} = 1+\frac{1}{\sqrt{2}}$ $\frac{dy}{dx} = \frac{\cos\frac{3\pi}{4}}{-2\sin\frac{3\pi}{2}} = -\frac{1}{2\sqrt{2}}$ Equation of tangent: $y-\left(1+\frac{1}{\sqrt{2}}\right) = -\frac{1}{2\sqrt{2}}(x-0)$ i.e $y = -\frac{1}{2\sqrt{2}}x + 1 + \frac{1}{\sqrt{2}}$
(d)	Area $= \int_{y_1}^{y_2} x\ dy$ $= \int_{\frac{5\pi}{4}}^{\frac{3\pi}{4}} \cos 2\theta \cos\theta\ d\theta$

	$=\frac{1}{2}\int_{\frac{5\pi}{4}}^{\frac{3\pi}{4}} \cos 3\theta + \cos\theta \, d\theta$ $=\frac{1}{2}\left[\frac{\sin 3\theta}{3} + \sin\theta\right]_{\frac{5\pi}{4}}^{\frac{3\pi}{4}}$ $=\frac{1}{2}\left[\frac{1}{3}\left(\frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}} - \left(-\frac{1}{3}\left(\frac{1}{\sqrt{2}}\right) - \frac{1}{\sqrt{2}}\right)\right]$ $=\frac{4}{3\sqrt{2}}$ unit^2 or $\frac{2\sqrt{2}}{3}$ unit^2
(e)	$\frac{x}{y^2} = \frac{\cos 2\theta}{(1+\sin\theta)^2}$ $\approx \frac{1 - \frac{(2\theta)^2}{2}}{(1+\theta)^2}$ since θ is sufficiently small $= (1-2\theta^2)(1+\theta)^{-2}$ $= (1-2\theta^2)(1-2\theta+3\theta^2 - ....)$ using Binomial Theorem $= 1 - 2\theta + 3\theta^2 - 2\theta^2 + ...$ $\approx 1 - 2\theta + \theta^2$ since θ is sufficiently small $= (1-\theta)^2$ (shown)