2023 H2 Further Math Promotional Examination Suggested Solutions

1. With respect to the origin as the pole, a curve C has polar equation of the form

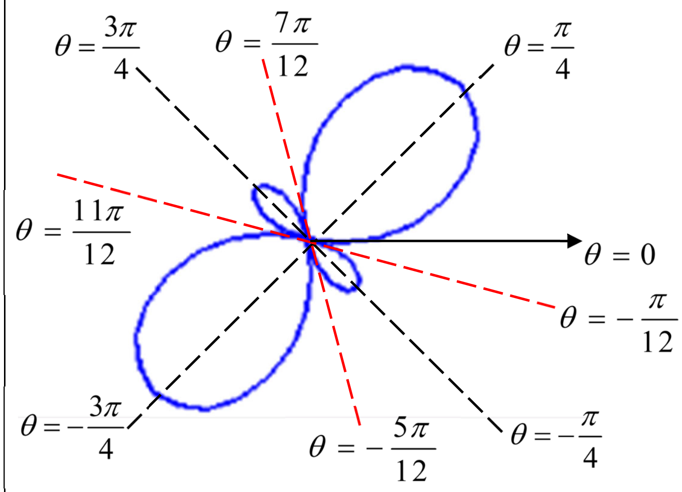
$$r = a + b\sin 2\theta \quad \text{for } -\pi < \theta \le \pi,$$

where a and b are constants, and b is non-zero.

(a) Find the range of values of $\frac{a}{b}$ so that there are tangents to C at the pole. [2]

(b) For the case where $a = 1$ and $b = 2$, sketch C, stating clearly the equations of the tangents to C at the pole, and the equations of the line of symmetry of C. [4]

	Solution
1.(a)	Let $a + b\sin 2\theta = 0$ $\sin 2\theta = -\frac{a}{b}$ For there to be solution, $-1 \le -\frac{a}{b} \le 1$ $-1 \le \frac{a}{b} \le 1$
1.(b)	Let $1 + 2\sin 2\theta = 0$ $\sin 2\theta = -\frac{1}{2}$ $2\theta = -\frac{5\pi}{6}, -\frac{\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$ $\theta = -\frac{5\pi}{12}, -\frac{\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}$ $\frac{\mathrm{d}r}{\mathrm{d}\theta} = 4\cos 2\theta = 0$ $2\theta = -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$ Lines of symmetry are $\theta = -\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}$

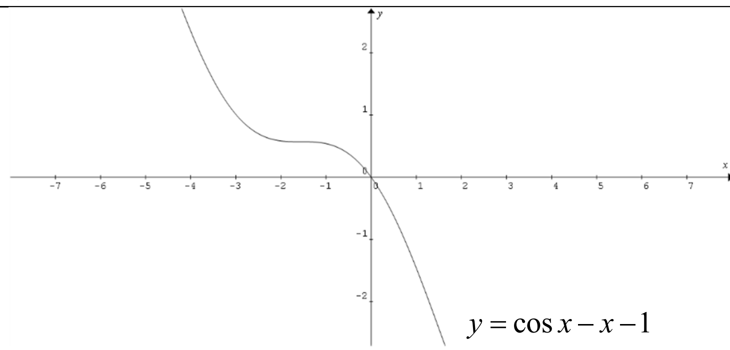


θ = 3π/4
θ = 7π/12
θ = π/4
θ = 11π/12
θ = 0
θ = −π/12
θ = −3π/4
θ = −5π/12
θ = −π/4

2. Consider the equation $\mathrm{f}(x)=0$, where $\mathrm{f}(x)=\cos x-x-1-\beta$ and β is very close to zero.

(a) By sketching the curve $y=\cos x-x-1$, show that $\mathrm{f}(x)=0$ has a single root α, and that α is very close to 0. [3]

(b) Use two iterations of the Newton-Raphson method, with initial approximation $x_0=0$, to show that $\alpha\approx-\beta+\frac{1}{2}\frac{\beta^2}{\beta-1}$. [You may assume for θ small, $\sin\theta\approx\theta$, $\cos\theta\approx1-\frac{\theta^2}{2}$.] [5]

	Solution
2.(a)	 $\mathrm{f}(x)=\cos x-x-1-\beta=0$ $\Rightarrow\cos x-x-1=\beta$ Graph of $y=\cos x-x-1$ passes through the origin, hence the horizontal line $y=\beta$, where β is very close to zero, will cut the graph of $y=\cos x-x-1$ near the origin. Hence the root α to the equation $\mathrm{f}(x)=\cos x-x-1-\beta=0$, which is the x-coordinate of the intersection point between $y=\cos x-x-1$ and $y=\beta$, is close to 0. Since $-1\le\sin x\le1$, $\frac{\mathrm{d}y}{\mathrm{d}x}=-\sin x-1\le0$ Hence graph of $y=\cos x-x-1$ is decreasing, therefore the horizontal line $y=\beta$ will cut the graph of $y=\cos x-x-1$ exactly once. Therefore $\mathrm{f}(x)=\cos x-x-1-\beta=0$ has a single root α which is close to 0.

2.(b)	$\mathrm{f}(x)=\cos x-x-1-\beta$ $\Rightarrow \mathrm{f}'(x)=-\sin x-1$ Newton-Raphson method $x_{n+1}=x_n-\dfrac{\mathrm{f}(x_n)}{\mathrm{f}'(x_n)}$ $x_0=0$, $\mathrm{f}\left(x_0\right)=-\beta$, $\mathrm{f}'\left(x_0\right)=-1$ $x_1=x_0-\dfrac{\mathrm{f}(x_0)}{\mathrm{f}'(x_0)}=-\beta$ $x_1=-\beta$, $\mathrm{f}\left(x_1\right)=\cos\left(-\beta\right)-1=\cos\beta-1$, $\mathrm{f}'\left(x_1\right)=-\sin\left(-\beta\right)-1=\sin\beta-1$ $x_2=x_1-\dfrac{\mathrm{f}(x_1)}{\mathrm{f}'(x_1)}=-\beta-\dfrac{\cos\beta-1}{\sin\beta-1}$ Since β is very small, $\cos\beta\approx 1-\dfrac{\beta^2}{2}$ and $\sin\beta\approx\beta$, and $x_2=-\beta-\dfrac{\cos\beta-1}{\sin\beta-1}$ $\approx-\beta-\dfrac{-\frac{\beta^2}{2}}{\beta-1}$ $=-\beta+\dfrac{1}{2}\dfrac{\beta^2}{\beta-1}$ (shown)

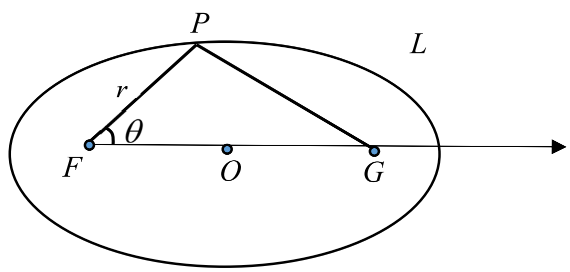
3. Points F and G lie on the x-axis at the points $(-c, 0)$ and $(c, 0)$ respectively, with $c > 0$. The curve L is defined as the locus of points P for which the total distance $FP + PG = 2a$, where $a > c$. It is given that $FP = r$ and $\measuredangle GFP = \theta$.

(a) Show that $r = \dfrac{l}{1 - k\cos\theta}$, giving the constants l and k in terms of a and c. [4]

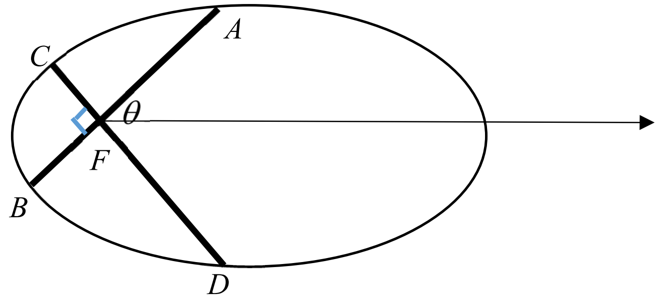
A, B, C, D are points on L such that the chords AB and CD both pass through F, and AB is perpendicular to CD.

(b) Find $\dfrac{1}{|FA|} + \dfrac{1}{|FB|}$ in terms of a and c. [2]

(c) Find $\dfrac{1}{|AB|} + \dfrac{1}{|CD|}$ in terms of a and c. [3]

	Solution
3.(a)	 For triangle PFG, $PG^2 = PF^2 + FG^2 - 2(PF)(FG)\cos\theta$ $(2a-r)^2 = r^2 + (2c)^2 - 2r(2c)\cos\theta$ $4cr\cos\theta - 4ar = 4c^2 - 4a^2$ $r = \dfrac{a^2 - c^2}{a - c\cos\theta}$ $r = \dfrac{a - \frac{c^2}{a}}{1 - \frac{c}{a}\cos\theta}$ Therefore $l = a - \dfrac{c^2}{a}$ $k = \dfrac{c}{a}$

3.(b)



Assuming Point A forms angle θ with axis.

$$\frac{1}{|FA|}+\frac{1}{|FB|}$$

$$=\frac{1-k\cos\theta}{l}+\frac{1-k\cos(\theta+\pi)}{l}$$

$$=\frac{2}{l}=\frac{2a}{a^2-c^2}$$

3.(c)

$$|AB|=|FA|+|FB|$$

$$=\frac{l}{1-k\cos\theta}+\frac{l}{1-k\cos(\theta+\pi)}$$

$$=\frac{l}{1-k\cos\theta}+\frac{l}{1+k\cos\theta}$$

$$=\frac{2l}{1-k^2\cos^2\theta}$$

$$|CD|=|FC|+|FD|$$

$$=\frac{l}{1-k\cos\left(\theta+\frac{\pi}{2}\right)}+\frac{l}{1-k\cos\left(\theta+\frac{3\pi}{2}\right)}$$

$$=\frac{l}{1+k\sin\theta}+\frac{l}{1-k\sin\theta}$$

$$=\frac{2l}{1-k^2\sin^2\theta}$$

$$\frac{1}{|AB|}+\frac{1}{|CD|}=\frac{1-k^2\sin^2\theta}{2l}+\frac{1-k^2\cos^2\theta}{2l}$$

$$=\frac{2-k^2}{2l}$$

$$=\frac{2a^2-c^2}{2a(a^2-c^2)}$$

4. A sequence $u_1,\ u_2,\ u_3,\ \ldots$ is such that $u_1 = \dfrac{2}{3}$ and $u_n = u_{n-1} - \dfrac{1}{4n^2-1}$, for all $n \geq 2$.

(a) Find the values of u_2, u_3 and u_4, giving each answer as a fraction in its lowest term. Make a conjecture for u_n in terms of n. [2]

(b) Use Mathematical Induction to prove your conjecture in **(i)** for all positive integers n. [4]

(c) Deduce the value of $\displaystyle\sum_{n=2}^{\infty} \frac{1}{4n^2-1}$. [3]

	Solution
4.(a)	$u_2 = u_1 - \dfrac{1}{4\left(2^2\right)-1} = \dfrac{2}{3} - \dfrac{1}{15} = \dfrac{3}{5}$ $u_3 = u_2 - \dfrac{1}{4\left(3^2\right)-1} = \dfrac{3}{5} - \dfrac{1}{35} = \dfrac{4}{7}$ $u_4 = u_3 - \dfrac{1}{4\left(4^2\right)-1} = \dfrac{4}{7} - \dfrac{1}{63} = \dfrac{5}{9}$ Conjecture: $u_n = \dfrac{n+1}{2n+1}$
4.(b)	Let P(n) be the statement: $u_n = \dfrac{n+1}{2n+1}$ for all $n \in \mathbb{Z}^+$. When $n = 1$, LHS = $u_1 = \dfrac{2}{3}$ RHS $= \dfrac{1+1}{2(1)+1} = \dfrac{2}{3} = \text{LHS}$ $\therefore$ P(1) is true. Assume P(k) is true for some $k \in \mathbb{Z}^+$, i.e. $u_k = \dfrac{k+1}{2k+1}$ To prove P(k+1) is also true, i.e. $u_{k+1} = \dfrac{k+2}{2k+3}$. LHS $= u_{k+1}$ $= u_k - \dfrac{1}{4(k+1)^2-1}$ (given) $= \dfrac{k+1}{2k+1} - \dfrac{1}{4k^2+8k+3}$ (by assumption)

	$= \dfrac{k+1}{2k+1} - \dfrac{1}{(2k+3)(2k+1)}$ $= \dfrac{(k+1)(2k+3)-1}{(2k+3)(2k+1)}$ $= \dfrac{2k^2+5k+2}{(2k+3)(2k+1)}$ $= \dfrac{(2k+1)(k+2)}{(2k+3)(2k+1)}$ $= \dfrac{k+2}{2k+3} = \text{RHS}$ Since P(1) is true, and P(k) is true $\Rightarrow$ P(k+1) is true, by Mathematical Induction, P(n) is true for all $n \in \mathbb{Z}^+$.
4.(c)	$\sum_{n=2}^{N} \dfrac{1}{4n^2-1}$ $= \sum_{n=2}^{N} (u_{n-1} - u_n)$ $= u_1 - u_2$ $+ u_2 - u_3$: : $+ u_{N-1} - u_N$ $= u_1 - u_N$ $= \dfrac{2}{3} - \dfrac{N+1}{2N+1}$ $= \dfrac{2}{3} - \dfrac{1}{2}\dfrac{(2N+1)+1}{2N+1}$ $= \dfrac{2}{3} - \dfrac{1}{2} - \dfrac{1}{2(2N+1)}$ $= \dfrac{1}{6} - \dfrac{1}{2(2N+1)}$ As $N \to \infty$, $\dfrac{1}{2(2N+1)} \to 0$. Hence $\sum_{n=1}^{\infty} \dfrac{1}{4n^2-1} = \dfrac{1}{6}$.

5. A function f is defined as $\mathrm{f}(x) = x^3 - x^{-1}$, $x \in \mathbb{R}$, $x > 1$.

(a) Using the trapezium rule with 3 ordinates, the value for $\int_2^a \mathrm{f}(x)\,\mathrm{d}x$, where $a \in \mathbb{R}$, $a > 2$, is approximately 5.56. Find the value of a correct to 1 decimal place. [3]

An approximate value for $\int_2^{2.5} \mathrm{f}(x)\,\mathrm{d}x$ is to be found using Simpson's rule.

(b) Explain why it is not possible to use Simpson's rule with 3 strips to find an approximate value for $\int_2^{2.5} \mathrm{f}(x)\,\mathrm{d}x$. [2]

(c) Estimate $\int_2^{2.5} \mathrm{f}(x)\,\mathrm{d}x$ using Simpson's rule with 4 strips. Leave your answer correct to 1 decimal place. [2]

(d) A student made the following claim:

'Since Simpson's rule will give an exact value for $\int_2^{2.5} \mathrm{f}(x)\,\mathrm{d}x$ if $\mathrm{f}(x)$ is of degree 3 or less, the value for $\int_2^{2.5} \mathrm{f}(x)\,\mathrm{d}x$ calculated in part **(c)** should be taken instead of the value for $\int_2^{2.5} \mathrm{f}(x)\,\mathrm{d}x$ given in part **(a)**.'

Comment on the student's claim. [2]

	Solution
5.(a)	Using trapezium rule with 3 ordinates, $\int_2^a x^3 - x^{-1}\,\mathrm{d}x \approx \frac{\frac{a-2}{2}}{2}\left[(2^3 - \frac{1}{2}) + 2[(\frac{a+2}{2})^3 - \frac{2}{a+2}] + (a^3 - \frac{1}{a})\right]$ $= \frac{a-2}{4}\left[\frac{15}{2} + \frac{(a+2)^3}{4} - \frac{4}{a+2} + a^3 - \frac{1}{a}\right]$ Given $\frac{a-2}{4}\left[\frac{15}{2} + \frac{(a+2)^3}{4} - (\frac{4}{a+2}) + a^3 - \frac{1}{a}\right] = 5.56$ $\Rightarrow \frac{15}{2} + \frac{(a+2)^3}{4} - (\frac{4}{a+2}) + a^3 - \frac{1}{a} - \frac{22.24}{a-2} = 0$ Using GC,

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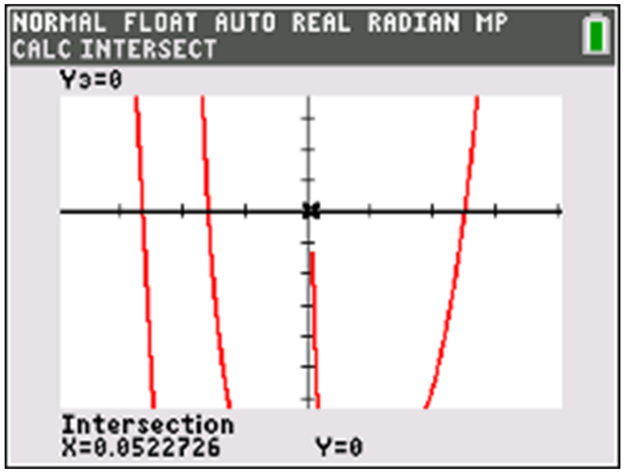

$a = -2.654242$ (rejected since $a > 2$)
or
$a = -1.606305$ (rejected since $a > 2$)
or
$a = 0.0522726$ (rejected since $a > 2$)
or
$a = 2.4988872 \approx 2.5$ (1 d.p.)
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<td>5.(b)</td>
<td>Simpson's rule requires number of strips (intervals) to be in pairs (3 ordinates form a pair) to form a parabola so that an estimate using the parabola can be carried out. If there are 3 strips, then 1 nor 2 parabolas cannot be formed and thus Simpson's rule cannot be carried out.</td>
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<td>5.(c)</td>
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$$\int_2^{2.5} \mathrm{f}(x)\,\mathrm{d}x \approx \frac{\frac{2.5-2}{4}}{3}\left[\begin{array}{l} 2^3 - \frac{1}{2} + 4[(\frac{17}{8})^3 - \frac{8}{17} + (\frac{19}{8})^3 - \frac{8}{19}] \\ +2[(\frac{18}{8})^3 - \frac{8}{18}] + (2.5)^3 - \frac{1}{2.5} \end{array}\right]$$

$$= 5.542481152$$

$$= 5.5 \quad \text{(1 d.p.)}$$

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<td>5.(d)</td>
<td>$\mathrm{f}(x)$ is <u>not</u> a polynomial of degree 3 or less, hence student's claim that Simpson's rule will give an exact value for $\int_2^{2.5} \mathrm{f}(x)\,\mathrm{d}x$ is not valid. Since both 5.6 given in part (a) and 5.5 found in part (c) are estimates, it is not meaningful to make comparisons and choose one over another.

Note:
A polynomial is an expression consisting of variables and coefficients where all the powers of the variables are <u>positive integers or zero</u>.</td>
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6. Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ be a function given by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x-y+5z+2k \\ x+3y+6z-k \\ y+z+3k \end{pmatrix},$$

where k is a constant.

(a) State the value of k for which T is a linear transformation, justifying your answer. [2]

For the rest of this question, use the value of k found in part **(a)**.

(b) Find the standard matrix representing T. [1]

(c) Find a basis for the nullspace of T and hence deduce the nullity and rank of T. [3]

(d) Find a basis for the range of T. [2]

(e) Find the values of a and b for which the set of vectors in $\mathbb{R}^3$ mapped to $\begin{pmatrix} 2a \\ b+1 \\ 0 \end{pmatrix}$ by T is a subspace of $\mathbb{R}^3$. [2]

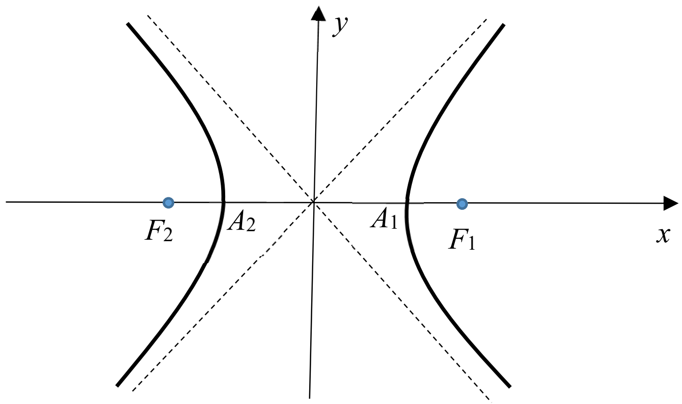
	Solution
6.(a)	$k=0$. This is because a linear transformation maps **0** to **0**, since $T(\mathbf{0}) = T(0\mathbf{v}) = \mathbf{0}$. We note that $T\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right) = k\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$, which justifies the above value of k.
6.(b)	$\begin{pmatrix} 2x-y+5z \\ x+3y+6z \\ y+z \end{pmatrix} = x\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} + z\begin{pmatrix} 5 \\ 6 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 5 \\ 1 & 3 & 6 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ Hence the standard matrix representing T is $\begin{pmatrix} 2 & -1 & 5 \\ 1 & 3 & 6 \\ 0 & 1 & 1 \end{pmatrix}$.
6.(c)	We form $\left(\begin{array}{ccc\|c} 2 & -1 & 5 & 0 \\ 1 & 3 & 6 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right)$.

	By G.C., the r.r.e.f. is $\left(\begin{array}{ccc\|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$. Hence, we have $x+3z=0$ and $y+z=0$, which means the nullspace of T is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = t\begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix}$, $t \in \mathbb{R}$. Thus, $\left\{\begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix}\right\}$ is a basis of the nullspace of T. The nullity of T is 1 and by the rank-nullity theorem, we have $\operatorname{rank}(T) = \dim(\mathbb{R}^3) - \operatorname{nullity}(T) = 3-1 = 2$
6.(d)	Since $-3\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 5 \\ 6 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$ and $\left\{\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 6 \\ 1 \end{pmatrix}\right\}$ is a linearly independent set, $\left\{\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 6 \\ 1 \end{pmatrix}\right\}$ is a basis for the range of T since $\operatorname{rank}(T) = 2$ by part **(c)**.
6.(e)	We perform row reduction on the augmented matrix. $\left(\begin{array}{ccc\|c} 2 & -1 & 5 & 2a \\ 1 & 3 & 6 & b+1 \\ 0 & 1 & 1 & 0 \end{array}\right) \xrightarrow[R_2:-3R_3+R_2]{R_1:R_3+R_1} \left(\begin{array}{ccc\|c} 2 & 0 & 6 & 2a \\ 1 & 0 & 3 & b+1 \\ 0 & 1 & 1 & 0 \end{array}\right)$ $\xrightarrow{R_1:\frac{1}{2}R_1} \left(\begin{array}{ccc\|c} 1 & 0 & 3 & a \\ 1 & 0 & 3 & b+1 \\ 0 & 1 & 1 & 0 \end{array}\right) \xrightarrow{R_2:-R_1+R_2} \left(\begin{array}{ccc\|c} 1 & 0 & 3 & a \\ 0 & 0 & 0 & -a+b+1 \\ 0 & 1 & 1 & 0 \end{array}\right)$ The system is consistent if and only if $a = b+1$ are equal real numbers. Thus assuming that $a=b+1$ are equal, we have

	$x = a - 3z,\ y = -z$ $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix},\ t \in \mathbb{R}$ Hence the set of vectors in $\mathbb{R}^3$ mapped to $\begin{pmatrix} 2a \\ b \\ 0 \end{pmatrix}$ is a subspace of $\mathbb{R}^3$ if and only if $a = 0$ and $b = -1$.

7. The diagram shows a hyperbola H with equation $x^2 - y^2 = 2$. Its foci are at $F_1(c,0)$ and $F_2(-c,0)$ respectively, and its vertices are at A_1 and A_2 respectively.

(a) Find the value of c and the coordinates of A_1 and A_2. [2]



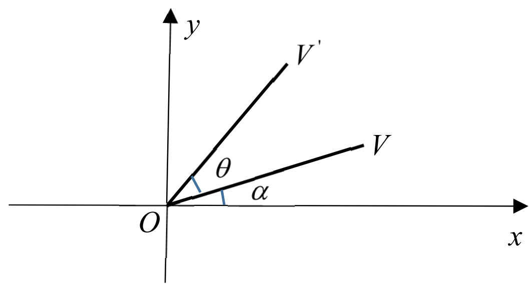
(b) With respect to the two-dimensional cartesian coordinate system, prove that if the position vector $\begin{pmatrix} V_x' \\ V_y' \end{pmatrix}$ is obtained by rotating the position vector $\begin{pmatrix} V_x \\ V_y \end{pmatrix}$ θ radian anticlockwise about the origin, then

$$\begin{pmatrix} V_x' \\ V_y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} V_x \\ V_y \end{pmatrix}.$$ [3]

(c) H is rotated $\frac{\pi}{3}$ radians anticlockwise about the origin to form the curve H'. Find the cartesian equation of H'. [3]

(d) Given that for any point P on H', $|BP - B'P|$ is a constant for points B and B'. Find the coordinates of B and B'. [2]

	Solution
7.(a)	$x^2 - y^2 = 2$ $\Rightarrow \frac{x^2}{2} - \frac{y^2}{2} = 1$ $c^2 = a^2 + b^2 = 2 + 2$ $\Rightarrow c = \sqrt{4} = 2$ $A_1(\sqrt{2},0),\ \ A_2(-\sqrt{2},0)$

7.(b)	 $OV = OV' = r$, $V_x = r\cos\alpha$, $V_y = r\sin\alpha$ $V_x' = r\cos(\theta+\alpha)$ $= (r\cos\alpha)\cos\theta - (r\sin\alpha)\sin\theta$ -------- (1) $= V_x\cos\theta - V_y\sin\theta$ $V_y' = r\sin(\theta+\alpha)$ $= (r\cos\alpha)\sin\theta + (r\sin\alpha)\cos\theta$ ------- (2) $= V_x\sin\theta + V_y\cos\theta$ Therefore $\begin{pmatrix} V_x' \\ V_y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} V_x \\ V_y \end{pmatrix}$
7.(b)	Alternative solution: $\begin{pmatrix} V_x \\ V_y \end{pmatrix} = V_x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + V_y\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ rotate θ radians anticlockwise about the origin will give $\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$. $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ rotate θ radians anticlockwise about the origin will give $\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$ $\begin{pmatrix} V_x' \\ V_y' \end{pmatrix} = V_x\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} + V_y\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$ $= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} V_x \\ V_y \end{pmatrix}$

7.(c)	For $\theta = \frac{\pi}{3}$, $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ $x' = \frac{1}{2}x - \frac{\sqrt{3}}{2}y \qquad (1)$ $y' = \frac{\sqrt{3}}{2}x + \frac{1}{2}y \qquad (2)$ (1) $\times\sqrt{3}$: $\sqrt{3}x' = \frac{\sqrt{3}}{2}x - \frac{3}{2}y \quad (3)$ (2) $\times\sqrt{3}$: $\sqrt{3}y' = \frac{3}{2}x + \frac{\sqrt{3}}{2}y \quad (4)$ (1) + (4): $x' + \sqrt{3}y' = 2x \qquad (5)$ (2) - (3): $-\sqrt{3}x' + y' = 2y \quad (6)$ Sub (5) and (6) into $x^2 - y^2 = 2$ $\left(\frac{\sqrt{3}y' + x'}{2}\right)^2 - \left(\frac{y' - \sqrt{3}x'}{2}\right)^2 = 2$ $\left(\sqrt{3}y' + x'\right)^2 - \left(y' - \sqrt{3}x'\right)^2 = 8$ $3(y')^2 + 2\sqrt{3}x'y' + (x')^2 - \left((y')^2 - 2\sqrt{3}x'y' + 3(x')^2\right) = 8$ $2(y')^2 + 4\sqrt{3}x'y' - 2(x')^2 = 8$ $(y')^2 + 2\sqrt{3}x'y' - (x')^2 = 4$ Therefore, the new cartesian equation of the hyperbola H' after the rotation is $y^2 - x^2 + 2\sqrt{3}xy = 4$.
7.(d)	B and B' will be foci for the graph of H' $\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}$

	$\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -2 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -\sqrt{3} \end{pmatrix}$ $B\left(1, \sqrt{3}\right),\ B'\left(-1, -\sqrt{3}\right)$

8. The population of pigeons in a town increases at a rate of x% per month, where $x > 0$. After numerous complaints from the town residents about the problems caused by the increasing number of pigeons, the town council takes action and sends a team to cull k pigeons at the end of every month starting from January 2023, with $k > 0$.

Let P_n be the pigeon population in the town n months after 31 December 2022. It is given that the pigeon population in the town on 31 December 2022 was 400.

(a) If $x = 2.5$, write down a recurrence relation to model the pigeon population n months after 31 December 2022, where $n \geq 1$. Solve the recurrence relation. [4]

(b) Using the result from part **(a)**, find the range of values of k so that the town council is able to resolve the complaints from residents regarding the pigeons. [2]

(c) State an assumption required for the model in part **(a)** to work. [1]

An environmentalist claimed that although the complaints from residents had to be addressed, the town also cannot exterminate all the pigeons.

(d) Find P_n in terms of x, k and n. Hence determine the relationship between x and k so that both the complaints from residents and the environmentalist's concern can be addressed. [4]

(e) State a possible reason why it is not advisable to exterminate all pigeons in the town. [1]

	Solution
8.(a)	$P_n = 1.025P_{n-1} - k,\ n \geq 1$
	Method 1: $P_n = 1.025P_{n-1} - k$ $= 1.025(1.025P_{n-2} - k) - k$ $= 1.025^2 P_{n-2} - 1.025(k) - k$ $= 1.025^2(1.025P_{n-3} - k) - 1.025(k) - k$ $= 1.025^3 P_{n-3} - 1.025^2(k) - 1.025(k) - k$ $\vdots$ $\vdots$ $= 1.025^n P_0 - 1.025^{n-1}(k) - 1.025^{n-2}(k) - \ldots - k$ $= 400(1.025^n) - k\dfrac{1.025^n - 1}{1.025 - 1}$ $= 400(1.025^n) - 40k(1.025^n) + 40k$ $= 1.025^n(400 - 40k) + 40k$

	Method 2: $P_n = 1.025P_{n-1} - k$ $P_n - \frac{k}{1.025-1} = 1.025P_{n-1} - k - \frac{k}{1.025-1}$ $P_n - 40k = 1.025P_{n-1} - 41k$ $= 1.025(P_{n-1} - 40k)$ So { $P_n - 40k$ } is a GP with first term $P_1 - 40k = 400(1.025) - k - 40k = 410 - 41k$ and common ratio 1.025 $P_n - 40k = 1.025^{n-1}(410 - 41k)$ $P_n = 1.025^{n-1}(410 - 41k) + 40k$
	Method 3: $P_n = A\left(1.025^n\right) + B$, where A, B are constants $P_0 = A + B = 400 \qquad ---(1)$ $P_1 = 1.025A + B = 400(1.025) - k = 410 - k \qquad ---(2)$ (2) – (1): $0.025A = 10 - k$ $A = 400 - 40k$ $B = 40k$ $\therefore P_n = 1.025^n(400 - 40k) + 40k$
8.(b)	If $(400 - 40k) > 0$, then $P_n \to \infty$ as $n \to \infty$. We require $400 - 40k \le 0$ $\Rightarrow k \ge 10$
8.(c)	The model assumes that: the rate of increase remains constant every month.
8.(d)	$P_n = \left(1+\frac{x}{100}\right)^n P_0 - k\left[\left(1+\frac{x}{100}\right)^{n-1} + \left(1+\frac{x}{100}\right)^{n-2} + \ldots + 1\right]$ $= \left(1+\frac{x}{100}\right)^n P_0 - k\frac{\left(1+\frac{x}{100}\right)^n - 1}{1+\frac{x}{100} - 1}$ $= \left(1+\frac{x}{100}\right)^n P_0 - \frac{100k}{x}\left[\left(1+\frac{x}{100}\right)^n - 1\right]$ $= \left(1+\frac{x}{100}\right)^n \left[P_0 - \frac{100k}{x}\right] + \frac{100k}{x}$

	For both complaints and concerns to be addressed, we require P_n to reach a nonzero equilibrium in the long run. $P_0 - \frac{100k}{x} = 0$ $xP_0 = 100k$ Since $P_0 = 400$, $x = \frac{k}{4}$
8.(e)	• Pigeons are part of the ecosystem in the town. By exterminating all pigeons, the balance in the ecosystem will be broken, leading to other problems. • Predators that prey on pigeons will die off. • Some pigeons keep the population of insects and worms under control. • Pigeons help in seed dispersal for some plants.

9. **(a)** Show that for a 3×3 upper triangular matrix

$$\mathbf{T}=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix},$$

its eigenvalues are 3 of its entries. Hence state its eigenvalues in the form of a_{ij}. [2]

It is given that $\mathbf{A}=\begin{pmatrix} 1 & a & b \\ 0 & 2 & c \\ 0 & 0 & 3 \end{pmatrix}$, $a,b,c\in\mathbb{R}$.

(b) Write down the eigenvalues of **A** and obtain their corresponding eigenvectors. [4]

It is given that **E** is a non-zero scalar matrix, i.e., $\mathbf{E}=\begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}$, $k\neq 0$.

(c) Write down $\mathbf{E}^{-1}$. [1]

The matrix **B** is given as $\mathbf{B}=\mathbf{E}^{-1}\mathbf{A}^{\mathrm{T}}$, where $\mathbf{A}^{\mathrm{T}}$ is the transpose of **A**.

(d) Show that if $\mathbf{e}_1$ is an eigenvector of **A**, then $\mathbf{e}_1$ is also an eigenvector of $\mathbf{B}^{\mathrm{T}}$. [2]

(e) Hence express $\mathbf{B}^{\mathrm{T}}$ in the form of

$$\mathbf{PDP}^{-1},$$

where **D** is a diagonal matrix, **P** and $\mathbf{P}^{-1}$ are invertible matrices to be determined. [4]

	Solution
9.(a)	Let **I** be the 3×3 identity matrix. Then,

	λ is an eigenvalue of $\mathbf{T}$ $\Leftrightarrow$ there exists $\mathbf{x} \neq \mathbf{0}$ such that $(\lambda\mathbf{I}-\mathbf{T})\mathbf{x}=\mathbf{0}$ $\Leftrightarrow \det\begin{pmatrix}\lambda-a_{11} & -a_{12} & -a_{13}\\ 0 & \lambda-a_{22} & -a_{23}\\ 0 & 0 & \lambda-a_{33}\end{pmatrix}=0$ $\Leftrightarrow (\lambda-a_{11})(\lambda-a_{22})(\lambda-a_{33})+(0)(0)(-a_{13})$ $+(0)(-a_{12})(-a_{23})-(0)(\lambda-a_{22})(-a_{13})$ $-(\lambda-a_{11})(-a_{23})(0)-(\lambda-a_{33})(-a_{12})(0)=0$ $\Leftrightarrow (\lambda-a_{11})(\lambda-a_{22})(\lambda-a_{33})=0$ $\Leftrightarrow \lambda=a_{11}$ or $\lambda=a_{22}$ or $\lambda=a_{33}$
9.(b)	Since **A** is an upper triangular matrix, its eigenvalues are $1,2,3$ by part **(a)**.
	For $\lambda=1$, $(\mathbf{I}-\mathbf{A})\mathbf{x}=\mathbf{0}$ $\Leftrightarrow \begin{pmatrix}1-1 & -a & -b\\ 0 & 1-2 & -c\\ 0 & 0 & 1-3\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$ $\Leftrightarrow \begin{pmatrix}0 & -a & -b\\ 0 & -1 & -c\\ 0 & 0 & -2\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$ $\Leftrightarrow ay+bz=0$ and $y+cz=0$ and $z=0$ $\Leftrightarrow ay+bz=0$ and $y=0$ and $z=0$ $\Leftrightarrow y=0$ and $z=0$ Thus $\begin{pmatrix}x\\ y\\ z\end{pmatrix}=x\begin{pmatrix}1\\ 0\\ 0\end{pmatrix}$ and so, $\begin{pmatrix}1\\ 0\\ 0\end{pmatrix}$ is an eigenvector corresponding to 1.
	For $\lambda=2$,

	$(2\mathbf{I}-\mathbf{A})\mathbf{x}=\mathbf{0}$ $\Leftrightarrow \begin{pmatrix} 2-1 & -a & -b \\ 0 & 2-2 & -c \\ 0 & 0 & 2-3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\Leftrightarrow \begin{pmatrix} 1 & -a & -b \\ 0 & 0 & -c \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\Leftrightarrow x-ay-bz=0$ and $cz=0$ and $z=0$ $\Leftrightarrow x=ay$ and $z=0$ Thus $\begin{pmatrix} x \\ y \\ z \end{pmatrix}=y\begin{pmatrix} a \\ 1 \\ 0 \end{pmatrix}$ and so, $\begin{pmatrix} a \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector corresponding to 2.
	For $\lambda=3$, $(3\mathbf{I}-\mathbf{A})\mathbf{x}=\mathbf{0}$ $\Leftrightarrow \begin{pmatrix} 3-1 & -a & -b \\ 0 & 3-2 & -c \\ 0 & 0 & 3-3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\Leftrightarrow \begin{pmatrix} 2 & -a & -b \\ 0 & 1 & -c \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\Leftrightarrow 2x-ay-bz=0$ and $y-cz=0$ $\Leftrightarrow 2x=ay+bz$ and $y=cz$ $\Leftrightarrow 2x=acz+bz$ and $y=cz$ $\Leftrightarrow x=\left(\frac{ac+b}{2}\right)z$ and $y=cz$ Thus $\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} \frac{ac+b}{2} \\ cz \\ z \end{pmatrix}=z\begin{pmatrix} \frac{ac+b}{2} \\ c \\ 1 \end{pmatrix}$ and so, $\begin{pmatrix} \frac{ac+b}{2} \\ c \\ 1 \end{pmatrix}$ is an eigenvector corresponding to 3.

9.(c)	$\mathbf{E}^{-1} = \begin{pmatrix} \frac{1}{k} & 0 & 0 \\ 0 & \frac{1}{k} & 0 \\ 0 & 0 & \frac{1}{k} \end{pmatrix}$
9.(d)	First, we note that $\mathbf{EB} = \mathbf{A}^T \Leftrightarrow \mathbf{B}^T\mathbf{E}^T = \mathbf{A} \Leftrightarrow k\mathbf{B}^T = \mathbf{A}$ Hence $\mathbf{Ae}_1 = \lambda\mathbf{e}_1 \Leftrightarrow k\mathbf{B}^T\mathbf{e}_1 = \lambda\mathbf{e}_1 \Leftrightarrow \mathbf{B}^T\mathbf{e}_1 = \frac{\lambda}{k}\mathbf{e}_1$
9.(e)	$\mathbf{B}^T = \begin{pmatrix} 1 & a & \frac{ac+b}{2} \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{k} & 0 & 0 \\ 0 & \frac{2}{k} & 0 \\ 0 & 0 & \frac{3}{k} \end{pmatrix} \begin{pmatrix} 1 & a & \frac{ac+b}{2} \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}^{-1}$ $\left(\begin{array}{ccc\|ccc} 1 & a & \frac{ac+b}{2} & 1 & 0 & 0 \\ 0 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$ $\xrightarrow[R_1:R_1-\frac{ac+b}{2}R_3]{R_2:R_2-cR_3} \left(\begin{array}{ccc\|ccc} 1 & a & 0 & 1 & 0 & -\frac{ac+b}{2} \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$ $\xrightarrow{R_1:R_1-aR_2} \left(\begin{array}{ccc\|ccc} 1 & 0 & 0 & 1 & -a & -\frac{ac+b}{2}+ac \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$ Hence $\mathbf{P} = \begin{pmatrix} 1 & a & \frac{ac+b}{2} \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$, $\mathbf{D} = \begin{pmatrix} \frac{1}{k} & 0 & 0 \\ 0 & \frac{2}{k} & 0 \\ 0 & 0 & \frac{3}{k} \end{pmatrix}$ and

	$\mathbf{P}^{-1} = \begin{pmatrix} 1 & -a & \dfrac{ac-b}{2} \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{pmatrix}$

10. A curve C is defined by the parametric equations

$$x = \sin t - t\cos t,\ y = \cos t + t\sin t\ ,$$

for $\frac{\pi}{6} \le t \le \pi a$, where a is a positive constant.

(a) It is given that C has a length of $\frac{\pi^2}{9}$ units. Show that $a = \frac{1}{2}$. [5]

Let $a = \frac{1}{2}$ for the rest of the question.

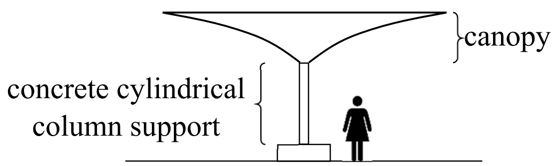


Figure 1

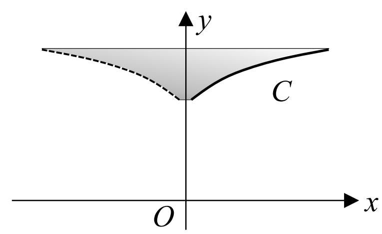


Figure 2

An architect designs a shelter for a public park. The shelter consists of a canopy supported by a concrete cylindrical column, with the diameter of the column the same as the width of the bottom of the canopy as shown in Figure 1. The canopy is formed (in suitable units) by rotating C completely about the y-axis. Coordinate axes have been superimposed onto C for ease of reference as shown in Figure 2.

(b) Determine the exact value of the diameter of the concrete cylindrical column. [2]

(c) The bottom surface area of the canopy, denoted by A, is to be coated with heat insulation paint. Show, with full working, that the exact value of A can be written in the form $p\pi^3 + q\pi^2 + r\pi$, where p , q and r are exact real numbers to be determined. [6]

(d) A reviewer of the design noted that the flat upper surface of the canopy (roof) may not be practical, even though it is easy to construct and cost less. Explain why he may think so. [1]

	Solution
10.(a)	$x = \sin t - t\cos t \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = \cos t - [\cos t - t\sin t] = t\sin t$ $y = \cos t + t\sin t \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t} = -\sin t + [\sin t + t\cos t] = t\cos t$

	Length of $C = \int_{\frac{\pi}{6}}^{\pi a} \sqrt{(\frac{\mathrm{d}x}{\mathrm{d}t})^2 + (\frac{\mathrm{d}y}{\mathrm{d}t})^2}\ \mathrm{d}t$ $= \int_{\frac{\pi}{6}}^{\pi a} \sqrt{(t\cos t)^2 + (t\sin t)^2}\ \mathrm{d}t$ $= \int_{\frac{\pi}{6}}^{\pi a} t\,\mathrm{d}t \quad (\text{since } t > 0)$ $= \left[\frac{t^2}{2}\right]_{\frac{\pi}{6}}^{\pi a}$ $= \frac{\pi^2 a^2}{2} - \frac{\pi^2}{72}$ Given length of $C = \frac{\pi^2}{9}$ $\therefore\ \frac{\pi^2 a^2}{2} - \frac{\pi^2}{72} = \frac{\pi^2}{9}$ $\frac{a^2}{2} = \frac{1}{9} + \frac{1}{72}$ $a^2 = \frac{18}{72} = \frac{1}{4}$ $\therefore\ a = \frac{1}{2} \quad (\text{reject } a = -\frac{1}{2} \text{ since } a > 0) \quad (\text{shown})$
10.(b)	When $t = \frac{\pi}{6}$, $x = \sin\frac{\pi}{6} - \frac{\pi}{6}\cos\frac{\pi}{6} = \frac{1}{2} - \frac{\pi}{6}\left(\frac{\sqrt{3}}{2}\right) = \frac{1}{2} - \frac{\sqrt{3}}{12}\pi$ $\therefore$ required diameter $= 2\left(\frac{1}{2} - \frac{\sqrt{3}}{12}\pi\right) = 1 - \frac{\sqrt{3}}{6}\pi$
10.(c)	$A = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 2\pi x \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t$ $= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 2\pi(\sin t - t\cos t)(t)\,\mathrm{d}t$ $= 2\pi\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} t\sin t\,\mathrm{d}t - 2\pi\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} t^2\cos t\,\mathrm{d}t$ $= 2\pi\left\{\left[t(-\cos t)\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} - \int_{\frac{\pi}{6}}^{\frac{\pi}{2}}(-\cos t)\,\mathrm{d}t\right\}$ $-2\pi\left\{\left[t^2(\sin t)\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} - \int_{\frac{\pi}{6}}^{\frac{\pi}{2}}(2t)(\sin t)\,\mathrm{d}t\right\}$

	$= 2\pi\left\{\frac{\sqrt{3}}{12}\pi + [\sin t]_{\frac{\pi}{6}}^{\frac{\pi}{2}}\right\}$ $- 2\pi\left\{\left(\frac{1}{4}\pi^2 - \frac{1}{72}\pi^2\right) - 2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} t\sin t\,\mathrm{d}t\right\}$ $= 2\pi\left(\frac{\sqrt{3}}{12}\pi + \frac{1}{2}\right) - 2\pi\left\{\frac{17}{72}\pi^2 - 2\left(\frac{\sqrt{3}}{12}\pi + \frac{3}{2}\right)\right\}$ $= \frac{\sqrt{3}}{6}\pi^2 + \pi - \frac{17}{36}\pi^3 + \frac{\sqrt{3}}{3}\pi^2 + 2\pi$ $= -\frac{17}{36}\pi^3 + \frac{\sqrt{3}}{2}\pi^2 + 3\pi$ where $p = -\frac{17}{36}$, $q = \frac{\sqrt{3}}{2}$, $r = 3$
10.(d)	Possible reasons and suggestions: • A canopy with flat roof does not allow rainwater to drain off easily. As such, the canopy may collapse due to the weight of rainwater in stormy weather. • A canopy with flat roof does not allow debri (dried leaves, dead branches etc.) to fall off from the roof easily. As such, it may be difficult to maintain the canopy.