FINAL EXAMINATION 2022
YEAR THREE EXPRESS
ADDITIONAL MATHEMATICS PAPER 2
Solutions with Markers' Comments

1 Given the function $f(x) = 2x^3 + ax^2 + bx - 12$ is divisible by $(x-3)$ and leaves a reminder of -14 when divided by $(2x+1)$. Find the value of a and of b. [5]

<u>Solutions</u>

Given $f(x) = 2x^3 + ax^2 + bx - 12$.

Factor Thm: $f(3) = 0$

$2(3)^3 + a(3)^2 + b(3) - 12 = 0$

$54 + 9a + 3b - 12 = 0$

$3a + b = -14$ -------(1)

Remainder Thm: $f\left(-\frac{1}{2}\right) = -14$

$2\left(-\frac{1}{2}\right)^3 + a\left(-\frac{1}{2}\right)^2 + b\left(-\frac{1}{2}\right) - 12 = -14$

$-\frac{1}{4} + \frac{a}{4} - \frac{b}{2} - 12 = -14$

$a = 2b - 7$ -------(1)

Sub (2) into (1):

$3(2b-7) + b = -14$

$6b - 21 + b = -14$

$\therefore b = 1$

$\therefore a = 2(1) - 7 = -5$

$\therefore f(x) = 2x^3 - 5x^2 + x - 12$

Markers' Comments:

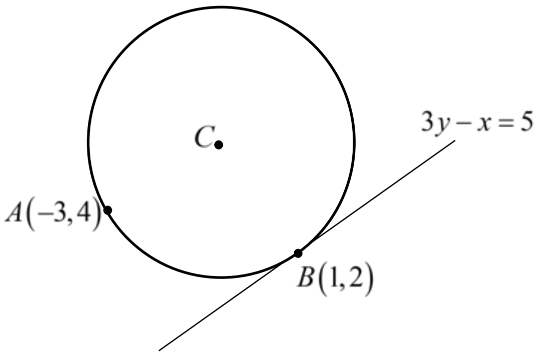
- Some students cannot differentiate between factor theorem and remainder theorem, and just equated the two equations they got after subbing in the roots of the divisor.
- Students are reminded to be careful and to check their workings. Quite a few students subbed $x = \frac{1}{2}$ instead of $x = -\frac{1}{2}$

2 In the diagram, the circle with centre C passes through the point $A(-3,4)$ and touches the line $3y-x=5$ at the point $B(1,2)$. Find

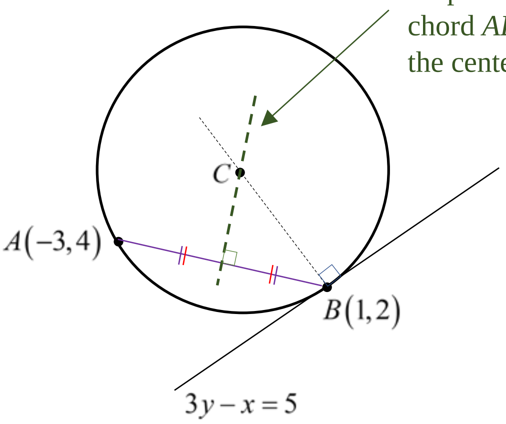
(a) the equation of the line BC, [3]

(b) the coordinates of C, [5]

(c) the equation of the circle. [2]



Solutions



(a) Given tangent: $y=\frac{1}{3}x+\frac{5}{3}$

$$\Rightarrow m_{BC}=-3$$

Equation of the line BC:

$$y-2=-3(x-1)$$

$$\therefore y=-3x+5 \text{ --------------(1)}$$

(b) Midpoint of AB $=\left(\frac{-3+1}{2},\frac{4+2}{2}\right)=(-1,3)$

Gradient of AB, $m_{AB}=\frac{4-2}{-3-1}=-\frac{1}{2} \quad \Rightarrow m_{\perp}=2$

Equation of the perpendicular bisector of AB :

$$y-3=2(x+1)$$

$$\therefore y=2x+5 \text{ -----(2)}$$

(1) = (2): $-3x+5=2x+5$

$\therefore x=0, \quad y=5$

$\therefore C(0,5)$

Markers' Comments:

- A lot of students were stuck at part b as they could not apply the property that perpendicular bisector of a circle cuts through the centre
- Students are reminded that they cannot assume properties of the diagram without being able to verify it, ie AC parallel to tangent

(c) Radius of circle $=\sqrt{(1-0)^2+(2-5)^2}=\sqrt{10}$ units

Or $=\sqrt{(-3-0)^2+(4-5)^2}=\sqrt{10}$ units

Equation of circle:

$x^2+(y-5)^2=10$ or $x^2+y^2-10y+15=0$

3 Given that $4x^3+16x^2+13x+3 \equiv (x-m)g(x)$, where $g(x)$ is a polynomial and m is an integer.

(a) Find the value of m. [2]

(b) Find $g(x)$. [2]

(c) Hence, or otherwise solve for $4x^3+16x^2+13x+3=0$. [1]

(d) Explain why the equation $4x^6+16x^4+13x^2+3=0$ has no real solutions. [2]

Solutions

(a) Let $f(x)=4x^3+16x^2+13x+3$.

Let $x=-3$:

$$f(x)=4(-3)^3+16(-3)^2+13(-3)+3=0$$

$$\Rightarrow (x+3) \text{ is a factor} \quad \therefore m=-3$$

(b) By synthetic/long division:

$$f(x)=(x+3)(4x^2+4x+1)$$
$$=(x+3)(2x+1)^2$$
$$\therefore g(x)=(2x+1)^2$$

(c) For $f(x)=0 \Rightarrow (x+3)(2x+1)^2=0$

$$\therefore x=-3, \quad -\frac{1}{2}$$

(d) In $4x^6+16x^4+13x^2+3=0$ replacing x with x^2:

$$\Rightarrow x^2=-3,-\frac{1}{2}$$

$\Rightarrow$ These equations have no real solutions

Markers' Comments:

- This question was not done well. Students could not tell that $(x-m)$ is a factor of $4x^3+16x^2+13x+3$ and they tried to find the value of m by subbing in values of m, ignoring $g(x)$.
  Eg, Sub $x=0$, $3=(0-m)g(0)$
  $m=-3$
- Some students did not show explicitly how to get the value of m – they should use factor theorem to verify that $(x+3)$ is a factor
- For part (d), it seems that many students are conditioned to think that b^2-4ac can be used in any situation to show that there are no real solutions

4 **(a)** Find the values of x and y which satisfy the equations,

$$9^x = 3^y,$$
$$2^x + 1 = 72\left(2^y\right).$$ [5]

(b) Solve the equation $1 + \log_4 a + \log_2 8 = \log_4 (a+3)$. [4]

Solutions

$$9^x = 3^y$$
$$\Rightarrow 3^{2x} = 3^y$$

$2x = y$ ------------------(1)

$2^x + 1 = 72\left(2^y\right)$ ----------(2)

Sub (1) into (2) :

$$2^x + 1 = 72\left(2^{2x}\right)$$
$$72\left(2^{2x}\right) - 2^x - 1 = 0$$
$$\left(9\times 2^x + 1\right)\left(8\times 2^x - 1\right) = 0$$
$$\therefore 2^x = 2^{-3} \text{ or } -\frac{1}{9}\ _{\text{(NA)}}$$
$$\therefore x = -3,\ y = -6$$

Let 2^x be a :

$$a + 1 = 72a^2$$
$$72a^2 - a - 1 = 0$$
$$(9a+1)(8a-1) = 0$$
$$\therefore a = \frac{1}{8} \text{ or } -\frac{1}{9}\ _{\text{(NA)}}$$
$$\therefore 2^x = \frac{1}{8} = 2^{-3}$$
$$\therefore x = -3,\ y = -6$$

Markers' Comments:

- For part (a), quite a few students forced their way to simplify equation 2 using laws of indices.
  For eg, $72(2^y) = 144^y$

4(b)

$$1 + \log_4 a + \log_2 8 = \log_4 (a+3)$$
$$1 + \log_4 a + 3 = \log_4 (a+3)$$
$$\log_4 (a+3) - \log_4 a = 4$$
$$\log_4 \left(\frac{a+3}{a}\right) = 4$$
$$\Rightarrow \frac{a+3}{a} = 4^4$$
$$a + 3 = 4^4 a$$
$$4^4 a - a = 3$$
$$a\left(4^4 - 1\right) = 3$$
$$\therefore a = \frac{3}{4^4 - 1} = \frac{1}{85}$$

Markers' Comments:

- For part (b), some students are not able to see that $log_2 8$ can be expressed as a constant and used 'change of base' law which could make the working more prone to mistakes.

- $log_4(a+3) \neq log_4 a + log_4 3$

5 **(a)** Given that $\dfrac{(\log_x y)^4}{\log_y x}+32=0$, express y in terms of x. [4]

(b) **(i)** Simplify $\dfrac{(9^n)(25^n)}{15^{2n}+(5^{2n+2})(3^{2n})}$. [4]

(ii) Hence find the value of x if $\dfrac{15^{2n}+(5^{2n+2})(3^{2n})}{(9^n)(25^n)}=3x-1$. [2]

Solutions

(a) $\dfrac{(\log_x y)^4}{\log_y x}+32=0$

$(\log_x y)^4(\log_x y)=-32$

$(\log_x y)^5=-32$

$\log_x y=-2$

$\therefore y=x^{-2}=\dfrac{1}{x^2}$

Markers' Comments:

- Students are generally inadept in using the Log Laws & changing the log base
  - Power Law: $(\log_x y)^4 \neq 4(\log_x y)$
  - Quotient Law: $\dfrac{(\log_x y)^4}{\log_y x}+32=0$

    (erroneous) $(\log_x y)^4-\log_y x+32=0$
  - Correct + efficient way of changing the base: $\dfrac{1}{\log_y x}=\log_x y$

(b) **(i)** $\dfrac{(9^n)(25^n)}{15^{2n}+(5^{2n+2})(3^{2n})}$

$=\dfrac{(3^{2n})(5^{2n})}{15^{2n}+(5^{2n}\times 5^2)(3^{2n})}$

$=\dfrac{15^{2n}}{15^{2n}+25(15^{2n})}$

$=\dfrac{15^{2n}}{15^{2n}(1+25)}$

$=\dfrac{1}{26}$

Markers' Comments:

- Many students did not cancel out the common term from *every* term ie

  $\dfrac{\cancel{(3^{2n})(5^{2n})}}{15^{2n}+(\cancel{5^{2n}}\times 5^2)\cancel{(3^{2n})}}=\dfrac{1}{15^{2n}+25}$,
- Students not able to recognize: $15^{2n}=3^{2n}\times 5^{2n}$

(ii) $26=3x-1$

$\therefore x=9$

6 **(a)** Prove that $\dfrac{\cos\theta}{1-\sin\theta}-\dfrac{1}{\cos\theta}=\tan\theta$. [3]

(b) Hence solve the equation $\dfrac{\cos\theta}{1-\sin\theta}-\dfrac{1}{\cos\theta}=3\cot\theta$ for $0^\circ\le\theta\le 360^\circ$. [3]

Solutions

(a) LHS: $\dfrac{\cos\theta}{1-\sin\theta}-\dfrac{1}{\cos\theta}$

$$=\frac{\cos^2\theta-(1-\sin\theta)}{(1-\sin\theta)(\cos\theta)}$$

$$=\frac{1-\sin^2\theta-1+\sin\theta}{(1-\sin\theta)(\cos\theta)}$$

$$=\frac{\sin\theta-\sin^2\theta}{(1-\sin\theta)(\cos\theta)}$$

$$=\frac{\sin\theta(1-\sin\theta)}{(1-\sin\theta)(\cos\theta)}$$

$$=\frac{\sin\theta}{\cos\theta}=\tan\theta=\text{RHS}$$

OR $\dfrac{\cos\theta}{1-\sin\theta}-\dfrac{1}{\cos\theta}$

$$=\frac{\cos\theta}{1-\sin\theta}\times\frac{1+\sin\theta}{1+\sin\theta}-\frac{1}{\cos\theta}$$

$$=\frac{\cos\theta(1+\sin\theta)}{1-\sin^2\theta}-\frac{1}{\cos\theta}$$

$$==\frac{\cos\theta(1+\sin\theta)}{\cos^2\theta}-\frac{1}{\cos\theta}$$

$$==\frac{1+\sin\theta}{\cos\theta}-\frac{1}{\cos\theta}$$

$$=\frac{1+\sin\theta-1}{\cos\theta}\frac{\sin\theta}{\cos\theta}=\tan\theta=\text{RHS}$$

(b) $\tan\theta=3\cot\theta$

$\tan^2\theta=3 \quad\Rightarrow\quad \tan\theta=\pm\sqrt{3}$

$\alpha=\tan^{-1}\sqrt{3}=60^\circ$

$\therefore\theta=60^\circ,120^\circ,240^\circ,300^\circ$

Markers' Comments:

- Many students 'forced' their answers in this proof question. Students are strongly reminded of the importance of integrity in obtaining their answers.

- Common errors to note:
  - $\cos\theta\ne 1-\sin\theta$
  - $\sec\theta\ne 1-\tan\theta$
  - $3\cot\theta\ne\dfrac{1}{3\tan\theta}$
- Quite a few students started the proof with *both* sides of the given equation. Students are reminded to start on one side of the identity (usually the more complex side), and use logical steps to transform it into the other side of the equation.
- Many students didn't solve for the negative root of the square eqn: $\tan^2\theta=3 \quad\Rightarrow\quad \tan\theta=\sqrt{3}$, obtaining only part of the solution set

7 **(a)** Obtain the first 3 terms in the expansion of $(2+x)^6$ in ascending powers of x. [2]

(b) Find the term in x in $\left(1+\frac{1}{x}\right)(2+x)^6$. [2]

Solutions

7a $(2+x)^6$

$= 2^6 + {}^6C_1(2)^5(x)^1 + {}^6C_2(2)^4(x)^2 + \ldots$

$= 64 + 192x + 240x^2 + \ldots$

7b $\left(1+\frac{1}{x}\right)(2+x)^6$

$= \left(1+\frac{1}{x}\right)(61 + 192x + 240x^2 + \ldots)$

$= \ldots + 192x + 240x + \ldots$

$= \ldots + 432x + \ldots$

Markers' Comments:

- Part (a): Some students struggled to apply the theorem correctly, not able to understand the pattern of the expansion and applied theorem using illogical deduction

- Students are reminded the final answer must be completely evaluated as shown, as some left their final answer in the theorem form.

- Part (b):
  - Some students used the General Term formula which in this case is not an efficient method and all have led to the wrong answer/incomplete answer
  - Many students stated the coefficient of x instead of the term in x.

8 The table below shows experimental values of two variables x and y. The variables x and y are related by the equation $y = px^q$, where p and q are constants.

x	10	50	300	800	3100
y	350	144.5	60.1	36.6	18.2

(a) On the grid opposite, draw a straight line graph of $\ln y$ against $\ln x$. [2]

(b) Use your graph to estimate the value of p and of q. [4]

An equation of $y = \frac{e^8}{x^2}$ was given.

(c) Draw the graph of $y = \frac{e^8}{x^2}$ on the same axes and find the value of x of the point of intersection. [3]

8(a)

$\ln x$	2.30	3.91	5.70	6.68	8.04
$\ln y$	5.86	4.97	4.10	3.60	2.90

Correct plotting of points
Line of best fit

(b) Use your graph to estimate the value of p and of q.

Linearizing:

$$y = px^q$$
$$\ln y = \ln\left(px^q\right)$$
$$\ln y = q\ln x + \ln p$$
$$Y = m\,X \;+\; c$$

From the graph:

$$\textbf{Gradient} \simeq \frac{6.5-3.2}{1-7.4} \simeq -0.516$$
$$\therefore q \simeq -0.516$$
$$(\text{to 3 sf})$$

$$\textbf{ln } \boldsymbol{y}\textbf{-intercept} \simeq 7.0$$
$$\therefore \ln p \simeq 7.0$$
$$\Rightarrow \quad p \simeq e^7$$
$$\simeq 1096.63..$$
$$\therefore p \simeq 1100(\text{to 3 sf})$$

(c) $y = \dfrac{e^8}{x^2} \Rightarrow \ln y = \ln\left(\dfrac{e^8}{x^2}\right)$

$$\ln y = \ln e^8 - \ln x^2 \Rightarrow \ln y = -2\ln x + 8$$

$\Rightarrow$ Insert: $Y = -2X + 8$ using any 2 points that satisfy the line eg $(4,0)$ and $(0,8)$

From the graph, the point of intersection is $(0.65, 6.65)$

$$\Rightarrow \ln x \approx 0.65$$
$$\therefore x \simeq e^{0.65}$$
$$\approx 1.92$$

Markers' Comments:

- For part (a)
  - the line of best fit drawn *must be extrapolated to intersect the vertical axis to capture vertical axis intercept, c.*
  - Points taken to calculate the gradient of the line must also be shown clearly on the graph or in your working. These points must not be any of the plotted points.
  - Students must go through the linearizing process to understand how the gradient and c will be used to solve for the unknown constants, p and q. Many students struggled with applying the laws of logarithm to transform the given exponential equation to a linear form required by the question.

- *Algebraic calculation* of c using the points given and equation of straight line will not be accepted; c must be obtained from the graph.

- For part (b), students need to explicitly state the value of p and of q as their final answer

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