National Junior College
2016 – 2017 H2 Further Mathematics
Topic F7: Matrices and Linear Spaces (Tutorial Set 3)

This tutorial set is for the following sections from the notes:
- §7 Row Space, Column Space and Null Space
- §8 Linear Transformations
- §9 Eigenvalues and Eigenvectors

Basic Mastery Questions

1 For each of the following matrices, find the bases for the row space, column space and null space.

(a) $\begin{pmatrix} 1 & 3 \\ -2 & 1 \\ 1 & -1 \end{pmatrix}$, **(b)** $\begin{pmatrix} 1 & 2 & -1 \\ -3 & -5 & 1 \\ 13 & 23 & -7 \end{pmatrix}$, **(c)** $\begin{pmatrix} 2 & 1 & 3 & 3 \\ 0 & -3 & 1 & -2 \\ 4 & 5 & 5 & 8 \end{pmatrix}$.

State the rank and the nullity for each of these matrices.

2 Determine whether each of the following is a linear transformation. Justify your answers.

(a) $\mathrm{T}_1 : \mathbb{R}^2 \to \mathbb{R}^2$. $\mathrm{T}_1\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} kx \\ y \end{pmatrix}$, where k is a nonzero real constant.

(b) $\mathrm{T}_2 : \mathbb{R}^2 \to \mathbb{R}^2$. $\mathrm{T}_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} |x+y| \\ |x-y| \end{pmatrix}$.

(c) $\mathrm{T}_3 : \mathbf{M}_{2,2}(\mathbb{R}) \to \mathbb{R}^2$, $\mathrm{T}_3\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+b \\ c-d \end{pmatrix}$.

(d) $\mathrm{T}_4 : \mathbf{P}_2 \to \mathbb{R}$, $\mathrm{T}_4\left(ax^2+bx+c\right) = b^2-4ac$.

(e) $\mathrm{T}_5 : \mathbb{R}^3 \to \mathbb{R}^4$, $\mathrm{T}_5(\mathbf{u}) = \mathbf{0}$ for all $\mathbf{u} \in \mathbb{R}^3$.

3 It is given that the linear transformation $\mathrm{T} : \mathbb{R}^2 \to \mathbb{R}^2$ is such that $\mathrm{T}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 11 \\ -13 \end{pmatrix}$ and $\mathrm{T}\begin{pmatrix} 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 8 \\ -11 \end{pmatrix}$.

(i) Find $\mathrm{T}\begin{pmatrix} 1 \\ -7 \end{pmatrix}$.

(ii) Find the 2×2 matrix $\mathbf{A}$ such that

$$\mathrm{T}(\mathbf{u}) = \mathbf{A}\mathbf{u}$$

for all $\mathbf{u} \in \mathbb{R}^2$.

4 The linear transformation $\mathrm{L}:\mathbb{R}^3 \to \mathbb{R}^3$ can be represented by the matrix

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & a+1 & -2 \\ 3 & 2a & a^2-4a \end{pmatrix}$$

It is given the rank of L is 2.

(i) State the nullity of L and find the value of a.
(ii) Find a basis for its range space.

5 Find the eigenvalues and eigenvectors of the following matrices.

(a) $\begin{pmatrix} 2 & -4 \\ -5 & 3 \end{pmatrix}$, **(b)** $\begin{pmatrix} -6 & 4 & -8 \\ -6 & -11 & -3 \\ 4 & -6 & 12 \end{pmatrix}$, **(c)** $\begin{pmatrix} 0 & 2 & 1 \\ -1 & 3 & 1 \\ 2 & -4 & -1 \end{pmatrix}$.

6 Let $\mathbf{A} = \begin{pmatrix} 2 & -4 \\ -5 & 3 \end{pmatrix}$.

(i) Find an invertible matrix **P** and a diagonal matrix **D** such that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$.
(ii) Find $\mathbf{A}^5$ without using a graphic calculator.

7 Show that $\mathbf{B}^n = \mathbf{B}$ for all positive integers n.

Practice Questions

8 The subspaces V and W of $\mathbb{R}^3$ are spanned by the sets

$$\left\{ \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix} \right\} \text{ and } \left\{ \begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ -3 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} \right\}$$

respectively.

(i) Find the dimensions of V and of W.
(ii) Given that $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in V$, obtain a linear relationship between x, y and z.
(iii) Find a 1×3 matrix **A** such that $\{\mathbf{X} : \mathbf{AX} = \mathbf{0}\} = W$.
(iv) Find a basis for the subspace $V \cap W$.

(1983 A Level / FM / Jun / P2)

9 **(a)** Given that $\begin{pmatrix} a & b & c \end{pmatrix}$ belongs to the row space of the matrix

$$\begin{pmatrix} 3 & 2 & 1 \\ -2 & -2 & 1 \\ 1 & -2 & 7 \end{pmatrix},$$

find a linear relation hat must be satisfied by a, b, c.

(b) Given that **P** and **Q** are 3×3 matrices,

(i) prove that the column space of **PQ** is a subspace of the column space of **P**,

(ii) state a similar result concerning the row space of **PQ**,

(iii) deduce that $\text{rank}(\mathbf{PQ})$ cannot exceed the smaller of $\text{rank}(\mathbf{P})$ and $\text{rank}(\mathbf{Q})$.

(1984 A Level / FM / Jun / P1)

10 The elements of the matrices **A** and **B** are given by

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}.$$

Write down in full the first column of the product **AB** and show that this can be put in the form $b_{11}\mathbf{c}_1 + b_{21}\mathbf{c}_2 + b_{31}\mathbf{c}_3$, where

$$\mathbf{c}_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix},\ \mathbf{c}_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix} \text{ and } \mathbf{c}_3 = \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix}.$$

Write down corresponding expressions for the second and third column of **AB**.

Hence show that the rank of **AB** cannot be greater than the rank of **A**.

For the case where

$$\mathbf{A} = \begin{pmatrix} 1 & \alpha & \beta \\ 2 & 2\alpha+\beta-1 & \alpha+2\beta \\ 5 & 5\alpha+3\beta-3 & 3\alpha+5\beta \end{pmatrix},\ \alpha, \beta \in \mathbb{R},$$

show that

(i) for all values of α and β the rank of **A** is not greater than 2,

(ii) if $\alpha = 0$ and $\beta = 1$, then, for all 3×3 matrix **B**, there are at least two linearly independent solutions for **x** of the equation

$$\mathbf{ABx} = \mathbf{0}\ ,\ \mathbf{x} \in \mathbb{R}^3.$$

(1993 A Level / FM / Jun / P1)

11 Determine the rank of the matrix

$$\mathbf{A}=\begin{pmatrix}1&1&2&3\\3&2&3&13\\4&4&9&7\\11&9&17&36\end{pmatrix}.$$

Deduce that if $\mathbf{x}$ is a solution of the equation

$$\mathbf{Ax}=p\begin{pmatrix}1\\3\\4\\11\end{pmatrix}+q\begin{pmatrix}1\\2\\4\\9\end{pmatrix}+r\begin{pmatrix}2\\3\\9\\17\end{pmatrix},$$

where p, q and r are given real numbers, then

$$\mathbf{x}=\begin{pmatrix}p-2\lambda\\q-11\lambda\\r+5\lambda\\\lambda\end{pmatrix}, \text{ where } \lambda\in\mathbb{R}.$$

Hence, or otherwise, for solution $\mathbf{x}=\begin{pmatrix}\alpha\\\beta\\\gamma\\\delta\end{pmatrix}$ of the equation $\mathbf{Ax}=\begin{pmatrix}4\\8\\17\\37\end{pmatrix}$,

(i) find $\mathbf{x}$ such that $\alpha=0$,

(ii) show that there is no $\mathbf{x}$ for which $\alpha^2+\beta^2+\gamma^2+\delta^2=1$.

(1992 A Level / FM / Nov / P1)

12 The linear transformations $\mathrm{T}_1:\mathbb{R}^4\to\mathbb{R}^4$, $\mathrm{T}_2:\mathbb{R}^4\to\mathbb{R}^4$ and $\mathrm{T}_3:\mathbb{R}^4\to\mathbb{R}^4$ are represented by the matrices $\mathbf{M}_1$, $\mathbf{M}_2$ and $\mathbf{M}_2\mathbf{M}_1$ respectively, where

$$\mathbf{M}_1=\begin{pmatrix}1&4&-5&8\\0&-4&1&-5\\-1&-3&0&-2\\0&1&1&0\end{pmatrix} \text{ and } \mathbf{M}_2=\begin{pmatrix}1&2&-1&3\\1&0&4&5\\3&2&7&13\\1&4&-6&1\end{pmatrix}.$$

(i) Show that the rank of $\mathbf{M}_1$ is equal to 3.

(ii) Write down a basis for R_1, the range space of T_1, and find a basis for the null space of T_1.

(iii) Find a basis for K_2, the null space of T_2, and hence show that K_2 is a subspace of R_1.

(iv) Hence, or otherwise, find three linearly independent vectors in the null space of T_3.

(1997 A Level / FM / Nov / P1)

13 Consider the equation $\mathbf{Ax} = \mathbf{b}$ where

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & 1 & 3 \\ 2 & 1 & 5 & -7 \\ 4 & -3 & 7 & -1 \\ 3 & 14 & 15 & -43 \end{pmatrix}, \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix},$$

and the elements of **x** and **b** are real. For the case where $b_1 = b_2 = b_3 = b_4 = 0$, the set of solutions of **x** is denoted by *K*, and for the case where $b_1 = 3$, $b_2 = 1$, $b_3 = 7$ and $b_4 = -11$, the set of solutions for **x** is denoted by *S*.

(i) Show that *K* is a vector space and find its dimension.

(ii) Show that *S* is not a vector space.

(iii) Given that $\mathbf{e}_1$ and $\mathbf{e}_2$ are two linearly independent vectors belonging to *K*, show that *S* is the set of vectors of the form $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \lambda\mathbf{e}_1 + \mu\mathbf{e}_2$, where λ and μ are real parameters.

(1991 A Level / FM / Jun / P1)

14 Show that the set *S* if vectors given by

$$S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

forms a basis for the linear space $\mathbb{R}^4$.

The linear transformation $\mathrm{L}: \mathbb{R}^3 \to \mathbb{R}^4$ is defined by

$$\mathrm{L}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y - z \\ x + z \\ x + y \\ 2x + y + z \end{pmatrix}.$$

Find the null space of L, and state its dimension. Show that

$$\mathrm{L}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} - 1\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

and express $\mathrm{L}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\mathrm{L}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ each as a linear combinations of the vectors of *S*.

(1985 A Level / FM / Jun / P1)

15 The linear transformation $\mathrm{T}:\mathbb{R}^4 \to \mathbb{R}^4$ is represented by the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -3 & -4 \\ 2 & 5 & -4 & -5 \\ 3 & a^2+5 & 2a-7 & 3a-9 \\ 6 & a^2+12 & 2a-14 & 3a-18 \end{pmatrix}.$$

Show that, provided $a \neq -1$ and $a \neq 2$, the dimension of the range space of T is 3.

In the case where $a = 2$,

(i) show that the dimension of *K*, the null space of T is 2,

(ii) show that there is a basis of *K*, which is to be found, of the form

$$\left\{ \begin{pmatrix} p \\ q \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} r \\ s \\ 0 \\ 1 \end{pmatrix} \right\},$$

where *p*, *q*, *r* and *s* are integers.

(iii) find a solution of

$$\mathbf{Ax} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 6 \end{pmatrix}$$

of the form $\begin{pmatrix} u \\ 0 \\ v \\ w \end{pmatrix}$, where *u*, *v* and *w* are nonzero integers.

(1995 A Level / FM / Nov / P1)

16 The vector **x** is an eigenvector of each of the matrices **A** and **B**, with corresponding eigenvalues λ and μ. Show that **x** is an eigenvector of **AB** with eigenvalue $\lambda\mu$.

Find the eigenvalues and corresponding eigenvectors of each of the matrices **C** and **D**, where

$$\mathbf{C} = \begin{pmatrix} 1 & 1 & 4 \\ 1 & 3 & -1 \\ 2 & 1 & 3 \end{pmatrix} \text{ and } \mathbf{D} = \begin{pmatrix} -2 & 1 & 1 \\ 0 & -1 & 4 \\ 0 & 0 & -3 \end{pmatrix}.$$

Hence, or otherwise, find *one* eigenvector of the matrix **CD** and its corresponding eigenvalue.

(1987 A Level / FM / Nov / P1)

17 A linear transformation $\mathrm{L}:\mathbb{R}^3 \to \mathbb{R}^3$ is defined by

$$\mathrm{L}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & -2 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Find a basis for the range of L and a basis for the null space of L.

(i) Find the image under L of the line $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix}$.

(ii) Find, in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, an equation of the line whose image under L is the point with position vector $\begin{pmatrix} 5 \\ 3 \\ -2 \end{pmatrix}$.

(iii) Show that the image under L of the plane with equation $x - y + z = 2$ is the plane $x - y + z = 0$.

(1983 A Level / FM / Nov / P2)

18 The linear transformation $\mathrm{T}:\mathbb{R}^4 \to \mathbb{R}^4$ is defined by

$$\mathrm{T}:\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \mapsto \mathbf{M}\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}, \text{ where } \mathbf{M} = \begin{pmatrix} 1 & 1 & 3 & -4 \\ -1 & 4 & 7 & -11 \\ 1 & -3 & -5 & 8 \end{pmatrix}.$$

(i) Show that the dimension of V, the range space of T, is 2.

(ii) Find a basis for V.

(iii) Show that the vector $\begin{pmatrix} -1 \\ 4 \\ 5 \end{pmatrix}$ does not belong to V.

(iv) Determine whether there is an element $\mathbf{x}$ of $\mathbb{R}^4$ such that $\mathbf{Mx} = \mathbf{y}$ in the following cases.

(a) $\mathbf{y} = \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}$, **(b)** $\mathbf{y} = \begin{pmatrix} 1 \\ 7 \\ 3 \end{pmatrix}$.

(1994 A Level / FM / Jun / P1)

19 Show that, for all real values of a, the rank of the matrix

$$\mathbf{M}=\begin{pmatrix} 1 & -2 & -3 & a \\ -1 & 3 & a+3 & -a+1 \\ 1 & -1 & a-3 & a+1 \\ 2 & -3 & a-6 & 2a+1 \end{pmatrix}$$

is equal to 2.

The null space of the linear transformation represented by $\mathbf{M}$ is denoted by K. The set $\{\mathbf{e}_1, \mathbf{e}_2\}$ is a basis for K, the vector $\mathbf{x}_0$ and $\mathbf{b}$ are such that $\mathbf{Mx}_0 = \mathbf{b}$.

(i) Show that if $\mathbf{x} = \mathbf{x}_0 + \lambda\mathbf{e}_1 + \mu\mathbf{e}_2$, with $\lambda, \mu \in \mathbb{R}$, then $\mathbf{Mx} = \mathbf{b}$.

(ii) Show that if $\mathbf{Mx} = \mathbf{b}$, then $\mathbf{x} - \mathbf{x}_0 \in K$, and deduce that $\mathbf{x}$ is of the form $\mathbf{x}_0 + \lambda\mathbf{e}_1 + \mu\mathbf{e}_2$.

(iii) Find the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ which are of the form

$$\begin{pmatrix} r \\ s \\ 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} t \\ u \\ 0 \\ 1 \end{pmatrix}$$

respectively, where r, s, t and u may depend on a.

(iv) Given that

$$\mathbf{x}_0 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} -1 \\ 2 \\ 0 \\ -1 \end{pmatrix},$$

find the values of a for which the equation $\mathbf{Mx} = \mathbf{b}$ has a solution of the form $\begin{pmatrix} v \\ v^{-1} \\ 1 \\ 1 \end{pmatrix}$.

(1993 A Level / FM / Nov / P1)

20 Show that if $\mathbf{e}$ is an eigenvector of a square matrix $\mathbf{A}$, with corresponding eigenvalue λ, then $\mathbf{e}$ is an eigenvector of a square matrix $\mathbf{A}^2$ with corresponding eigenvalue λ^2.

Find the eigenvalues and corresponding eigenvectors of the matrix

$$\mathbf{B}=\begin{pmatrix} 2 & -5 & 6 \\ 2 & 3 & 2 \\ -1 & 5 & -5 \end{pmatrix}.$$

Find a matrix $\mathbf{Q}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{B}^2 = \mathbf{QDQ}^{-1}$.

(1989 A Level / FM / Jun / P1)

21 The vector **x** is an eigenvector of the 3×3 matrix **A**, with corresponding eigenvalue λ.

Show that if **A** is non-singular, then

(i) $\lambda \neq 0$,

(ii) the vector **x** is an eigenvector of the matrix $\mathbf{A}^{-1}$. with corresponding eigenvalue λ^{-1}.

Find the eigenvalues and corresponding eigenvectors of the matrices **A** and **B**, where

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 4 \\ 0 & 2 & 8 \\ 0 & 0 & -3 \end{pmatrix} \text{ and } \mathbf{B} = \left(\mathbf{A} + 5\mathbf{I}\right)^{-1} .$$

(1990 A Level / FM / Jun / P1)

22 The matrix **A** has eigenvalues $\lambda_1, \lambda_2, \lambda_3$ and corresponding eigenvectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ respectively. The matrix **B** has eigenvalues μ_1, μ_2, μ_3 for which the corresponding eigenvectors are also $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ respectively. Show that the matrix $\mathbf{A}+\mathbf{B}$ has eigenvalues $\lambda_1+\mu_1, \lambda_2+\mu_2, \lambda_3+\mu_3$ with the corresponding eigenvectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

The matrix

$$\mathbf{A} = \begin{pmatrix} 0 & -1 & 0 \\ -4 & -9 & -6 \\ 5 & 11 & 7 \end{pmatrix}$$

has eigenvectors $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$. Find the corresponding eigenvalues.

The matrix

$$\mathbf{B} = \begin{pmatrix} -4 & -16 & -11 \\ -9 & -27 & -19 \\ 14 & 44 & 31 \end{pmatrix}$$

has eigenvalues $1, 2, -3$, for which corresponding eigenvectors $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ respectively.

(i) Evaluate the inverse of the matrix $\begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \\ 1 & -2 & -3 \end{pmatrix}$.

(ii) Hence find matrices **R** ans **S** and a diagonal matrix **D** such that $\mathbf{M}^5 = \mathbf{RDS}$, where $\mathbf{M} = \mathbf{A} + \mathbf{B}$.

(1996 A Level / FM / Nov / P1)

23 The matrix **A** has λ as an eigenvalue with corresponding eigenvector **x**. The non-singular matrix **E** is of the same order (or size) as **A**. Show that $\mathbf{Ex}$ is an eigenvector of the matrix **B**, where $\mathbf{B} = \mathbf{EAE}^{-1}$, and that λ is the corresponding eigenvalue.

For the case where

$$\mathbf{A} = \begin{pmatrix} 1 & a & b \\ 0 & -2 & c \\ 0 & 0 & 3 \end{pmatrix},\ a, b, c \in \mathbb{R}, \text{ and } \mathbf{E} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

(i) write down the eigenvalues of **A** and obtain corresponding eigenvectors,

(ii) find the eigenvalues and corresponding eigenvectors of **B**,

(iii) find a non-singular matrix **Q** and a diagonal matrix **D** such that

$$\mathbf{B}^n = \mathbf{QDQ}^{-1}$$

where n is a positive integer. (Note that the evaluation of $\mathbf{Q}^{-1}$ is not required.)

(1992 A Level / FM / Nov / P1)

24 The matrices **A** and **B** are given by

$$\mathbf{A} = \begin{pmatrix} 5 & 2 & -4 \\ 7 & -8 & -7 \\ -4 & 12 & 5 \end{pmatrix} \text{ and } \mathbf{B} = \mathbf{A} + k\mathbf{I}.$$

where **I** is the identity matrix and $k \in \mathbb{R}$.

(i) Find the eigenvalues and a corresponding set of eigenvectors of **A**.

(ii) Hence find the eigenvalues of **B** in terms of *k*, and show that there are corresponding eigenvectors of **B** which are independent of *k*.

(iii) Deduce that there is a non-singular matrix **Q**, independent of *k*, and a diagonal matrix **D**, whose elements are to be determined, such that $\mathbf{B}^2 = \mathbf{QDQ}^{-1}$.
(Note that the evaluation of $\mathbf{Q}^{-1}$ is not required.)

(1992 A Level / FM / Jun / P1)

25 It is given that the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of the matrix

$$\mathbf{M} = \begin{pmatrix} a + \frac{3}{8} & -\frac{1}{8} & \frac{1}{8} \\ -\frac{1}{4} & a + \frac{1}{2} & \frac{1}{4} \\ -\frac{1}{8} & \frac{1}{8} & a + \frac{5}{8} \end{pmatrix}$$

are the roots of the equation

$$32(\lambda - a)^3 - 48(\lambda - a)^2 + 22(\lambda - a) - 3 = 0.$$

Find $\lambda_1, \lambda_2, \lambda_3$ in terms of a.

Find matrices **Q** and **D** such that $\mathbf{M} = \mathbf{QDQ}^{-1}$, where the elements of **Q** are independent of a, and **D** is a diagonal matrix. (The evaluation of $\mathbf{Q}^{-1}$ is not required.)

Find the set of values of a such that all the elements of $\mathbf{M}^n$ tend to zero as $n \to \infty$.

(1995 A Level / FM / Jun / P1)

26 Show that if **e** is an eigenvector of the matrix **M**, with corresponding eigenvalue λ, then **e** is an eigenvector of the matrix $k\mathbf{M}$ with the corresponding eigenvalue $k\lambda$.

Given that the eigenvalue of the matrix

$$\mathbf{A} = \begin{pmatrix} a+1 & a-1 & 1-a \\ a+1 & a-1 & -1-a \\ 2 & -2 & 0 \end{pmatrix},$$

where $a^2 \neq 1$, are 2, -2 and $2a$, show that there is a non-singular matrix **Q**, whose elements are independent of a, and a diagonal matrix **D** such that $\mathbf{A} = \mathbf{QDQ}^{-1}$.

Express each of the matrices

(i) $\begin{pmatrix} 101 & 99 & -99 \\ 101 & 99 & -101 \\ 2 & -2 & 0 \end{pmatrix}$, **(ii)** $\begin{pmatrix} 0.501 & 0.499 & -0.499 \\ 0.501 & 0.499 & -0.501 \\ 0.002 & -0.002 & 0 \end{pmatrix}$.

in the form $\mathbf{QDQ}^{-1}$, where the non-singular matrix **Q** and the diagonal matrix **D** are to be found. (You are not required to find $\mathbf{Q}^{-1}$.)

(1991 A Level / FM / Jun / P1)

27 The eigenvalues of the matrix

$$\mathbf{A} = \begin{pmatrix} -2 & 1 & -1 \\ -4 & -7 & 4 \\ 0 & -1 & -1 \end{pmatrix}$$

are denoted by $\lambda_1, \lambda_2, \lambda_3$, where $\lambda_1 < \lambda_2 < \lambda_3$. Find $\lambda_1, \lambda_2, \lambda_3$ and corresponding eigenvectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

Show that if **e** is an eigenvector of a matrix **M**, with corresponding eigenvalue λ, then **e** is an eigenvector of the matrix $k_1(\mathbf{M} + k_2\mathbf{I})$, with corresponding eigenvalue $k_1(\lambda + k_2)$.

Hence find a matrix **B** with the same eigenvectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ as **A** and with corresponding eigenvalues $0, \frac{2}{3}, 1$.

Find a matrix **P** and the least integer n for which

$$\mathbf{P}^{-1}\mathbf{B}^n\mathbf{P} = \mathbf{D},$$

where **D** is of the form

$$\mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and $0 < \beta < 0.001$.

(1997 A Level / FM / Nov / P1)

Application Problems

28 (Linear Transformations in $\mathbb{R}^2$)

It is given that the position vector of a point Q is $\begin{pmatrix} x \\ y \end{pmatrix}$ in $\mathbb{R}^2$.

(a) What is the geometrical interpretation for each of the following linear transformations?

(i) $\mathrm{L}_1\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$,

(ii) $\mathrm{L}_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 3y \end{pmatrix}$.

Find the corresponding 2×2 matrices of these linear transformations.

(b) Find a 2×2 matrix that corresponds to each of linear / graph transformations.

(i) Reflect Q in the line $y = x$.

(ii) Rotate Q through $90°$ about the origin in clockwise direction.

(c) Explain why $\mathrm{L}_3\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+2 \\ y \end{pmatrix}$ (translation), is not a linear transformation.

(d) Let the matrix $\mathbf{P} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

(i) Describe the geometric interpretation of the linear transformation

$$\mathrm{L}_4\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{P}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Justify your answer.

(ii) Show that $\mathbf{P}^T = \mathbf{P}^{-1}$.

29 (Rotation of Conics in $\mathbb{R}^2$)

Verify that the equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

can be rewritten as

$$\mathbf{X}^T\mathbf{M}\mathbf{X} + \begin{pmatrix} d & e \end{pmatrix}\mathbf{X} + f = 0,$$

where $\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\mathbf{M} = \begin{pmatrix} a & \frac{1}{2}b \\ \frac{1}{2}b & c \end{pmatrix}$.

The matrix **M** is called the ***matrix of the quadratic form***.

(i) Show that **M** has two distinct eigenvalues if $b \neq 0$.

Given that the conic section S has equation,

$$36x^2 + 96xy + 64y^2 + 20x - 15y + 25 = 0$$

(ii) Express its matrix of the quadratic form as $\mathbf{PDP}^{-1}$, where **D** is a diagonal matrix, and **P** is a 2×2 matrix in the form $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, where θ is to be determined exactly in the form $\tan^{-1} a$.

(iii) Using the result $\mathbf{P}^T = \mathbf{P}^{-1}$, show that this equation can be rewritten as

$$\mathbf{Y}^T\mathbf{D}\mathbf{Y} + \begin{pmatrix} 20 & -15 \end{pmatrix}\mathbf{P}\mathbf{Y} + 25 = 0.$$

Let $\mathbf{Y} = \begin{pmatrix} x' \\ y' \end{pmatrix}$, restate this equation in terms of x' and y'.

(iv) The equation in **(iii)** represents another conic section S'. Describe the graph transformation from S to S'.

(v) State the coordinates of the focus (or foci) of S', and the equation(s) of the directrix (or directrices) of S'.

Hence, find the coordinates of the focus (or foci) of S, and the equation(s) of the directrix (or directrices) of S.

Numerical Answers

Basic Mastery Questions

1 **(a)** $\{(1\ \ 3),(0\ \ 1)\}$, $\left\{\begin{pmatrix}1\\-2\\1\end{pmatrix},\begin{pmatrix}3\\1\\-1\end{pmatrix}\right\}$, $\left\{\begin{pmatrix}0\\0\end{pmatrix}\right\}$, rank $=2$, nullity $=0$.

(b) $\{(1\ \ 2\ \ -1),(0\ \ 1\ \ -2)\}$, $\left\{\begin{pmatrix}1\\-3\\13\end{pmatrix},\begin{pmatrix}2\\-5\\23\end{pmatrix}\right\}$, $\left\{\begin{pmatrix}-3\\2\\1\end{pmatrix}\right\}$, rank $=2$, nullity $=1$.

(c) $\{(2\ \ 1\ \ 3\ \ 3),(0\ \ -3\ \ 1\ \ -2)\}$, $\left\{\begin{pmatrix}2\\0\\4\end{pmatrix},\begin{pmatrix}1\\-3\\5\end{pmatrix}\right\}$, $\left\{\begin{pmatrix}-5\\1\\3\\0\end{pmatrix},\begin{pmatrix}-7\\-4\\0\\6\end{pmatrix}\right\}$, rank $=2$, nullity $=2$.

2 **(a)** Yes **(b)** No **(c)** Yes **(d)** No **(e)** Yes.

3 **(i)** $\begin{pmatrix}-3\\2\end{pmatrix}$. **(ii)** $\begin{pmatrix}4 & 1\\-5 & -1\end{pmatrix}$.

4 **(i)** nullity $=1$, $a=1$. **(ii)** $\left\{\begin{pmatrix}1\\2\\3\end{pmatrix},\begin{pmatrix}2\\2\\2\end{pmatrix}\right\}$.

5 **(a)** $\mathbf{e}_1=\begin{pmatrix}-4\\5\end{pmatrix}$ with $\lambda_1=7$, $\mathbf{e}_2=\begin{pmatrix}1\\1\end{pmatrix}$ with $\lambda_2=-2$.

(b) $\mathbf{e}_1=\begin{pmatrix}4\\-3\\-2\end{pmatrix}$ with $\lambda_1=-5$, $\mathbf{e}_2=\begin{pmatrix}1\\0\\-2\end{pmatrix}$ with $\lambda_2=10$, $\mathbf{e}_3=\begin{pmatrix}-1\\3\\1\end{pmatrix}$ with $\lambda_3=-10$.

(c) $\mathbf{e}_1=\begin{pmatrix}1\\0\\1\end{pmatrix}$ and $\mathbf{e}_2=\begin{pmatrix}2\\1\\0\end{pmatrix}$ with $\lambda_1=\lambda_2=1$, $\mathbf{e}_3=\begin{pmatrix}1\\1\\-2\end{pmatrix}$ with $\lambda_3=0$.

6 **(i)** $\mathbf{P}=\begin{pmatrix}-4 & 1\\5 & 1\end{pmatrix}$ and $\mathbf{D}=\begin{pmatrix}7 & 0\\0 & 2\end{pmatrix}$. **(ii)** $\begin{pmatrix}7452 & -7484\\-9355 & 9323\end{pmatrix}$.

Practice Questions

8 **(i)** 2, 2 **(ii)** $5x-3y-2z=0$ **(iii)** $(3\ \ -3\ \ -2)$ **(iv)** $\{(0\ \ 2\ \ -3)\}$

9 **(a)** $4a-5b-2c=0$

11 $\text{rank}(\mathbf{A})=3$, $\mathbf{x}=(0\ \ -4.5\ \ 3.5\ \ 0.5)^T$

12 **(ii)** $\left\{\begin{pmatrix}1\\0\\-1\\0\end{pmatrix},\begin{pmatrix}8\\-5\\-2\\0\end{pmatrix},\begin{pmatrix}-5\\1\\0\\1\end{pmatrix}\right\}$; $\left\{\begin{pmatrix}1\\-1\\1\\1\end{pmatrix}\right\}$ **(iii)** $\left\{\begin{pmatrix}-5\\1\\0\\1\end{pmatrix},\begin{pmatrix}8\\-5\\-2\\0\end{pmatrix}\right\}$ **(iv)** $\begin{pmatrix}1\\-1\\1\\1\end{pmatrix},\begin{pmatrix}0\\0\\0\\1\end{pmatrix},\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$.

14 $\left\{\lambda\begin{pmatrix}-1\\1\\1\end{pmatrix}\right\}$, 1, $\mathrm{L}\begin{pmatrix}1\\0\\1\end{pmatrix}=1\begin{pmatrix}1\\0\\1\\0\end{pmatrix}-2\begin{pmatrix}1\\1\\0\\0\end{pmatrix}+4\begin{pmatrix}0\\1\\0\\1\end{pmatrix}-1\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$, $\mathrm{L}\begin{pmatrix}0\\1\\1\end{pmatrix}=1\begin{pmatrix}1\\0\\1\\0\end{pmatrix}-1\begin{pmatrix}1\\1\\0\\0\end{pmatrix}+2\begin{pmatrix}0\\1\\0\\1\end{pmatrix}+0\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$.

15 **(iii)** $\begin{pmatrix}u\\0\\v\\w\end{pmatrix}$.

16 **C** has eigenvalues -1, 3, 5 with corresponding eigenvectors $\begin{pmatrix}-17\\6\\7\end{pmatrix},\begin{pmatrix}1\\-2\\1\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix}$.

D has eigenvalues -3, -2, -1 with corresponding eigenvectors $\begin{pmatrix}1\\-2\\1\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}$.

CD has an eigenvector $\begin{pmatrix}1\\-2\\1\end{pmatrix}$ with corresponding eigenvalue -9.

17 $\{(1\ \ 1\ \ 0),(0\ \ 1\ \ 1)\}$; $\{(2\ \ -2\ \ 1)\}$; **(i)** $\mathbf{r}=\begin{pmatrix}5\\3\\-2\end{pmatrix}+\lambda\begin{pmatrix}1\\0\\1\end{pmatrix}$; **(ii)** $\mathbf{r}=\begin{pmatrix}5\\-2\\0\end{pmatrix}+\lambda\begin{pmatrix}2\\-2\\1\end{pmatrix}$.

18 **(ii)** $\left\{\begin{pmatrix}1\\-1\\1\end{pmatrix},\begin{pmatrix}1\\4\\-3\end{pmatrix}\right\}$. **(iv)** **(a)** It has a solution. **(b)** It has no solution.

19 **(iii)** $\mathbf{e}_1=\begin{pmatrix}3-2a\\-a\\1\\0\end{pmatrix}$, $\mathbf{e}_2=\begin{pmatrix}-2-a\\-1\\0\\1\end{pmatrix}$. **(iv)** $a=-\frac{1}{3}$ or $a=1$.

20 Eigenvalues are 1, 3, -4 with corresponding eigenvectors $\begin{pmatrix}11\\-5\\-6\end{pmatrix},\begin{pmatrix}5\\-7\\-5\end{pmatrix},\begin{pmatrix}1\\0\\-1\end{pmatrix}$.

$\mathbf{Q}=\begin{pmatrix}11&5&1\\-5&-7&0\\-6&-5&-1\end{pmatrix}$, $\mathbf{D}=\begin{pmatrix}1&0&0\\0&9&0\\0&0&16\end{pmatrix}$

21 **A** has eigenvalues 1, 2, -3 with corresponding eigenvectors $\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}3\\1\\0\end{pmatrix},\begin{pmatrix}-1\\8\\-5\end{pmatrix}$.

B has eigenvalues $\frac{1}{6},\frac{1}{7},\frac{1}{2}$ with corresponding eigenvectors $\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}3\\1\\0\end{pmatrix},\begin{pmatrix}-1\\8\\-5\end{pmatrix}$.

22 The corresponding eigenvalues are 1, -1, -2.

(i) $\begin{pmatrix}1&1&1\\-1&-4&-3\\1&3&2\end{pmatrix}$. **(ii)** $\mathbf{R}=\begin{pmatrix}1&1&1\\-1&1&2\\1&-2&-3\end{pmatrix},\mathbf{S}=\begin{pmatrix}1&1&1\\-1&-4&-3\\1&3&2\end{pmatrix},\mathbf{D}=\begin{pmatrix}32&0&0\\0&1&0\\0&0&-\frac{3}{25}\end{pmatrix}$.

23 **(i)** **A** has eigenvalues 1, -2, 3 with corresponding eigenvectors $\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}-a\\3\\0\end{pmatrix},\begin{pmatrix}ac+5b\\2c\\10\end{pmatrix}$.

(ii) **B**: eigenvalues 1, -2, 3 with corresponding eigenvectors $\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}-a\\3\\-a\end{pmatrix},\begin{pmatrix}ac+5b\\2c\\ac+5b+10\end{pmatrix}$.

(iii) $\mathbf{Q}=\begin{pmatrix}1&-a&ac+5b\\0&3&2c\\1&-a&ac+5b+10\end{pmatrix}$, $\mathbf{D}=\begin{pmatrix}1&0&0\\0&(-2)^n&0\\0&0&3^n\end{pmatrix}$

24 **(i)** Eigenvalues are 1, 2, -1 with corresponding eigenvectors $\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}54\\7\\44\end{pmatrix},\begin{pmatrix}-3\\1\\-4\end{pmatrix}$.

(ii) Eigenvalues are $1+k, 2+k, -1+k$.

25 $a+\frac{1}{4}, a+\frac{1}{2}, a+\frac{3}{4}$. $\mathbf{Q}=\begin{pmatrix}1&1&0\\1&0&1\\0&1&1\end{pmatrix}$, $\mathbf{D}=\begin{pmatrix}a+\frac{1}{4}&0&0\\0&a+\frac{1}{2}&0\\0&0&a+\frac{3}{4}\end{pmatrix}$. $\{a\in\mathbb{R}:-\frac{5}{4}<a<\frac{1}{4}\}$.

26 $\mathbf{Q}=\begin{pmatrix}1&0&1\\0&1&1\\1&1&0\end{pmatrix}$, $\mathbf{D}=\begin{pmatrix}2&0&0\\0&-2&0\\0&0&2a\end{pmatrix}$. **(i)** $a=100$, $\mathbf{A}=\begin{pmatrix}101&99&-99\\101&99&-101\\2&-2&0\end{pmatrix}$.

(ii) $a=500$, $k=0.001$, $k\mathbf{A}=\begin{pmatrix}0.501&0.499&-0.499\\0.501&0.499&-0.501\\0.002&-0.002&0\end{pmatrix}$, $\mathbf{D}=\begin{pmatrix}0.002&0&0\\0&-0.002&0\\0&0&1\end{pmatrix}$.

27 $\lambda_i=-5,-3,-2$ with $\mathbf{e}_i=\begin{pmatrix}1\\-4\\-1\end{pmatrix},\begin{pmatrix}1\\-2\\-1\end{pmatrix},\begin{pmatrix}-1\\4\\4\end{pmatrix}$. $\mathbf{B}=\frac{1}{3}\begin{pmatrix}3&1&-1\\-4&-2&4\\0&-1&4\end{pmatrix}$. $\mathbf{P}=\begin{pmatrix}3&1&-1\\-4&-2&4\\0&-1&4\end{pmatrix}$,

least $n=18$.

Application Problems

28 **(a)** **(i)** Reflect Q about the x-axis; $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

(ii) Scaling Q parallel to the y-axis by a factor 3; $\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$.

(b) **(i)** $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. **(ii)** $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

(d) Rotate Q through an angle θ about the origin in anticlockwise direction.

29 **(ii)** $\mathbf{D}=\begin{pmatrix} 0 & 0 \\ 0 & 100 \end{pmatrix}$, $\theta=\tan^{-1}\left(-\frac{3}{4}\right)$. **(iii)** $100y'^2+25x'+25=0$.

(iv) Rotate S about O through an angle $\tan^{-1}\frac{3}{4}$ in anticlockwise direction.

(v) $\left(-\frac{17}{16},0\right)$, $x'=-\frac{15}{16}$; $\left(-\frac{17}{20},\frac{51}{80}\right)$, $64x-48y+75=0$.