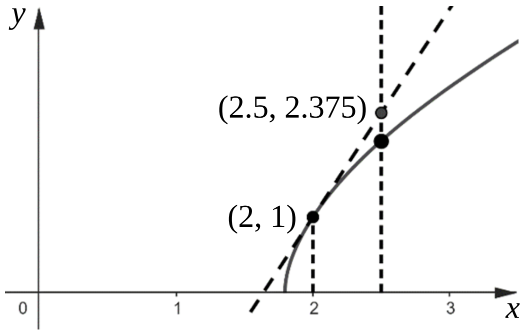
Q1	Solution
	$\frac{\mathrm{d}x}{\mathrm{d}t}=4t^3-\frac{4}{t},\frac{\mathrm{d}y}{\mathrm{d}t}=8t$ $\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2=\left(4t^3-\frac{4}{t}\right)^2+(8t)^2$ $=16t^6-32t^2+\frac{16}{t^2}+64t^2$ $=16t^6+32t^2+\frac{16}{t^2}$ $=16\left(t^6+2t^2+\frac{1}{t^2}\right)$ $=16\left(t^3+\frac{1}{t}\right)^2$ $=\frac{16\left(t^4+1\right)^2}{t^2}$ Surface area $=\int_1^k 2\pi\left(4t^2\right)\frac{4\left(t^4+1\right)}{t}\,\mathrm{d}t$ $=32\pi\int_1^k t^5+t\,\mathrm{d}t$ $=32\pi\left[\frac{1}{6}t^6+\frac{1}{2}t^2\right]_1^k$ $=32\pi\left(\frac{1}{6}k^6+\frac{1}{2}k^2-\frac{2}{3}\right)$ $=\frac{16}{3}\pi\left(k^6+3k^2-4\right)$ Solving $\frac{16}{3}\pi\left(k^6+3k^2-4\right)=384\pi$, (or $k^6+3k^2-76=0$) we get $k=\pm 2$. Since $k>1$, $k=2$.

Q2	Solution
	Multiply throughout by n^{n-1} $n^n x_n=4(n-1)^{n-1}x_{n-1}+2$ Let $n^n x_n=y_n$, $y_n=4y_{n-1}+2$ where $y_1=1$ $4y_{n-1}+2$ $=4^2 y_{n-2}+4\times 2+2$ $=4^3 y_{n-3}+4^2\times 2+4\times 2+2$ $=\cdots$ $=4^{n-1}y_1+4^{n-2}\times 2+4^{n-3}\times 2+\ldots+4\times 2+2$

Q2	Solution
	$= 4^{n-1} + \dfrac{2(1-4^{n-1})}{1-4}$ $= 4^{n-1} - \dfrac{2}{3}\left(1-4^{n-1}\right)$ $= \dfrac{-2+5\cdot 4^{n-1}}{3}$ Hence $x_n = \dfrac{-2+5\cdot 4^{n-1}}{3n^n}$.
	Alternative 1 $y_n = 4y_{n-1} + 2$ where $y_1 = 1$ The general solution is of the form $y_n = A\cdot 4^n + B$. $y_1 = 1 \Rightarrow 4A + B = 1$ $y_2 = 6 \Rightarrow 16A + B = 6$ Using GC, $A = \dfrac{5}{12}, B = -\dfrac{2}{3}$ Hence $y_n = \dfrac{5}{12}\cdot 4^n - \dfrac{2}{3}$.
	Alternative 2 $y_n = 4y_{n-1} + 2$ where $y_1 = 1$ $y_n + \dfrac{2}{3} = 4\left(y_{n-1} + \dfrac{2}{3}\right)$ $y_n + \dfrac{2}{3} = 4^{n-1}\left(y_1 + \dfrac{2}{3}\right)$ $y_n = \dfrac{5}{3}\cdot 4^{n-1} - \dfrac{2}{3}$

Q3	Solution
(a)	Let $y = ux \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x} = u + x\dfrac{\mathrm{d}u}{\mathrm{d}x}$ $x^2 y\dfrac{\mathrm{d}y}{\mathrm{d}x} = x^3 + x^2 y - y^3$ $\Rightarrow x^2(ux)\left(u + x\dfrac{\mathrm{d}u}{\mathrm{d}x}\right) = x^3 + x^2(ux) - (ux)^3$ $\Rightarrow u^2x^3 + ux^4\dfrac{\mathrm{d}u}{\mathrm{d}x} = x^3 + ux^3 - u^3x^3$ $\Rightarrow ux\dfrac{\mathrm{d}u}{\mathrm{d}x} = 1 + u - u^2 - u^3$ $\Rightarrow \dfrac{u}{u^3+u^2-u-1}\dfrac{\mathrm{d}u}{\mathrm{d}x} = -\dfrac{1}{x}$ $\Rightarrow \displaystyle\int \dfrac{u}{(u-1)(u+1)^2}\,\mathrm{d}u = \int -\dfrac{1}{x}\,\mathrm{d}x$ Let $\dfrac{u}{(u-1)(u+1)^2} \equiv \dfrac{A}{u-1} + \dfrac{B}{u+1} + \dfrac{C}{(u+1)^2}$

Q3	Solution
	$\Rightarrow u \equiv A(u+1)^2 + B(u+1)(u-1) + C(u-1)$ Let $u=1$: $1 = 4A \Rightarrow A = \frac{1}{4}$ Let $u=-1$: $-1 = -2C \Rightarrow C = \frac{1}{2}$ Let $u=0$: $0 = \frac{1}{4} - B - \frac{1}{2} \Rightarrow B = -\frac{1}{4}$ $\Rightarrow \int \frac{1}{4(u-1)} - \frac{1}{4(u+1)} + \frac{1}{2(u+1)^2}\,\mathrm{d}u = -\ln\lvert x\rvert + c$ $\Rightarrow \frac{1}{4}\ln\lvert u-1\rvert - \frac{1}{4}\ln\lvert u+1\rvert - \frac{1}{2(u+1)} = -\ln\lvert x\rvert + c$ $\Rightarrow \frac{1}{4}\ln\left\lvert\frac{u-1}{u+1}\right\rvert + \ln\lvert x\rvert = \frac{1}{2(u+1)} + c$ $\Rightarrow \frac{1}{4}\ln\left\lvert\frac{x^4(u-1)}{u+1}\right\rvert = \frac{1}{2(u+1)} + c$ $\Rightarrow \ln\left\lvert\frac{x^4(u-1)}{u+1}\right\rvert = \frac{2}{u+1} + c'$ (where $c' = 4c$) Sub. $u = \frac{y}{x}$: $\ln\left\lvert\frac{x^4\left(\frac{y}{x}-1\right)}{\frac{y}{x}+1}\right\rvert = \frac{2}{\frac{y}{x}+1} + c'$ $\Rightarrow \ln\left\lvert\frac{x^4(y-x)}{y+x}\right\rvert = \frac{2x}{y+x} + c$ Since $x > y$, $y - x < 0$, so the general solution is $\underline{\ln\frac{x^4(x-y)}{y+x} = \frac{2x}{y+x} + c}$
(b)	$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^3 + x^2y - y^3}{x^2y}$ Let $\mathrm{f}(x,y) = \frac{x^3 + x^2y - y^3}{x^2y} = \frac{x}{y} + 1 - \frac{y^2}{x^2}$, $x_0 = 2$, $y_0 = 1$. Using Euler method with step size 0.5, $y_1 = y_0 + 0.5\mathrm{f}(x_0, y_0)$ $= 1 + 0.5\left(\frac{2}{1} + 1 - \frac{1^2}{2^2}\right)$ $= \underline{2.375}$

Q3	Solution
(c)	 The approximation is an over-estimate.

Q4 Solution

(i) Volume of revolution about y-axis, V

$$= \int_0^{a\left(\frac{3\pi}{2}+1\right)} 2\pi xy \, dx$$

$$= \int_0^{\frac{3\pi}{2}} 2\pi\left[a(t-\sin t)\right]\left[a(1-\cos t)\right]\frac{dx}{dt}\,dt$$

$$= \int_0^{\frac{3\pi}{2}} 2\pi\left[a(t-\sin t)\right]\left[a(1-\cos t)\right]^2\,dt$$

$$= 2\pi a^3 \int_0^{\frac{3\pi}{2}} (t-\sin t)(1-\cos t)^2\,dt \quad \text{(Shown)}$$

(ii) V

$$= 2\pi a^3 \int_0^{\frac{3\pi}{2}} (t-\sin t)(1-\cos t)^2\,dt$$

$$= 2\pi a^3 \int_0^{\frac{3\pi}{2}} (t-\sin t)(1-2\cos t+\cos^2 t)\,dt$$

$$= 2\pi a^3 \int_0^{\frac{3\pi}{2}} (t-\sin t)(1-2\cos t+\frac{\cos 2t+1}{2})\,dt$$

$$= 2\pi a^3 \int_0^{\frac{3\pi}{2}} (t-\sin t)(\frac{3}{2}-2\cos t+\frac{\cos 2t}{2})\,dt$$

$$= 2\pi a^3 \int_0^{\frac{3\pi}{2}} \frac{3}{2}t-\frac{3}{2}\sin t+2\cos t\sin t-\frac{\cos 2t\sin t}{2}+t\left(\frac{\cos 2t}{2}-2\cos t\right)\,dt$$

$$= 2\pi a^3 \int_0^{\frac{3\pi}{2}} \frac{3}{2}t-\frac{3}{2}\sin t+\sin 2t-\frac{\sin 3t-\sin t}{4}+t\left(\frac{\cos 2t}{2}-2\cos t\right)\,dt$$

$$= 2\pi a^3 \int_0^{\frac{3\pi}{2}} \frac{3t}{2}-\frac{5\sin t}{4}+\sin 2t-\frac{\sin 3t}{4}+t\left(\frac{\cos 2t}{2}-2\cos t\right)\,dt$$

$$= 2\pi a^3 \left[\frac{3t^2}{4} + \frac{5\cos t}{4} - \frac{\cos 2t}{2} + \frac{\cos 3t}{12} + t\left(\frac{\sin 2t}{4} - 2\sin t \right) \right]_0^{\frac{3\pi}{2}} - \int_0^{\frac{3\pi}{2}} \left(\frac{\sin 2t}{4} - 2\sin t \right) \mathrm{d}t$$

$$= 2\pi a^3 \left[\frac{3t^2}{4} + \frac{5\cos t}{4} - \frac{\cos 2t}{2} + \frac{\cos 3t}{12} + t\left(\frac{\sin 2t}{4} - 2\sin t \right) \right]_0^{\frac{3\pi}{2}} - \left[-\frac{\cos 2t}{8} + 2\cos t \right]_0^{\frac{3\pi}{2}}$$

$$= 2\pi a^3 \left[\frac{3t^2}{4} - \frac{3\cos t}{4} - \frac{3\cos 2t}{8} + \frac{\cos 3t}{12} + t\left(\frac{\sin 2t}{4} - 2\sin t \right) \right]_0^{\frac{3\pi}{2}}$$

$$= 2\pi a^3 \left[\frac{3}{4}\left(\frac{9\pi^2}{4} - 0 \right) - \frac{3}{4}(0-1) - \frac{3}{8}(-1-1) + \frac{1}{12}(0-1) + \left(\frac{3\pi}{2} \right)\left(\frac{1}{4}(0) - 2(-1) \right) \right]$$

$$= 2\pi a^3 \left(\frac{27\pi^2}{16} + 3\pi + \frac{17}{12} \right)$$

$$= \pi a^3 \left(\frac{27\pi^2}{8} + 6\pi + \frac{17}{6} \right)$$

$$= \frac{1}{24}\pi a^3 \left(81\pi^2 + 72\pi + 34 \right)$$

<u>Alternative</u>

$$V = 2\pi a^3 \int_0^{\frac{3\pi}{2}} \left[(t - 2t\cos t + t\cos^2 t) + (-\sin t)(1-\cos t)^2 \right] \mathrm{d}t$$

Let $I = \int t\cos t \, \mathrm{d}t = t\sin t + \cos t + c$ ----(1)

Let $J = \int t\cos^2 t \, \mathrm{d}t$

$$u = t \Rightarrow \frac{\mathrm{du}}{\mathrm{d}t} = 1$$

$$\frac{\mathrm{dv}}{\mathrm{d}t} = \cos^2 t \Rightarrow v = \int \tfrac{1}{2}(1+\cos 2t)\,\mathrm{d}t = \tfrac{1}{2}t + \tfrac{1}{4}\sin 2t$$

$$J = \tfrac{1}{2}t^2 + \tfrac{1}{4}t\sin 2t - \tfrac{1}{2}\int (t + \tfrac{1}{2}\sin 2t)\,\mathrm{d}t$$

$$J = \tfrac{1}{2}t^2 + \tfrac{1}{4}t\sin 2t - \tfrac{1}{2}\left[\tfrac{1}{2}t^2 - \tfrac{1}{4}\cos 2t \right] + c$$

$$J = \tfrac{1}{2}\left[\tfrac{1}{2}t^2 + \tfrac{1}{2}t\sin 2t + \tfrac{1}{4}\cos 2t \right] + c$$

Let $K = \int (-\sin t)(1-\cos t)^2 \, \mathrm{d}t$

	$K=\int(-\sin t)(1-\cos t)^2\,\mathrm{d}t=-\frac{1}{3}(1-\cos t)^3+c$ V $=2\pi a^3\int_0^{\frac{3\pi}{2}}\left[(t-2t\cos t+t\cos^2 t)+(-\sin t)(1-\cos t)^2\right]\mathrm{d}t$ $=2\pi a^3\begin{bmatrix}\frac{1}{2}t^2-2(t\sin t+\cos t)\\+\frac{1}{2}\left[\frac{1}{2}t^2+\frac{1}{2}t\sin 2t+\frac{1}{4}\cos 2t\right]\\-\frac{1}{3}(1-\cos t)^3\end{bmatrix}_0^{\frac{3\pi}{2}}$ $=2\pi a^3\begin{bmatrix}\frac{3}{4}t^2-2t\sin t-2\cos t\\+\frac{1}{4}t\sin 2t+\frac{1}{8}\cos 2t\\-\frac{1}{3}(1-\cos t)^3\end{bmatrix}_0^{\frac{3\pi}{2}}$ $=2\pi a^3\begin{bmatrix}\frac{27}{16}\pi^2-0+3\pi\\+0-\frac{1}{8}\\-\frac{1}{3}\end{bmatrix}-2\pi a^3\begin{bmatrix}0-2-0\\+0+\frac{1}{8}\\-0\end{bmatrix}$ $=2\pi a^3\left[\frac{27}{16}\pi^2+3\pi+\frac{17}{12}\right]$

Q5	Solution
(i)	$\det(\mathbf{A})=\det(\mathbf{A}^{\mathrm{T}})=\det(-\mathbf{A})=(-1)^3\det(\mathbf{A})=-\det(\mathbf{A})$ since **A** is skew symmetric $\therefore 2\det(\mathbf{A})=0 \Rightarrow \det(\mathbf{A})=0$.
(ii) (a)	Method 1 $\mathbf{a}\times\mathbf{x}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\times\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$ $=\begin{pmatrix}a_2x_3-a_3x_2\\a_3x_1-a_1x_3\\a_1x_2-a_2x_1\end{pmatrix}$ $=\begin{pmatrix}0&-a_3&a_2\\a_3&0&-a_1\\-a_2&a_1&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$ $\therefore \mathbf{M}=\begin{pmatrix}0&-a_3&a_2\\a_3&0&-a_1\\-a_2&a_1&0\end{pmatrix}$ Method 2 $\mathbf{a}\times\mathbf{i}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\times\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}0\\a_3\\-a_2\end{pmatrix}$ $\mathbf{a}\times\mathbf{j}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\times\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}-a_3\\0\\a_1\end{pmatrix}$ $\mathbf{a}\times\mathbf{k}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\times\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}a_2\\-a_1\\0\end{pmatrix}$ $\therefore \mathbf{M}=\begin{pmatrix}0&-a_3&a_2\\a_3&0&-a_1\\-a_2&a_1&0\end{pmatrix}$
(ii) (b)	**M** is skew symmetric and so by **(i)** $\det(\mathbf{M})=0$ Hence **M** is not invertible.
(ii) (c)	$\ker(T)=\{k\mathbf{a} \mid k\in\mathbb{R}\}$, the set of vectors parallel to **a** or line through origin and parallel to **a** $\mathrm{R}(T)=\{\mathbf{v}\in\mathbb{R}^3 \mid \mathbf{a}\cdot\mathbf{v}=0\}$, the set of vectors perpendicular to **a** or plane through origin and perpendicular to **a**

Q6	Solution
(i)	$\mathbf{T}_\theta\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}=\begin{pmatrix}\cos 2\theta & \sin 2\theta\\ \sin 2\theta & -\cos 2\theta\end{pmatrix}\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$ $=\begin{pmatrix}\cos 2\theta\cos\theta+\sin 2\theta\sin\theta\\ \sin 2\theta\cos\theta-\cos 2\theta\sin\theta\end{pmatrix}$ $=\begin{pmatrix}\cos(2\theta-\theta)\\ \sin(2\theta-\theta)\end{pmatrix}$ $=\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$ $\mathbf{T}_\theta\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}=\begin{pmatrix}\cos 2\theta & \sin 2\theta\\ \sin 2\theta & -\cos 2\theta\end{pmatrix}\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}$ $=\begin{pmatrix}-\cos 2\theta\sin\theta+\sin 2\theta\cos\theta\\ -\sin 2\theta\sin\theta-\cos 2\theta\cos\theta\end{pmatrix}$ $=\begin{pmatrix}\sin(2\theta-\theta)\\ -\cos(2\theta-\theta)\end{pmatrix}$ $=-\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}$ The eigenvalues are 1 and -1 with eigenvector $\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$ and $\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}$ respectively.
(ii)	$\mathbf{T}_\theta=\mathbf{R}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\mathbf{R}^{-1}$
(iii)	$\theta=\frac{\pi}{3}$ $\mathbf{T}_\theta=\begin{pmatrix}-\frac{1}{2} & \frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix}$ $\mathbf{T}_\theta\begin{pmatrix}2\\ 1\end{pmatrix}=\begin{pmatrix}-\frac{1}{2} & \frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}2\\ 1\end{pmatrix}=\begin{pmatrix}-1+\frac{\sqrt{3}}{2}\\ \sqrt{3}+\frac{1}{2}\end{pmatrix}$
(iv)	$\mathbf{T}_\alpha\mathbf{T}_\beta$ $=\begin{pmatrix}\cos 2\alpha & \sin 2\alpha\\ \sin 2\alpha & -\cos 2\alpha\end{pmatrix}\begin{pmatrix}\cos 2\beta & \sin 2\beta\\ \sin 2\beta & -\cos 2\beta\end{pmatrix}$ $=\begin{pmatrix}\cos 2\alpha\cos 2\beta+\sin 2\alpha\sin 2\beta & \cos 2\alpha\sin 2\beta-\sin 2\alpha\cos 2\beta\\ \sin 2\alpha\cos 2\beta-\cos 2\alpha\sin 2\beta & \sin 2\alpha\sin 2\beta+\cos 2\alpha\cos 2\beta\end{pmatrix}$ $=\begin{pmatrix}\cos 2(\alpha-\beta) & -\sin 2(\alpha-\beta)\\ \sin 2(\alpha-\beta) & \cos 2(\alpha-\beta)\end{pmatrix}$ which is a rotation through an angle of $2(\alpha-\beta)$

Q7	Solution
(i)	$\text{f}(x)=\dfrac{1}{x^2}-\sqrt{x-2}$ $\text{f}(2)=\dfrac{1}{2^2}-\sqrt{2-2}=\dfrac{1}{4}>0$ $\text{f}(3)=\dfrac{1}{3^2}-\sqrt{3-2}=-\dfrac{8}{9}<0$ Since $y=\text{f}(x)$ is a continuous curve over the interval $[2,3]$ with $\text{f}(2)\cdot\text{f}(3)<0$, therefore there exists a root α in the interval.
(ii)	Let $x_1=\dfrac{2\lvert\text{f}(3)\rvert+3\lvert\text{f}(2)\rvert}{\lvert\text{f}(3)\rvert+\lvert\text{f}(2)\rvert}=\dfrac{91}{41}$ Note that $\text{f}\left(\dfrac{91}{41}\right)<0$. $\therefore \alpha\in\left[2,\dfrac{91}{41}\right]$ $\beta=\dfrac{2\lvert\text{f}(\frac{91}{41})\rvert+\frac{91}{41}\lvert\text{f}(2)\rvert}{\lvert\text{f}(\frac{91}{41})\rvert+\lvert\text{f}(2)\rvert}=2.106450495$ $\beta=2.11$ (3.s.f)
(iii)	$\text{f}(x)=\dfrac{1}{x^2}-\sqrt{x-2}$ $\text{f}'(x)=-\dfrac{2}{x^3}-\dfrac{1}{2}(x-2)^{-\frac{1}{2}}<0$ for $2\le \text{x}\le\dfrac{91}{41}$ $\text{f}''(x)=\dfrac{6}{x^4}+\dfrac{1}{4}(x-2)^{-\frac{3}{2}}>0$ for $2\le \text{x}\le\dfrac{91}{41}$ $\text{f}(x)=\dfrac{1}{x^2}-\sqrt{x-2}$ is concave upward and its gradient negative over the interval $2\le \text{x}\le\dfrac{91}{41}$, β is an over-estimate of the root.
(iv)	$\text{f}(x)=\dfrac{1}{x^2}-\sqrt{x-2}$, $\text{f}'(x)=-\dfrac{2}{x^3}-\dfrac{1}{2\sqrt{x-2}}$ Applying $x_{n+1}=x_n-\dfrac{\text{f}(x_n)}{\text{f}'(x_n)}$ with $x_0=2.5$, $x_1=x_0-\dfrac{\text{f}(x_0)}{\text{f}'(x_0)}=1.844866$ $\text{f}'(x_1)=-\dfrac{2}{(x_1)^3}-\dfrac{1}{2\sqrt{x_1-2}}$ is undefined for $x_1=1.84486$. Therefore the Newton-Raphson method failed.
(v)	$x_{n+1}=2+\dfrac{1}{(x_n)^4}$ $x_0=3$

	$x_1 = 2 + \frac{1}{(x_0)^4} = 2.012$ $x_2 = 2 + \frac{1}{(x_1)^4} = 2.061$ $x_3 = 2 + \frac{1}{(x_2)^4} = 2.055$ γ is 2.06 correct to 3 s.f. Since $\text{f}(2.055) = 0.00227 > 0$ and $\text{f}(2.065) = -0.02044 < 0$, $\gamma = 2.06$ is sufficiently accurate correct to 3 significant figures.

Q8	Solution
(a)	$m\frac{\mathrm{d}v}{\mathrm{d}t}=C-kv$ $\Rightarrow \frac{1}{C-kv}\frac{\mathrm{d}v}{\mathrm{d}t}=\frac{1}{m}$ $\Rightarrow \int \frac{1}{C-kv}\,\mathrm{d}v=\int \frac{1}{m}\,\mathrm{d}t$ $\Rightarrow -\frac{1}{k}\ln\lvert C-kv\rvert=\frac{t}{m}+c$ (for arbitrary constant c) $\Rightarrow \ln\lvert C-kv\rvert=-\frac{kt}{m}-kc$ $\Rightarrow C-kv \quad =\pm\mathrm{e}^{-\frac{kt}{m}-kc}$ $\qquad\qquad = A\mathrm{e}^{-\frac{kt}{m}}$ (where $A=\pm\mathrm{e}^{-kc}$) When $t=0$, $v=v_0$: $A=C-kv_0$ $\Rightarrow C-kv=\left(C-kv_0\right)\mathrm{e}^{-\frac{kt}{m}}$ $\Rightarrow \underline{v=\frac{C-\left(C-kv_0\right)\mathrm{e}^{-\frac{kt}{m}}}{k}}$
(b)	As $t\to\infty$, $\mathrm{e}^{-\frac{kt}{m}}\to 0$, so $v\to\frac{C}{k}$. After a long time, the velocity of the falling body approaches the constant value $\frac{C}{k}$, which is independent of the initial velocity v_0.
(c)(i)	$m\frac{\mathrm{d}v}{\mathrm{d}t}=C-qt-kv$ $\Rightarrow \frac{\mathrm{d}v}{\mathrm{d}t}+\frac{k}{m}v=\frac{C-qt}{m}$ Multiply both sides by integrating factor $\mathrm{e}^{\int\frac{k}{m}\mathrm{d}t}=\mathrm{e}^{\frac{kt}{m}}$: $\mathrm{e}^{\frac{kt}{m}}\frac{\mathrm{d}v}{\mathrm{d}t}+\frac{k}{m}\mathrm{e}^{\frac{kt}{m}}v=\mathrm{e}^{\frac{kt}{m}}\left(\frac{C-qt}{m}\right)$ $\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{\frac{kt}{m}}v\right)=\frac{\mathrm{e}^{\frac{kt}{m}}}{m}\left(C-qt\right)$ $\Rightarrow \mathrm{e}^{\frac{kt}{m}}v=\int\frac{\mathrm{e}^{\frac{kt}{m}}}{m}\left(C-qt\right)\,\mathrm{d}t$ $\qquad =\frac{\mathrm{e}^{\frac{kt}{m}}}{k}\left(C-qt\right)-\int\frac{\mathrm{e}^{\frac{kt}{m}}}{k}\left(-q\right)\,\mathrm{d}t$ $\qquad =\frac{\mathrm{e}^{\frac{kt}{m}}}{k}\left(C-qt\right)+\frac{q}{k}\int\mathrm{e}^{\frac{kt}{m}}\,\mathrm{d}t$

Q8	Solution
	$= \frac{e^{\frac{kt}{m}}}{k}(C-qt)+\frac{q}{k}\left(\frac{me^{\frac{kt}{m}}}{k}\right)+c$ (arbitrary constant c) $\Rightarrow v=\frac{C-qt}{k}+\frac{qm}{k^2}+ce^{-\frac{kt}{m}}$ When $t=0$, $v=0$: $\frac{C}{k}+\frac{qm}{k^2}+c=0 \Rightarrow c=-\frac{C}{k}-\frac{qm}{k^2}$ $\therefore\ v=\frac{C-qt}{k}+\frac{qm}{k^2}-\left(\frac{C}{k}+\frac{qm}{k^2}\right)e^{-\frac{kt}{m}}$ $\Rightarrow \underline{v=\left(\frac{C}{k}+\frac{qm}{k^2}\right)\left(1-e^{-\frac{kt}{m}}\right)-\frac{qt}{k}}$ (shown)
(c)(ii)	$\mathrm{F}(t)=C-qt=0 \Rightarrow qt=C \Rightarrow t=\frac{C}{q}$ Substitute into equation for v: $v=\left(\frac{C}{k}+\frac{qm}{k^2}\right)\left(1-e^{-\frac{Ck}{qm}}\right)-\frac{C}{k}$ $=\frac{qm}{k^2}-\left(\frac{qm}{k^2}+\frac{C}{k}\right)e^{-\frac{Ck}{qm}}$ Since $\frac{Ck}{qm}>0$, $e^{-\frac{Ck}{qm}}<\frac{1}{1+\frac{Ck}{qm}}=\frac{qm}{qm+Ck}$ $\Rightarrow v > \frac{qm}{k^2}-\left(\frac{qm+Ck}{k^2}\right)\left(\frac{qm}{qm+Ck}\right)$ $=\frac{qm}{k^2}-\frac{qm}{k^2}$ $=0$ $\therefore$ v is positive when $\mathrm{F}(t)=0$. (shown)

Q9	Solution
(a) (i)	$4u_r=2u_{r-1}-u_{r-2}$ $4\lambda^2-2\lambda+1=0$ $\lambda=\frac{2\pm\sqrt{-12}}{8}=\frac{1}{4}(1\pm\sqrt{3}\mathrm{i})$ $u_r=\left(\frac{1}{2}\right)^r\left(A\sin\frac{r\pi}{3}+B\cos\frac{r\pi}{3}\right)$

Q9	Solution
(a) (ii)	$u_{k+3}=\left(\frac{1}{2}\right)^{k+3}\left(A\sin\frac{(k+3)\pi}{3}+B\cos\frac{(k+3)\pi}{3}\right)$ $=\left(\frac{1}{2}\right)^{k+3}\left(-A\sin\frac{k\pi}{3}-B\cos\frac{k\pi}{3}\right)$ $=-\frac{1}{8}\left(\frac{1}{2}\right)^{k}\left(A\sin\frac{k\pi}{3}+B\cos\frac{k\pi}{3}\right)$ $=-\frac{1}{8}u_k$ $\therefore \alpha=-\frac{1}{8}$ $\sum_{r=1}^{\infty}u_{3r-2}$ $=\frac{u_1}{1-\left(-\frac{1}{8}\right)}$ $=\frac{8}{9}u_1$
(b) (i)	Case 1: First signal requires 1 microsecond There are 2 different signals that require 1 microsecond and for the remaining $n-1$ microseconds, a_{n-1} messages can be transmitted, so total number of messages is $2a_{n-1}$ Case 2: First signal requires 2 microseconds There are 3 different signals that require 2 microsecond and for the remaining $n-2$ microseconds, a_{n-2} messages can be transmitted, so total number of messages is $3a_{n-2}$ Combining the two cases above, the recurrence relation is $a_n=2a_{n-1}+3a_{n-2}$
(ii)	$\lambda^2-2\lambda-3=0$ $\lambda=-1$ or $\lambda=3$ Hence $a_n=A(-1)^n+B(3)^n$ Note that $a_1=2$ and $a_2=7$ $-A+3B=2$ $A+9B=7$ $A=\frac{1}{4},B=\frac{3}{4}$ $\therefore a_n=\frac{1}{4}(-1)^n+\frac{3}{4}(3)^n$

Q10	Solution
(i)	When $\theta = 0, r = \dfrac{a}{1+e}$. When $\theta = \pi, r = \dfrac{a}{1-e}$. $\dfrac{a}{1+e} + \dfrac{a}{1-e} = \dfrac{2a}{1-e^2} = 5.9$ $\dfrac{\frac{a}{1-e}}{\frac{a}{1+e}} = 2.15 \Rightarrow \dfrac{1+e}{1-e} = 2.15$ $\Rightarrow e = 0.3650793 \approx 0.365$ $\dfrac{2a}{1-e^2} = 5.9 \Rightarrow \dfrac{2a}{1-0.3650793^2} = 5.9$ $\Rightarrow a = 2.55682 \approx 2.56$
(ii)	$\dfrac{\text{Required time}}{\text{orbital period}} = \dfrac{\text{Area from } \theta = \frac{\pi}{2} \text{ to } \theta = 3}{2(\text{Area from } \theta = 0 \text{ to } \theta = \pi)}$ $\dfrac{\text{Required time}}{130} = \dfrac{0.5\int_{\frac{\pi}{2}}^{3}\left(\frac{a}{1+e\cos\theta}\right)^2 d\theta}{2\left(0.5\int_0^{\pi}\left(\frac{a}{1+e\cos\theta}\right)^2 d\theta\right)}$ Required time $= \dfrac{8.11021694}{25.45271596} \times 130 = 41.423 \approx 41.4$ days
(iii)	$\dfrac{\text{Additional time required}}{\text{orbital period}} = \dfrac{\text{Area from } \theta = 3 \text{ to } \theta = \pi}{2(\text{Area from } \theta = 0 \text{ to } \theta = \pi)}$ $\dfrac{\text{Additional time required}}{130} = \dfrac{0.5\int_{3}^{\pi}\left(\frac{a}{1+e\cos\theta}\right)^2 d\theta}{2\left(0.5\int_0^{\pi}\left(\frac{a}{1+e\cos\theta}\right)^2 d\theta\right)}$ Additional time required $= \dfrac{1.143695252}{25.45271596} \times 130 = 5.8414$ ≈ 5.84 days
	$25x^2 + 4y^2 - 50x_0x - 8y_0y + 25{x_0}^2 + 4{y_0}^2 - 100 = 0$ $25\left(x^2 - 2x_0x + {x_0}^2\right) + 4\left(y^2 - 2yy_0 + {y_0}^2\right) = 100$ $25\left(x - x_0\right)^2 + 4\left(y - y_0\right)^2 = 100$ $\dfrac{\left(x-x_0\right)^2}{2^2} + \dfrac{\left(y-y_0\right)^2}{5^2} = 1$ Thus, length of semi-major axis of Q is 5.

Q10	Solution
	$\dfrac{(\text{Period of } Q)^2}{(\text{Period of } P)^2} = \dfrac{5^3}{(5.9/2)^3}$ Period of $Q = \sqrt{\dfrac{5^3}{(5.9/2)^3} \times 130^2} = 286.8569 \approx 287$ days