Tampines Meridian Junior College
2024 H2 Mathematics (9758)
Chapter 5 Vectors
Learning Package

Resources

- ☐ Core Concept Notes
- ☐ Discussion Questions
- ☐ Extra Practice Questions
- ☐ Exploration Activity

SLS Resources

- ☐ Recordings on Core Concepts
- ☐ Quick Concept Checks
- ☐ Exploration Activity

Reflection or Summary Page

H2 Mathematics (9758)
Chapter 5 Vectors
Core Concept Notes

Success Criteria:

Surface Learning	**Deep Learning**	**Transfer Learning**
☐ Identify position vectors, displacement vectors and direction vectors. ☐ Convert a vector in cartesian form $x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ into column vector form $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$. ☐ Express a displacement vector in terms of the position vectors of its end points. (e.g. $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}$) ☐ Carry out addition and subtraction of vectors, multiplication of a vector by a scalar, and interpret these operations in geometrical terms. ☐ Calculate the magnitude of a vector. ☐ Find the unit vector of a given vector. ☐ Determine if 2 given vectors are parallel. ☐ Calculate dot (scalar) product of two vectors in component form. ☐ Find angle between 2 vectors using the definition of dot product $\mathbf{a}\bullet\mathbf{b}=\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\theta$. ☐ Calculate cross (vector) product of two vectors in component form. ☐ Find a vector perpendicular to two non-parallel vectors using cross product. ☐ Use of properties of dot and cross product of equal vectors. (i.e. $\mathbf{a}\bullet\mathbf{a}=\lvert\mathbf{a}\rvert^2$, $\mathbf{a}\times\mathbf{a}=\mathbf{0}$)	☐ Find a vector of a specified magnitude that is parallel to a given vector. ☐ Apply ratio theorem to find vectors. ☐ Determine whether three points with given coordinates are collinear. ☐ Determine if 2 given vectors are perpendicular using dot product (i.e. $\mathbf{a}\bullet\mathbf{b}=0$). ☐ Find length of projection of a vector **a** onto a vector **b** ☐ Find projection vector of a vector **a** onto a vector **b**. ☐ Find the perpendicular/shortest distance from a point to a line. ☐ Use cross product to find area of triangle and parallelogram. ☐ Use properties of dot and cross product to solve problems.	☐ Use ratio theorem or collinearity in geometrical applications. ☐ Visualise using a diagram and use vector concepts to solve problems. ☐ Give geometrical interpretation of $\lvert\mathbf{a}\bullet\hat{\mathbf{b}}\rvert$, $\lvert\mathbf{a}\times\hat{\mathbf{b}}\rvert$ and $\lvert\mathbf{a}\times\mathbf{b}\rvert$. ☐ Solve vector related questions involving unknowns.

§1 Introduction to Vectors

At the 'O' Level, you have learnt about vectors in 2-dimensional space. We will be now exposed to vectors in 3-dimensional space, which is more realistic since we live in a 3-dimensional world. All operations in 2-D also apply in 3-D.

Basic Vector Operations (3 dimensional)

(i) $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \pm \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 \pm x_2 \\ y_1 \pm y_2 \\ z_1 \pm z_2 \end{pmatrix}$

(ii) $\lambda \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \lambda x \\ \lambda y \\ \lambda z \end{pmatrix}, \quad \lambda \in \mathbb{R}$

(iii) If $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$,

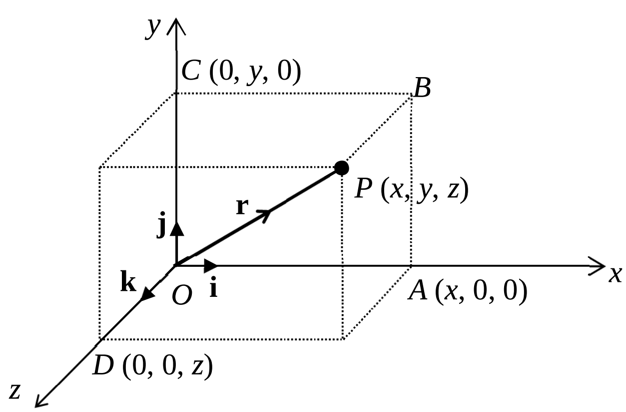
then $x_1 = x_2$, $y_1 = y_2$ and $z_1 = z_2$.

1.1 3-Dimensional Vectors in Cartesian Form

In the 3-dimensional Cartesian space with the *x*, *y*, *z* axes, **i**, **j** and **k** are **mutually perpendicular unit vectors** along the *x*, *y* and *z* axes respectively and i.e.

$$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |\mathbf{i}| = 1; \qquad \mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |\mathbf{j}| = 1; \qquad \mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad |\mathbf{k}| = 1.$$

Let $P(x, y, z)$ be any point in space, and let **r** be its position vector.



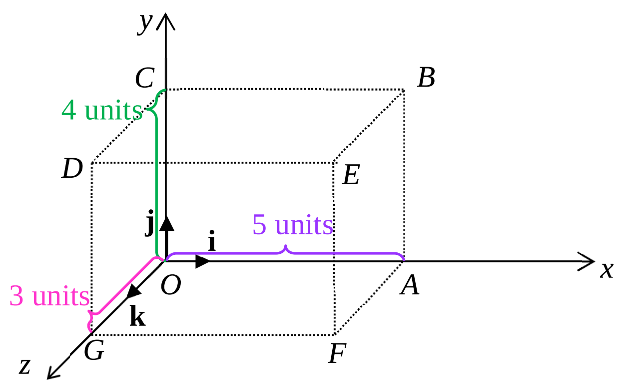
In **Cartesian** form, the vector $\overrightarrow{OP}$ is given by

$$\begin{aligned}\overrightarrow{OP} = \mathbf{r} &= \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BP} \\ &= \overrightarrow{OA} + \overrightarrow{OC} + \overrightarrow{OD} \\ &= x\mathbf{i} + y\mathbf{j} + z\mathbf{k}\end{aligned}$$

In column vector form,

$$\begin{aligned}\overrightarrow{OP} = \mathbf{r} &= x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \\ &= x\begin{pmatrix}1\\0\\0\end{pmatrix} + y\begin{pmatrix}0\\1\\0\end{pmatrix} + z\begin{pmatrix}0\\0\\1\end{pmatrix}. \\ &= \begin{pmatrix}x\\y\\z\end{pmatrix}\end{aligned}$$

Exercise: Given the diagram below such that *OA*, *OC* and *OG* are 5, 4 and 3 units respectively, write down, in column vector form, the position vectors of the points *A*, *B*, *D* and *E*.



Solution:

$$\overrightarrow{OA} = \begin{pmatrix}5\\0\\0\end{pmatrix}, \quad \overrightarrow{OB} = \begin{pmatrix}5\\4\\0\end{pmatrix}, \quad \overrightarrow{OD} = \begin{pmatrix}0\\4\\3\end{pmatrix}, \quad \overrightarrow{OE} = \begin{pmatrix}5\\4\\3\end{pmatrix}$$

The magnitude of the vector $\overrightarrow{OP}$ is given by

$$|\mathbf{r}| = \sqrt{x^2 + y^2 + z^2} \,.$$

The unit vector in the direction of **r** is

$$\hat{\mathbf{r}} = \frac{\mathbf{r}}{|\mathbf{r}|} = \frac{1}{\sqrt{x^2+y^2+z^2}}(x\mathbf{i}+y\mathbf{j}+z\mathbf{k}) = \frac{1}{\sqrt{x^2+y^2+z^2}}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Example 1

Find the unit vector in the direction of $\overrightarrow{OD}$.

Solution:

$\overrightarrow{OD} = \begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix}$, $\left|\overrightarrow{OD}\right| = \sqrt{0^2 + 4^2 + 3^2} = \sqrt{25} = 5$

Hence, the unit vector in the direction of $\overrightarrow{OD} = \frac{1}{5}\begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix}$.

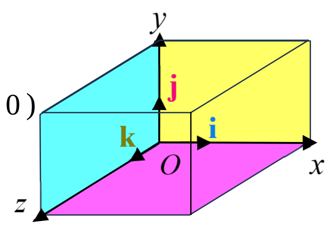
Exercise: Write down a unit vector parallel to $\overrightarrow{OD}$.

$\frac{1}{5}\begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix}$ or $-\frac{1}{5}\begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix}$.

Note:

(a) If P is a point in the x-y plane, then $\overrightarrow{OP} = x\mathbf{i} + y\mathbf{j} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$ (i.e. $z = 0$)

(b) If Q is a point in the x-z plane, then $\overrightarrow{OQ} = x\mathbf{i} + z\mathbf{k} = \begin{pmatrix} x \\ 0 \\ z \end{pmatrix}$ (i.e. $y = 0$)



(c) If R is a point in the y-z plane, then $\overrightarrow{OR} = y\mathbf{j} + z\mathbf{k} = \begin{pmatrix} 0 \\ y \\ z \end{pmatrix}$ (i.e. $x = 0$)

Example 2

The position vectors of points *A*, *B* and *C* are $2\mathbf{i} - \mathbf{j} + \mathbf{k}$, $3\mathbf{i} + 5\mathbf{j} - 4\mathbf{k}$, and $4\mathbf{i} + \mathbf{j}$ respectively. Find

Convert to column vectors first

(i) the length of $\overrightarrow{AB}$,

(ii) the unit vector in the direction of $\overrightarrow{AC}$,

(iii) the position vector of *D* so that *ABCD* is a parallelogram,

(iv) the position vector of *E* such that $\overrightarrow{BE} = 2\overrightarrow{AC}$.

Solution:

(i) $\overrightarrow{AB} = \begin{pmatrix} 3 \\ 5 \\ -4 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \\ -5 \end{pmatrix}$

Recall that:
$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$\left|\overrightarrow{AB}\right| = \sqrt{\left(1^2 + 6^2 + (-5)^2\right)} = \sqrt{62}$

(ii) $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$

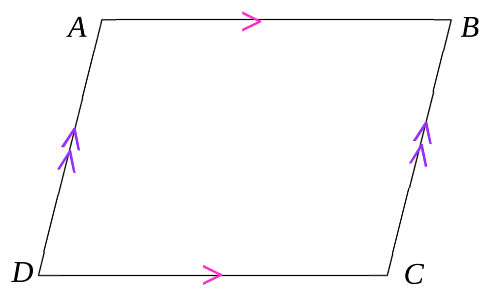
Unit vector in the direction of $\overrightarrow{AC} = \dfrac{1}{\sqrt{\left(2^2 + 2^2 + (-1)^2\right)}} \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$

$= \dfrac{1}{3} \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$

(iii) If *ABCD* is a parallelogram, then $\overrightarrow{AB} = \overrightarrow{DC}$ $\left(\text{or } \overrightarrow{AD} = \overrightarrow{BC}\right)$

$\Rightarrow \begin{pmatrix} 1 \\ 6 \\ -5 \end{pmatrix} = \overrightarrow{OC} - \overrightarrow{OD} = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} - \overrightarrow{OD}$

$\Rightarrow \overrightarrow{OD} = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 6 \\ -5 \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \\ 5 \end{pmatrix}$



(iv) $\overrightarrow{BE} = 2\overrightarrow{AC}$

$\overrightarrow{OE} - \overrightarrow{OB} = \overrightarrow{OE} - \begin{pmatrix} 3 \\ 5 \\ -4 \end{pmatrix} = 2\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$

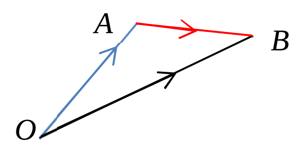
$\overrightarrow{OE} = \begin{pmatrix} 4 \\ 4 \\ -2 \end{pmatrix} + \begin{pmatrix} 3 \\ 5 \\ -4 \end{pmatrix} = \begin{pmatrix} 7 \\ 9 \\ -6 \end{pmatrix}$

§2 Basic Vector Algebra

2.1 Triangle Law of Vector Addition

Consider three vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{AB}$ formed by the points *O*, *A*, and *B*. Observe that these vectors make up the sides of a triangle as shown and that

$$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$$



This is known as the triangle law of addition.

An extremely useful result:
$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

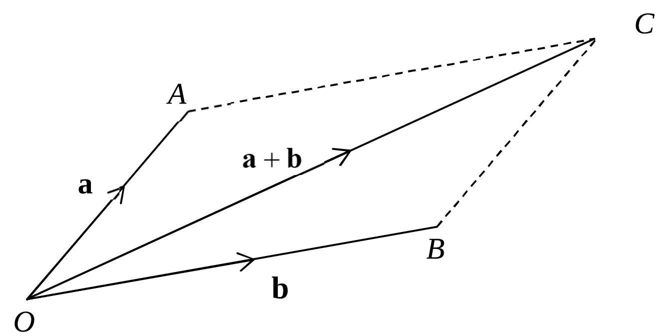
For any two column vectors, $\mathbf{a} = \begin{pmatrix} p \\ q \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} x \\ y \end{pmatrix}$, $\mathbf{a} + \mathbf{b} = \begin{pmatrix} p+x \\ q+y \end{pmatrix}$.

Example: $\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} -5 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$

2.2 Parallelogram Law of Vector Addition

Consider the parallelogram *OBCA* below.

If $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, then $\overrightarrow{OC} = \mathbf{a} + \mathbf{b}$, where *OC* is a diagonal of the parallelogram *OBCA.*



Proof:

$$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC}$$
$$= \mathbf{a} + \mathbf{b}$$

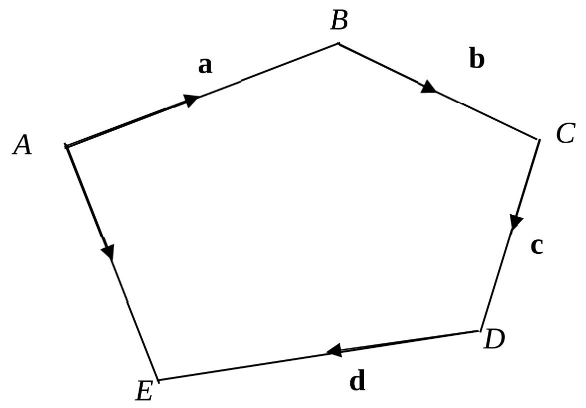
Note: $\overrightarrow{AC} = \overrightarrow{OB} = \mathbf{b}$ as $\overrightarrow{AC}$ and $\overrightarrow{OB}$ are equal vectors.
They have same magnitude and are in the same direction.

Exercise: Write down $\overrightarrow{AB}$ (the other diagonal) in terms of **a** and **b**.

$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$$
$$= -\overrightarrow{OA} + \overrightarrow{OB}$$
$$= \mathbf{b} - \mathbf{a}$$

2.3 Polygon Law of Vector Addition

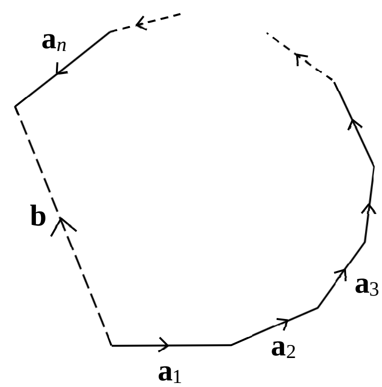
In the polygon *ABCDE*, let $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$ and $\overrightarrow{DE}$ represent the vectors **a**, **b**, **c** and **d** respectively.



The polygon law of vector addition allows us to find the vector $\overrightarrow{AE}$ by using the following:

$$\overrightarrow{AE} = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} = \mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}$$

In general, this can be extended to any polygon like the one below.



In this case:

$$\mathbf{b} = \mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 + ... + \mathbf{a}_n$$

Note:
The polygon does not have to be a plane figure (a two-dimensional figure).

2.4 Laws of Vector Algebra

(i) Commutative law of vector addition: $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$

(ii) Associative law of vector addition: $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$

(iii) Commutative law of scalar multiplication: $\lambda\mathbf{a} = \mathbf{a}\lambda$, $\lambda \in \mathbb{R}$

(iv) Associative law of scalar multiplication: $\lambda(\mu\mathbf{a}) = (\lambda\mu)\mathbf{a} = \mu(\lambda\mathbf{a})$, $\lambda, \mu \in \mathbb{R}$

(v) Distributive law: $(\lambda + \mu)\mathbf{a} = \lambda\mathbf{a} + \mu\mathbf{a}$

and $\lambda(\mathbf{a} + \mathbf{b}) = \lambda\mathbf{a} + \lambda\mathbf{b}$, $\lambda, \mu \in \mathbb{R}$

§3 Application to Geometry

3.1 Parallel Vectors

If **a** and **b** are **non-zero** vectors, then **a** and **b** are **parallel** if and only if $\mathbf{b} = \lambda \mathbf{a}$ for some $\lambda \in \mathbb{R}$, $\lambda \neq 0$.

i.e. $\mathbf{a} // \mathbf{b} \Leftrightarrow \mathbf{b} = \lambda \mathbf{a}$ for some $\lambda \in \mathbb{R}$, $\lambda \neq 0$.
or $\mathbf{a} = \mu \mathbf{b}$ for some $\mu \in \mathbb{R}$, $\mu \neq 0$.

3.2 Non-Parallel Vectors

Let **a** and **b** be **non-zero** and **non-parallel** vectors.

If $\lambda \mathbf{a} = \mu \mathbf{b}$ for some $\lambda, \mu \in \mathbb{R}$, then $\lambda = \mu = 0$.

Investigation!

Let **a** and **b** be **non-zero** and **non-parallel** vectors.

If $\alpha \mathbf{a} + \beta \mathbf{b} = \gamma \mathbf{a} + \delta \mathbf{b}$ for $\alpha, \beta, \gamma, \delta \in \mathbb{R}$, can you find relationships between α, β, γ and δ?

$$\alpha \mathbf{a} + \beta \mathbf{b} = \gamma \mathbf{a} + \delta \mathbf{b}$$
$$(\alpha - \gamma)\ \mathbf{a} = (\delta - \beta)\ \mathbf{b}$$

Since **a** and **b** are non-zero and non-parallel vectors,

$\alpha - \gamma = 0$ and $\delta - \beta = 0$.

Hence we have $\alpha = \gamma$ and $\beta = \delta$.

3.3 Collinearity of Points

3 points *A*, *B*, and *C* are collinear if and only if *A*, *B*, and *C* lie on the same straight line
This also implies that $\overrightarrow{AB}$ parallel to $\overrightarrow{AC}$ with *A* as a common point.

Points *A*, *B*, and *C* are collinear $\Leftrightarrow$ $\overrightarrow{AB} = \lambda\overrightarrow{AC}$, for some $\lambda \in \mathbb{R}, \lambda \neq 0$.

Given that points *A*, *B*, and *C* are collinear, are the following statements true?

(1) $\overrightarrow{AB} = \mu\overrightarrow{BC}$, for some $\mu \in \mathbb{R}, \mu \neq 0$. True / ~~False~~

(2) $\overrightarrow{AC} = \gamma\overrightarrow{BC}$, for some $\gamma \in \mathbb{R}, \gamma \neq 0$. True / ~~False~~

Example 3

The position vectors of points *A*, *B*, and *C* are $\mathbf{a} = 4\mathbf{i} - 9\mathbf{j} - \mathbf{k}$, $\mathbf{b} = \mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$ and $\mathbf{c} = p\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ respectively. Find the value of *p* such that *A*, *B*, and *C* are collinear.

Solution:

Since *A*, *B* and *C* are collinear, there exists some $\lambda \in \mathbb{R}, \lambda \neq 0$ such that $\overrightarrow{AC} = \lambda\overrightarrow{AB}$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} - \begin{pmatrix} 4 \\ -9 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ 12 \\ 6 \end{pmatrix}$$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix} p-4 \\ 8 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} p-4 \\ 8 \\ 4 \end{pmatrix} = \lambda \begin{pmatrix} -3 \\ 12 \\ 6 \end{pmatrix}$$

$p - 4 = -3\lambda$ ---(1)

$12\lambda = 8,$ ---(2)

$6\lambda = 4$ ---(3)

Solving (1), (2) and (3), $p = 2,\ \lambda = \dfrac{2}{3}$.

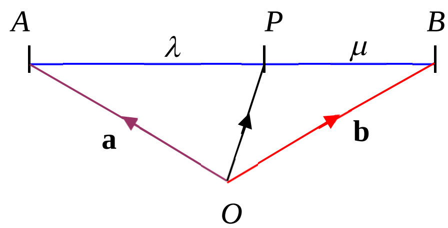
3.4 <u>Ratio Theorem</u>

Let $\overrightarrow{OA}$ and $\overrightarrow{OB}$ be two non-zero and non-parallel vectors and suppose that the point P divides AB internally in the ratio $\lambda : \mu$, i.e, $AP : PB = \lambda : \mu$.

Then:

$$\overrightarrow{OP} = \frac{\mu\overrightarrow{OA} + \lambda\,\overrightarrow{OB}}{\mu + \lambda}$$

i.e. $\overrightarrow{OP} = \dfrac{\mu\,\mathbf{a} + \lambda\,\mathbf{b}}{\mu + \lambda}$ **[in MF27]**



Proof:

Given $\dfrac{AP}{PB} = \dfrac{\lambda}{\mu}$, then $\mu AP = \lambda PB$

Since $\overrightarrow{AP} // \overrightarrow{PB}$,

$$\mu\overrightarrow{AP} = \lambda\overrightarrow{PB}$$

$$\mu\left(\overrightarrow{OP} - \overrightarrow{OA}\right) = \lambda\left(\overrightarrow{OB} - \overrightarrow{OP}\right)$$

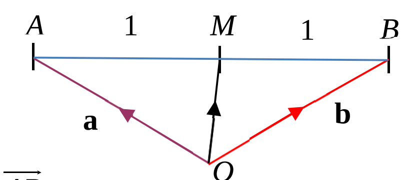
$$(\mu + \lambda)\overrightarrow{OP} = \mu\overrightarrow{OA} + \lambda\overrightarrow{OB}$$

$$\overrightarrow{OP} = \frac{\mu\overrightarrow{OA} + \lambda\overrightarrow{OB}}{(\mu + \lambda)}$$

In particular, if M is the midpoint of AB, then $\overrightarrow{OM} = \dfrac{\overrightarrow{OA} + \overrightarrow{OB}}{2}$.

Thinking point:

Why is it **incorrect** to write $\overrightarrow{OM} = \dfrac{1}{2}\overrightarrow{AB}$?

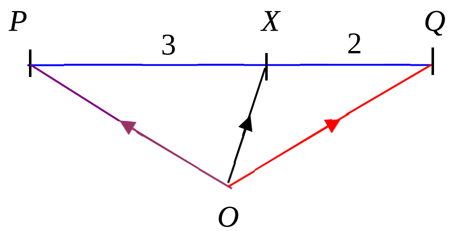


$\overrightarrow{OM}$ and $\overrightarrow{AB}$ are not parallel so we cannot write $\overrightarrow{OM} = \dfrac{1}{2}\overrightarrow{AB}$.

Example 4

The coordinates of the points P and Q are (3, 6, 0) and (2, 4, 2) respectively. Determine the coordinates of the point X which divides the line segment PQ in the ratio 3 : 2.

Solution:



Using Ratio Theorem,

$$\overrightarrow{OX}=\frac{3\overrightarrow{OQ}+2\overrightarrow{OP}}{5}=\frac{3\begin{pmatrix}2\\4\\2\end{pmatrix}+2\begin{pmatrix}3\\6\\0\end{pmatrix}}{5}=\frac{1}{5}\begin{pmatrix}12\\24\\6\end{pmatrix}$$

Coordinates of the point X are $\left(\frac{12}{5},\frac{24}{5},\frac{6}{5}\right)$.

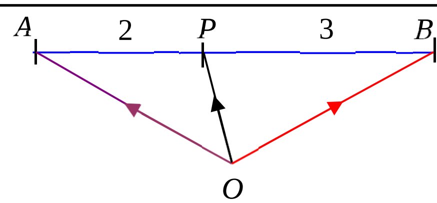
Example 5

Write down the vector $\overrightarrow{OP}$ in terms of $\overrightarrow{OA}$ and $\overrightarrow{OB}$ in the following cases:

(1) P lies on $\boldsymbol{AB}$ such that $5AP = 2AB$.

Using Ratio Theorem,

$$\overrightarrow{OP}=\frac{3\overrightarrow{OA}+2\overrightarrow{OB}}{5}$$

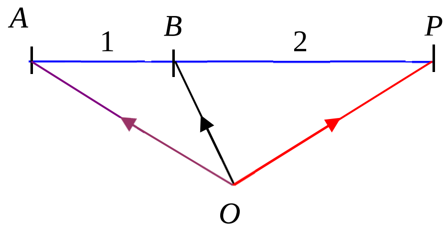


(2) P lies on $\boldsymbol{AB}$ **produced** such that $AP = 3AB$.

Using Ratio Theorem,

$$\overrightarrow{OB}=\frac{2\overrightarrow{OA}+\overrightarrow{OP}}{3}$$

$$\Rightarrow \overrightarrow{OP}=3\overrightarrow{OB}-2\overrightarrow{OA}$$

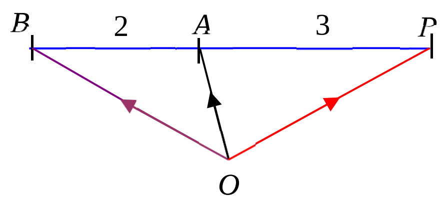


(3) P lies on $\boldsymbol{BA}$ **produced** such that $2AP = 3AB$.

Using Ratio Theorem,

$$\overrightarrow{OA}=\frac{3\overrightarrow{OB}+2\overrightarrow{OP}}{5}$$

$$\Rightarrow \overrightarrow{OP}=\frac{5\overrightarrow{OA}-3\overrightarrow{OB}}{2}$$

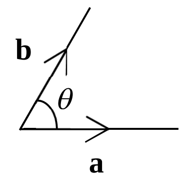


§4 Scalar Product (Dot Product) of Vectors

4.1 Definition

The **scalar product** or **dot product** of two non-zero vectors **a** and **b**, is defined to be

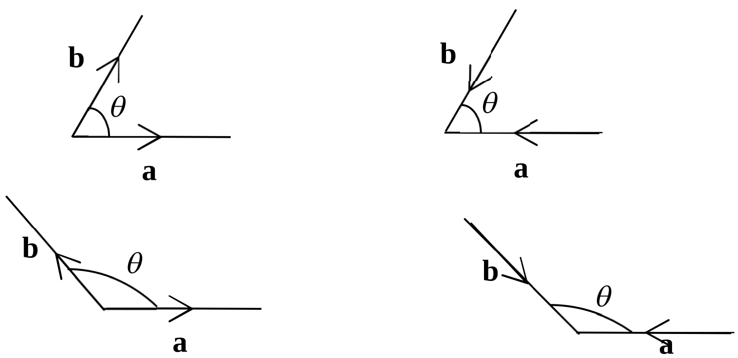
$$\mathbf{a}\cdot\mathbf{b}=|\mathbf{a}||\mathbf{b}|\cos\theta\,,$$



where θ is the angle between **a** and **b**. ($0°\le\theta\le180°$)

Note:

- The angle between **a** and **b** (θ)**,** is defined as the angle between the directions of the two vectors when they **both converge to one point or diverge from a point**.



4.2 Some Important Results of Scalar (Dot) Product

Investigation: What do you observe when θ takes on the following values?

θ	**Result**
$0°$	**a** is parallel to **b** and in the same direction $\Leftrightarrow \mathbf{a}\cdot\mathbf{b}=\|\mathbf{a}\|\|\mathbf{b}\|$ In particular: $\mathbf{a}\cdot\mathbf{a}=\|\mathbf{a}\|\|\mathbf{a}\|\cos0°=\|\mathbf{a}\|^2$
$90°$	**a** is perpendicular to **b** $\Leftrightarrow$ $\mathbf{a}\cdot\mathbf{b}=0$
$180°$	**a** is parallel to **b** but in opposite direction $\Leftrightarrow \mathbf{a}\cdot\mathbf{b}=-\|\mathbf{a}\|\|\mathbf{b}\|$ (since $\cos180°=-1$)
acute ($0°<\theta<90°$)	$\mathbf{a}\cdot\mathbf{b}>0$ (as $\cos\theta>0$)
obtuse ($90°<\theta<180°$)	$\mathbf{a}\cdot\mathbf{b}<0$ (as $\cos\theta<0$)

4.3 Basic Properties of Scalar (Dot) Product

(i) $\mathbf{a}\cdot\mathbf{b}$ is a scalar

E.g. $\begin{pmatrix}2\\1\\-3\end{pmatrix}\cdot\begin{pmatrix}3\\1\\1\end{pmatrix}=4$

(ii) $\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a}$

E.g. $\begin{pmatrix}2\\1\\-3\end{pmatrix}\cdot\begin{pmatrix}3\\1\\1\end{pmatrix}=\begin{pmatrix}3\\1\\1\end{pmatrix}\cdot\begin{pmatrix}2\\1\\-3\end{pmatrix}=4$

(iii) $(\lambda\mathbf{a})\cdot\mathbf{b}=\lambda(\mathbf{a}\cdot\mathbf{b})=\mathbf{a}\cdot(\lambda\mathbf{b})$, where $\lambda\in\mathbb{R}$

E.g. $\left(5\begin{pmatrix}2\\1\\-3\end{pmatrix}\right)\cdot\begin{pmatrix}3\\1\\1\end{pmatrix}=5\left(\begin{pmatrix}2\\1\\-3\end{pmatrix}\cdot\begin{pmatrix}3\\1\\1\end{pmatrix}\right)=\begin{pmatrix}2\\1\\-3\end{pmatrix}\cdot\left(5\begin{pmatrix}3\\1\\1\end{pmatrix}\right)=20$

(iv) $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})=\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}$ [Distributive Law]

E.g. $\begin{pmatrix}2\\1\\-3\end{pmatrix}\cdot\left(\begin{pmatrix}3\\1\\1\end{pmatrix}+\begin{pmatrix}0\\4\\3\end{pmatrix}\right)=\begin{pmatrix}2\\1\\-3\end{pmatrix}\cdot\begin{pmatrix}3\\5\\4\end{pmatrix}=\begin{pmatrix}2\\1\\-3\end{pmatrix}\cdot\begin{pmatrix}3\\1\\1\end{pmatrix}+\begin{pmatrix}2\\1\\-3\end{pmatrix}\cdot\begin{pmatrix}0\\4\\3\end{pmatrix}=4+(-5)=-1$

(v) $(\mathbf{a}+\mathbf{b})\cdot(\mathbf{c}+\mathbf{d})=(\mathbf{a}+\mathbf{b})\cdot\mathbf{c}+(\mathbf{a}+\mathbf{b})\cdot\mathbf{d}=\mathbf{a}\cdot\mathbf{c}+\mathbf{b}\cdot\mathbf{c}+\mathbf{a}\cdot\mathbf{d}+\mathbf{b}\cdot\mathbf{d}$ [Distributive Law]

E.g. $\left(\begin{pmatrix}2\\1\\-3\end{pmatrix}+\begin{pmatrix}3\\1\\1\end{pmatrix}\right)\cdot\left(\begin{pmatrix}0\\4\\3\end{pmatrix}+\begin{pmatrix}1\\2\\1\end{pmatrix}\right)=\begin{pmatrix}2\\1\\-3\end{pmatrix}\cdot\begin{pmatrix}0\\4\\3\end{pmatrix}+\begin{pmatrix}3\\1\\1\end{pmatrix}\cdot\begin{pmatrix}0\\4\\3\end{pmatrix}+\begin{pmatrix}2\\1\\-3\end{pmatrix}\cdot\begin{pmatrix}1\\2\\1\end{pmatrix}+\begin{pmatrix}3\\1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\2\\1\end{pmatrix}$

(vi) $\mathbf{a}\cdot\mathbf{0}=0$

E.g. $\begin{pmatrix}2\\1\\-3\end{pmatrix}\cdot\begin{pmatrix}0\\0\\0\end{pmatrix}=0$

It can be easily proven (Refer to **Annex *A***) that

$$\mathbf{a}\cdot\mathbf{b}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\cdot\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}=a_1b_1+a_2b_2+a_3b_3$$

Using the operation above, we obtain

$$\mathbf{a}\cdot\mathbf{a}=|\mathbf{a}|^2$$

Proof: $\mathbf{a}\cdot\mathbf{a}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\cdot\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}=a_1^2+a_2^2+a_3^2=|\mathbf{a}|^2$

Example 6

Given $\mathbf{a} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -1 \\ 8 \\ 1 \end{pmatrix}$, find $\mathbf{a} \cdot \mathbf{b}$.

Solution:

$$\mathbf{a} \cdot \mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 8 \\ 1 \end{pmatrix} = 2(-1) + (-1)(8) + 3(1) = -7$$

Note:

- $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$ **does not** imply that $\mathbf{b} = \mathbf{c}$

  E.g. $\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = 4 = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ but $\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \neq \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$.

No division of vectors is allowed, that is,

$\mathbf{a} \cdot \mathbf{b} = \mathbf{c} \cdot \mathbf{d}$ **does not** imply that $\mathbf{a} = \dfrac{\mathbf{c} \cdot \mathbf{d}}{\mathbf{b}}$

4.4 Uses of Scalar (Dot) Product

4.4.1 Perpendicular vectors

For two **non-zero** vectors **a** and **b**,

$$\mathbf{a} \text{ is perpendicular to } \mathbf{b} \Leftrightarrow \mathbf{a}\cdot\mathbf{b}=0$$

Example 7

Let $\overrightarrow{OA}$ be $2\mathbf{i}+(\lambda+2)\mathbf{j}+4\mathbf{k}$ and $\overrightarrow{OB}$ be $6\mathbf{i}+2\lambda\mathbf{j}+\lambda\mathbf{k}$. If $\overrightarrow{OA}$ and $\overrightarrow{AB}$ are perpendicular vectors, find the value(s) of λ.

Solution:

$$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\begin{pmatrix}6\\2\lambda\\\lambda\end{pmatrix}-\begin{pmatrix}2\\\lambda+2\\4\end{pmatrix}=\begin{pmatrix}4\\\lambda-2\\\lambda-4\end{pmatrix}$$

Since $\overrightarrow{OA}$ and $\overrightarrow{AB}$ are perpendicular vectors,

$$\overrightarrow{OA}\cdot\overrightarrow{AB}=0$$

$$\begin{pmatrix}2\\\lambda+2\\4\end{pmatrix}\cdot\begin{pmatrix}4\\\lambda-2\\\lambda-4\end{pmatrix}=0$$

$$8+\lambda^2-4+4\lambda-16=0$$

$$\lambda^2+4\lambda-12=0$$

$$(\lambda+6)(\lambda-2)=0$$

$$\therefore \lambda=-6 \text{ or } \lambda=2.$$

4.4.2 <u>Angle between Two Vectors</u>

Let θ be the angle between the two non-zero vectors **a** and **b**.

Since $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$, then

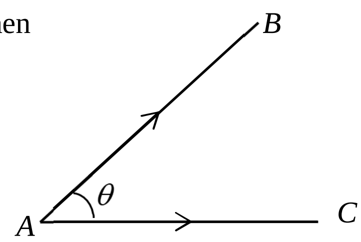
$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

Note: $\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{\mathbf{a}}{|\mathbf{a}|}\cdot\frac{\mathbf{b}}{|\mathbf{b}|} = \hat{\mathbf{a}}\cdot\hat{\mathbf{b}}$.

If θ is the angle between the two vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$, then

$$\cos\theta = \frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|}$$

i.e. $$\cos\angle BAC = \frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|}$$



Example 8

Points *A*, *B*, *C* have position vectors $3\mathbf{i}+4\mathbf{j}+2\mathbf{k}$, $3\mathbf{j}+\mathbf{k}$, $8\mathbf{i}-2\mathbf{j}+6\mathbf{k}$ respectively. Find angle *BAC*.

Solution:

For $\angle BAC$, use vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

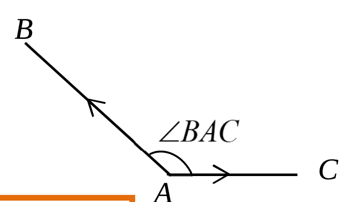
$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix}-3\\-1\\-1\end{pmatrix}$ and $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix}5\\-6\\4\end{pmatrix}$

$$\cos\angle BAC = \frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|}$$

$$= \frac{\begin{pmatrix}-3\\-1\\-1\end{pmatrix}\cdot\begin{pmatrix}5\\-6\\4\end{pmatrix}}{\sqrt{11}\sqrt{77}}$$

$$= \frac{-13}{\sqrt{11}\sqrt{77}}$$

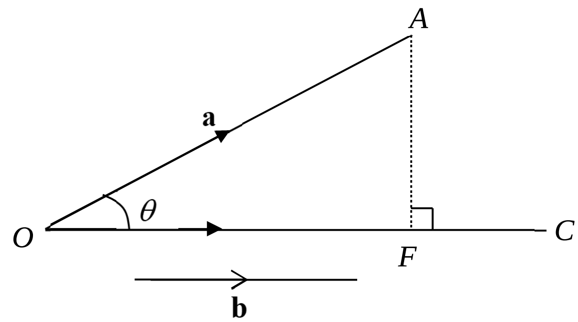
$\angle BAC = 116.5^\circ$ (1 d.p)



Since $\cos\angle BAC$ is negative, $\angle BAC$ must be obtuse.

4.4.3 Length of Projection and Projection Vector

Let θ be the angle between $\overrightarrow{OA}$ and $\overrightarrow{OC}$ (see diagram below).
Let F be the foot of the perpendicular from the point A to OC.
Let $\mathbf{b}$ be a vector parallel to $\overrightarrow{OC}$.



Since $\quad \mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|},$$

using ΔOAF, $\quad \cos\theta = \dfrac{OF}{OA} = \dfrac{OF}{|\mathbf{a}|}$

$$\therefore \frac{OF}{|\mathbf{a}|} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

$$\Rightarrow OF = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}$$

Note: Value of $\mathbf{a}\cdot\mathbf{b}$ is *negative* when θ is obtuse.
However, since length OF must be positive,

$$OF = \frac{|\mathbf{a}\cdot\mathbf{b}|}{|\mathbf{b}|} \text{ or } \left|\mathbf{a}\cdot\hat{\mathbf{b}}\right|$$

Geometrical interpretation of OF: OF is the **length of projection** of **a** onto **b**.

Another result:

$$\overrightarrow{OF} = \left(\mathbf{a}\cdot\hat{\mathbf{b}}\right)\hat{\mathbf{b}} \quad \text{or equivalently} \quad \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\mathbf{b}$$

Geometrical interpretation of $\overrightarrow{OF}$: $\overrightarrow{OF}$ is the **projection vector** of **a** onto **b**.

Using the above result, to find $\overrightarrow{FA}$:

$$\overrightarrow{FA} = \overrightarrow{OA} - \overrightarrow{OF} = \mathbf{a} - \left(\mathbf{a}\cdot\hat{\mathbf{b}}\right)\hat{\mathbf{b}}$$

Example 9

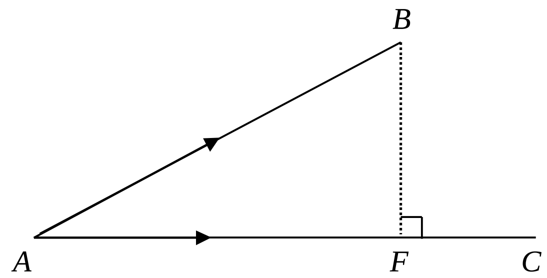
The points *A*, *B* and *C* have coordinates $(1,2,-1)$, $(3,0,1)$ and $(0,2,-5)$. Find

(i) the length of the projection of $\overrightarrow{AB}$ onto $\overrightarrow{AC}$;

(ii) the projection vector of $\overrightarrow{AB}$ onto $\overrightarrow{AC}$.

Solution:

Let *F* be the foot of the perpendicular from *B* to *AC*.



$$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\begin{pmatrix}2\\-2\\2\end{pmatrix} \quad \text{and} \quad \overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=\begin{pmatrix}-1\\0\\-4\end{pmatrix}$$

(i) Length of projection of $\overrightarrow{AB}$ onto $\overrightarrow{AC}$

$$= AF$$

$$=\frac{\left|\overrightarrow{AB}\cdot\overrightarrow{AC}\right|}{\left|\overrightarrow{AC}\right|}$$

$$=\frac{\left|\begin{pmatrix}2\\-2\\2\end{pmatrix}.\begin{pmatrix}-1\\0\\-4\end{pmatrix}\right|}{\sqrt{17}}$$

$$=\frac{|-10|}{\sqrt{17}}=\frac{10\sqrt{17}}{17}\text{ units}$$

(ii) Projection vector of $\overrightarrow{AB}$ onto $\overrightarrow{AC}$

$$=\overrightarrow{AF}$$

$$=\left(-\frac{10}{\sqrt{17}}\right)\frac{\overrightarrow{AC}}{\left|\overrightarrow{AC}\right|}$$

$$=\frac{-10}{\left(\sqrt{17}\right)^2}\begin{pmatrix}-1\\0\\-4\end{pmatrix}=\frac{10}{17}\begin{pmatrix}1\\0\\4\end{pmatrix}$$

§5 Vector Product (Cross Product) of Vectors

5.1 Definition

The **vector product** (or **cross product)** of two non-zero vectors **a** and **b** is defined to be

$$\mathbf{a} \times \mathbf{b} = (|\mathbf{a}||\mathbf{b}|\sin\theta)\,\hat{\mathbf{n}},$$

where

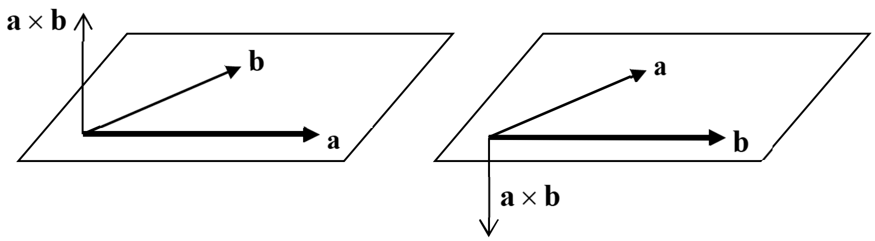
(i) θ is the angle between **a** and **b** $(0° \le \theta \le 180°)$,

(ii) $\hat{\mathbf{n}}$ is a **unit vector perpendicular** to both **a** and **b** taken in the direction of a right-handed screw turned from **a** to **b**. Hence, $\mathbf{a} \times \mathbf{b}$ is a vector perpendicular to both **a** and **b** taken in the direction of a right-handed screw turned from **a** to **b**.

Note: (1) $|\hat{\mathbf{n}}| = 1$

(2) If $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$, then $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.

For each of the following pairs of vectors **a** and **b**, indicate the direction of $\mathbf{a} \times \mathbf{b}$.



$$\mathbf{a} \times \mathbf{b} \neq \mathbf{b} \times \mathbf{a}$$

[unlike scalar (dot) product where $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$]

5.2 Some Results of Vector (Cross) Product

Investigation: What do you observe when θ takes on the following values?

θ	**Result**
0°	**a** is parallel to **b** and in same direction $\Rightarrow \mathbf{a} \times \mathbf{b} = \mathbf{0}$ *Think about:* What is $\mathbf{a} \times \mathbf{a}$? In particular: $\mathbf{i} \times \mathbf{i} = \mathbf{0}$, $\mathbf{j} \times \mathbf{j} = \mathbf{0}$, $\mathbf{k} \times \mathbf{k} = \mathbf{0}$
180°	**a** is parallel to **b** and in opposite direction $\Rightarrow \mathbf{a} \times \mathbf{b} = \mathbf{0}$ (since $\sin 180^\circ = 0$)

Remarks:

(1) From **3.1**: Two non-zero vectors **a** and **b** are **parallel** $\Leftrightarrow$ $\mathbf{b} = \lambda\mathbf{a}$ or $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ for some $\lambda \in \mathbb{R}$, $\lambda \neq 0$.

(2) From **4.4.1**: Two non-zero vectors **a** and **b** are **perpendicular** $\Leftrightarrow$ $\mathbf{a} \cdot \mathbf{b} = 0$.

It can be easily proven (Refer to **Annex *B***) that

$$\mathbf{a}\times\mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ -(a_1b_3 - a_3b_1) \\ a_1b_2 - a_2b_1 \end{pmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix} \quad \textbf{[in MF27]}$$

where $\mathbf{a}\times\mathbf{b}$ is a vector perpendicular to both **a** and **b** taken in the direction of a right-handed screw turned from **a** to **b**.

Example 10

Given $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = \mathbf{j} + 3\mathbf{k}$ **,** find $\mathbf{a}\times\mathbf{b}$.

Solution:

$$\mathbf{a}\times\mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} (1)(3)-(1)(-2) \\ -((2)(3)-(0)(-2)) \\ (2)(1)-(0)(1) \end{pmatrix} = \begin{pmatrix} 3-(-2) \\ -(6-0) \\ 2-0 \end{pmatrix} = \begin{pmatrix} 5 \\ -6 \\ 2 \end{pmatrix}$$

Checking the answer for $\mathbf{a}\times\mathbf{b}$:
Recall $\mathbf{a}\times\mathbf{b}$ is perpendicular to both **a** and **b**. Hence, $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a} = 0$ and $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b} = 0$.

$$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a} = \begin{pmatrix} 5 \\ -6 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = 0 \ \checkmark \qquad (\mathbf{a}\times\mathbf{b})\cdot\mathbf{b} = \begin{pmatrix} 5 \\ -6 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} = 0 \ \checkmark$$

<u>Extension</u>

Given $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = \mathbf{j} + 3\mathbf{k}$ **,** find $\mathbf{b}\times\mathbf{a}$. [Answer: $\begin{pmatrix} -5 \\ 6 \\ -2 \end{pmatrix}$]

Compare your answer in Example 10, what can be said about $\mathbf{a}\times\mathbf{b}$ and $\mathbf{b}\times\mathbf{a}$?

5.3 Basic Properties of the Vector (Cross) Product

(i) $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$

E.g. $\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} \times \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -11 \\ -1 \end{pmatrix}$; $\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} = \begin{pmatrix} -3 \\ 11 \\ 1 \end{pmatrix} = -\begin{pmatrix} 3 \\ -11 \\ -1 \end{pmatrix}$

(ii) $(\lambda\mathbf{a}) \times (\mu\mathbf{b}) = (\mu\mathbf{a}) \times (\lambda\mathbf{b}) = (\lambda\mu)(\mathbf{a} \times \mathbf{b}), \quad \lambda, \mu \in \mathbb{R}$

E.g. $5\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} \times 7\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = 7\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} \times 5\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = (35)\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} \times \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$

(iii) $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$ [Distributive Law]

E.g. $\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} \times \left(\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix} \right) = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} \times \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} \times \begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix}$

(iv) $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$ [Distributive Law]

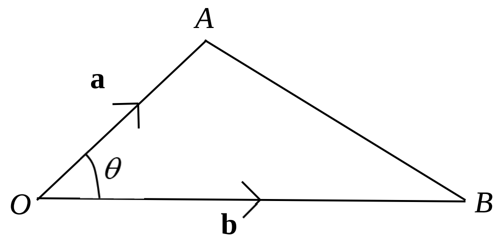
E.g. $\left(\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \right) \times \begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} \times \begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix}$

5.4 Uses of Vector (Cross) Product

5.4.1 Finding Area of Triangle

From $\mathbf{a}\times\mathbf{b} = \left(|\mathbf{a}||\mathbf{b}|\sin\theta\right)\hat{\mathbf{n}}$,

we observe that $|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$ since $0^\circ \le \theta \le 180^\circ$ and $|\hat{\mathbf{n}}| = 1$.



Recall at 'O' Level:

Area of $\Delta OAB = \frac{1}{2}|\mathbf{a}||\mathbf{b}|\sin\theta$ (since $0 < \theta < 180^\circ$ in a triangle)

$= \frac{1}{2}|\mathbf{a}\times\mathbf{b}|$

In general,

$$\text{Area of } \Delta ABC = \frac{1}{2}\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|\sin\theta = \frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|$$

Example 11

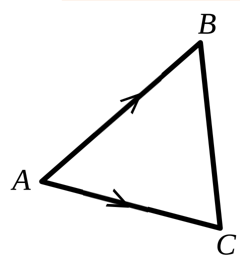
A triangle has vertices at the points $A(1,2,1)$, $B(1,0,3)$ and $C(-1,2,-1)$. Find the area of the triangle *ABC*.

Solution:

Area of $\Delta ABC = \frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|$

Use any 2 sides of triangle.

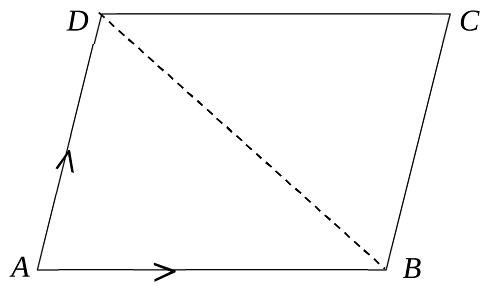
$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix}0\\-2\\2\end{pmatrix}$ $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix}-2\\0\\-2\end{pmatrix}$



$\overrightarrow{AB}\times\overrightarrow{AC} = \begin{pmatrix}0\\-2\\2\end{pmatrix}\times\begin{pmatrix}-2\\0\\-2\end{pmatrix} = \begin{pmatrix}0\\2\\-2\end{pmatrix}\times\begin{pmatrix}2\\0\\2\end{pmatrix} = \begin{pmatrix}4\\-4\\-4\end{pmatrix}$

$\therefore$ Area of $\Delta ABC = \frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right| = \frac{1}{2}\left|\begin{pmatrix}4\\-4\\-4\end{pmatrix}\right| = \frac{4}{2}\left|\begin{pmatrix}1\\-1\\-1\end{pmatrix}\right| = 2\sqrt{1+1+1} = 2\sqrt{3}$ units^2

5.4.2 <u>Area of Parallelogram</u>



Area of parallelogram $ABCD$

$= 2(\text{Area of triangle } ABD)$

$= 2\left[\frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AD}\right|\right]$

$= \left|\overrightarrow{AB} \times \overrightarrow{AD}\right|$

In general,

$$\text{Area of parallelogram } ABCD = \left|\overrightarrow{AB} \times \overrightarrow{AD}\right|$$

Use any 2 adjacent sides

Discussion: Are there any other ways to obtain the area of the parallelogram above?
Yes! $\left|\overrightarrow{DB} \times \overrightarrow{DA}\right|$ or $\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$, etc.

Example 12

The points A, B and C have coordinates $(-3,\ 0,\ 1)$, $(-1,\ 1, -1)$ and $(2,\ 1, -2)$. Find the exact area of parallelogram $OABC$ with reference to the origin O.

Solution:

Sketch the parallelogram to help identify the adjacent sides

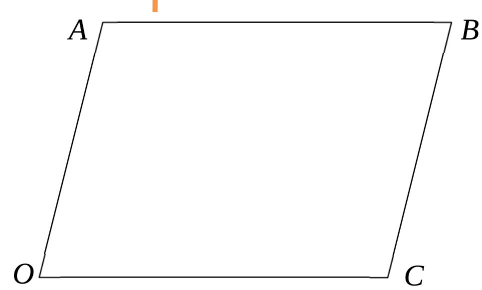
Area of parallelogram $OABC$

$= \left|\overrightarrow{OA} \times \overrightarrow{OC}\right|$

$= \left|\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}\right| = \left|\begin{pmatrix} -1 \\ -4 \\ -3 \end{pmatrix}\right|$

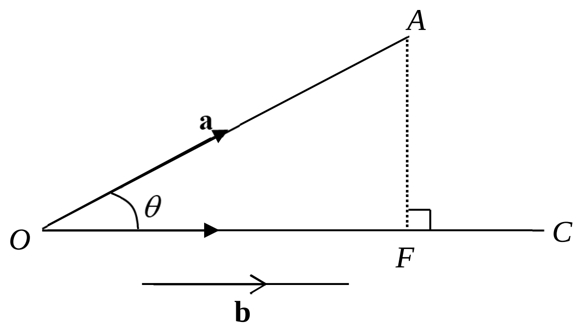
$= \sqrt{(-1)^2 + (-4)^2 + (-3)^2}$

$= \sqrt{26} \text{ units}^2$



5.4.3 Perpendicular Distance

Let θ be the angle between $\overrightarrow{OA}$ and $\overrightarrow{OC}$ (see diagram below).
Let F be the foot of the perpendicular from the point A to OC.
Let $\mathbf{b}$ be a vector parallel to $\overrightarrow{OC}$.



Since $\mathbf{a}\times\mathbf{b} = \left(|\mathbf{a}||\mathbf{b}|\sin\theta\right)\hat{\mathbf{n}}$,

$|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$ since $0^\circ \le \theta \le 180^\circ$.

$$\therefore \sin\theta = \frac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}||\mathbf{b}|}$$

Using ΔOAF, $\sin\theta = \dfrac{FA}{OA} = \dfrac{FA}{|\mathbf{a}|}$,

$$\therefore \frac{FA}{|\mathbf{a}|} = \frac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}||\mathbf{b}|} \Rightarrow FA = \frac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{b}|}$$

Thus we have the following result:

$$FA = \frac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{b}|} \text{ or } \left|\,\mathbf{a}\times\hat{\mathbf{b}}\,\right|$$

Geometrical interpretation of FA: **Perpendicular/shortest distance** from point A to OC.

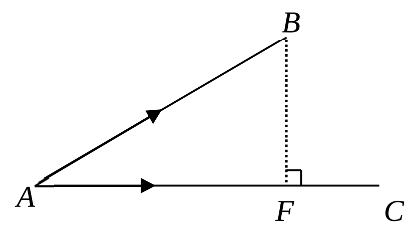
Example 13

The points A, B and C have coordinates $(1,2,-1)$, $(3,0,1)$ and $(0,2,-5)$. Find the shortest distance from B to AC.

Solution:

Let F be the foot of the perpendicular from B to AC.

$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\begin{pmatrix}2\\-2\\2\end{pmatrix}$ and $\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=\begin{pmatrix}-1\\0\\-4\end{pmatrix}$



Shortest distance from B to AC

$= BF$

$$=\frac{\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|}{\left|\overrightarrow{AC}\right|}$$

$$=\frac{\left|\begin{pmatrix}2\\-2\\2\end{pmatrix}\times\begin{pmatrix}-1\\0\\-4\end{pmatrix}\right|}{\sqrt{17}}$$

$$=\frac{\left|\begin{pmatrix}8\\6\\-2\end{pmatrix}\right|}{\sqrt{17}}=\frac{\sqrt{104}}{\sqrt{17}}=\frac{2\sqrt{442}}{17}\text{ units}$$

Scalar & Vector Products Summary

Scalar (Dot) Product

Definition: $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta \quad \left(0^\circ \le \theta \le 180^\circ\right)$

Proving two non-zero vectors are **perpendicular**

$$\mathbf{a} \perp \mathbf{b} \quad \Leftrightarrow \quad \mathbf{a}\cdot\mathbf{b} = 0$$

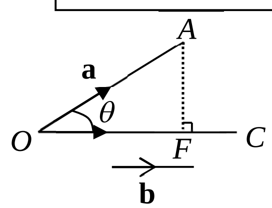
Angle between Two Vectors

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

In particular, **direction cosines** are the angles that a vector makes with the positive *x*-, *y*-, *z*- axis resp.

Length of projection of **a** onto **b:**

$$OF = \left|\mathbf{a}\cdot\hat{\mathbf{b}}\right|$$



Projection vector of **a** onto **b:**

$$\overrightarrow{OF} = \left(\mathbf{a}\cdot\hat{\mathbf{b}}\right)\hat{\mathbf{b}}$$

$$\therefore \overrightarrow{FA} = \overrightarrow{OA} - \overrightarrow{OF} = \mathbf{a} - \left(\mathbf{a}\cdot\hat{\mathbf{b}}\right)\hat{\mathbf{b}}$$

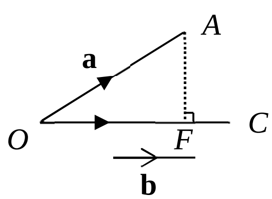
Vector (Cross) Product

Definition: $\mathbf{a}\times\mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\ \hat{\mathbf{n}}\ (0^\circ \le \theta \le 180^\circ)$

So $|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$

Normal Vector

$\mathbf{a}\times\mathbf{b}$ is a vector perpendicular to both **a** and **b**



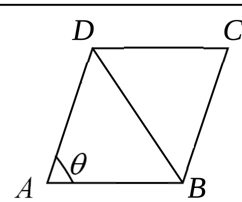
Perpendicular distance FA

$$FA = \left|\mathbf{a}\times\hat{\mathbf{b}}\right|$$

Area of ΔABD

$$= \frac{1}{2}\left|\overrightarrow{AB}\right|\left|\overrightarrow{AD}\right|\sin\theta$$

$$= \frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AD}\right|$$

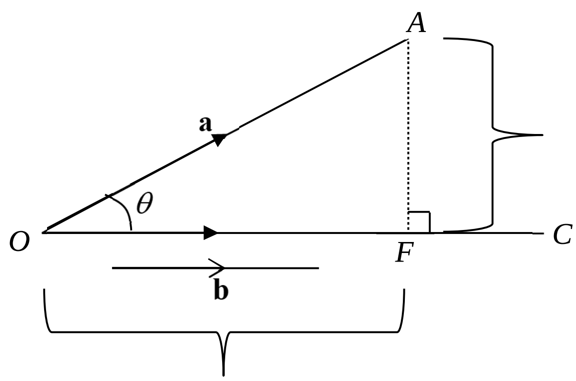


Area of parallelogram *ABCD*
= 2 (Area of ΔABD)

$$= \left|\overrightarrow{AB}\times\overrightarrow{AD}\right|$$

§6 Summary and geometrical interpretation of Scalar and Vector Products

Let θ be the angle between $\overrightarrow{OA}$ and $\overrightarrow{OC}$ (see diagram below).
Let F be the foot of the perpendicular from the point A to OC.
Let $\mathbf{b}$ be a vector parallel to $\overrightarrow{OC}$.



$$FA = \frac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{b}|} \text{ or } |\mathbf{a}\times\hat{\mathbf{b}}|$$

FA is the **perpendicular/shortest distance** from point A to OC.

$$\overrightarrow{FA} = \overrightarrow{OA} - \overrightarrow{OF} = \mathbf{a} - (\mathbf{a}\cdot\hat{\mathbf{b}})\hat{\mathbf{b}}$$

$$OF = \frac{|\mathbf{a}\cdot\mathbf{b}|}{|\mathbf{b}|} \text{ or } |\mathbf{a}\cdot\hat{\mathbf{b}}|$$

OF is the **length of projection** of **a** onto **b**.

$$\overrightarrow{OF} = (\mathbf{a}\cdot\hat{\mathbf{b}})\hat{\mathbf{b}} \quad \text{or equivalently} \quad \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\mathbf{b}$$

$\overrightarrow{OF}$ is the **projection vector** of **a** onto **b**.

Annex *A* - Evaluating the Scalar Product of 2 Vectors

Recall that, since **i**, **j**, **k** are mutually perpendicular vectors,

$$\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1, \qquad \text{since } \mathbf{i}\cdot\mathbf{i} = |\mathbf{i}|^2 = 1^2 = 1, \text{ etc.}$$

$$\mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = 0.$$

Let $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ be two vectors. Then,

$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= \left(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}\right)\cdot\left(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}\right)\\ &= a_1b_1(\mathbf{i}\cdot\mathbf{i}) + a_1b_2(\mathbf{i}\cdot\mathbf{j}) + a_1b_3(\mathbf{i}\cdot\mathbf{k}) + a_2b_1(\mathbf{j}\cdot\mathbf{i}) + a_2b_2(\mathbf{j}\cdot\mathbf{j})\\ &\quad + a_2b_3(\mathbf{j}\cdot\mathbf{k}) + a_3b_1(\mathbf{k}\cdot\mathbf{i}) + a_3b_2(\mathbf{k}\cdot\mathbf{j}) + a_3b_3(\mathbf{k}\cdot\mathbf{k})\\ &= a_1b_1 + a_2b_2 + a_3b_3\end{aligned}$$

In column vector form, we have

$$\mathbf{a}\cdot\mathbf{b} = \begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\cdot\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix} = a_1b_1 + a_2b_2 + a_3b_3$$

Annex *B* - Evaluating the Vector Product of 2 Vectors

Let $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ be two vectors.

$$\begin{aligned}\mathbf{a}\times\mathbf{b} &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})\times(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})\\ &= a_1b_1\,(\mathbf{i}\times\mathbf{i}) + a_1b_2\,(\mathbf{i}\times\mathbf{j}) + a_1b_3\,(\mathbf{i}\times\mathbf{k})\\ &\quad + a_2b_1\,(\mathbf{j}\times\mathbf{i}) + a_2b_2\,(\mathbf{j}\times\mathbf{j}) + a_2b_3\,(\mathbf{j}\times\mathbf{k})\\ &\quad + a_3b_1\,(\mathbf{k}\times\mathbf{i}) + a_3b_2\,(\mathbf{k}\times\mathbf{j}) + a_3b_3\,(\mathbf{k}\times\mathbf{k})\\ &= (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}\end{aligned}$$

Thus

$$\mathbf{a}\times\mathbf{b} = \begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\times\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix} = \begin{pmatrix}a_2b_3 - a_3b_2\\ -\left(a_1b_3 - a_3b_1\right)\\ a_1b_2 - a_2b_1\end{pmatrix}$$

H2 Mathematics (9758)
Chapter 5 Vectors
Exploration Activity

Activity 1

Here is a challenge for you to try!
Step 1. Print out the triangle found on the page 3 and cut it out using a pair of scissors.
Step 2. Without drawing anything on the triangle, attempt to balance the triangle on one finger. Keep trying until you succeed.
Step 3. Draw a small cross on the triangle where you successfully balanced the triangle.

§1 Definitions

A ***median*** (not to be confused with meridian) of a triangle is a line segment joining a vertex (corner of a triangle) to the midpoint of the opposing side.
This means that a triangle has exactly three medians, one from each vertex.

Activity 2

Let's see what happens when we draw the medians onto the triangle.
Step 1. On the same side of the triangle where you drew the cross, draw the three medians.
Step 2. Attempt to balance the triangle again, this time using the intersection of the three medians.
How close was your cross to the intersection? ☺

§2 Centroid of a triangle

A ***centroid*** of a triangle is the centre of mass of the 2-dimension triangle. Each median of the triangle will pass through the centroid, which explains why all three intersect at the same point.

§3 Did you know?

Using the triangle after activity 2, measure the distance from any vertex to the centroid and note it down. Now, measure the distance from same vertex to the midpoint of the vertex's opposing side. What do you realise?

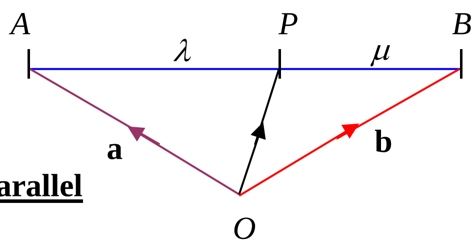
You would have noticed that the distance from the vertex to the centroid is two-thirds of the distance from the vertex to the midpoint of the opposing side.

But can we prove this statement?

§4 Vector concepts needed to prove statement

1. Ratio Theorem

$$\overrightarrow{OP} = \frac{\mu\overrightarrow{OA} + \lambda\overrightarrow{OB}}{\mu + \lambda}$$



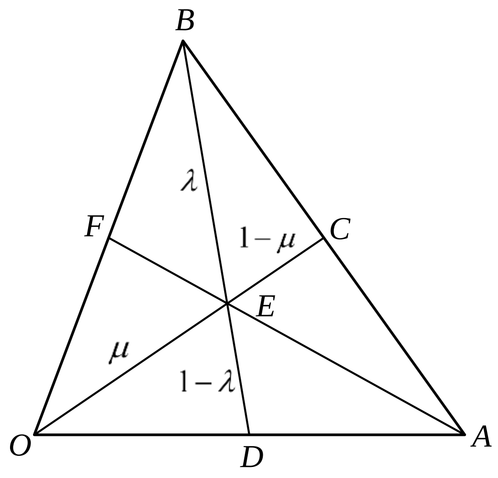
2. Equal vectors that are **non-zero** and **non-parallel**

$\alpha\mathbf{a} + \beta\mathbf{b} = \gamma\mathbf{a} + \delta\mathbf{b} \Rightarrow \alpha = \gamma$ and $\beta = \delta$

§5 The Proof

Here is how we prove it:

First, define a triangle as shown in the diagram.



Objective: Prove that $\overrightarrow{OE}=\frac{2}{3}\overrightarrow{OC}$,

$\overrightarrow{AE}=\frac{2}{3}\overrightarrow{AF}$ and $\overrightarrow{BE}=\frac{2}{3}\overrightarrow{BD}$.

Let $\overrightarrow{OA}=\mathbf{a}$ and $\overrightarrow{OB}=\mathbf{b}$,

$\frac{OE}{EC}=\frac{\mu}{1-\mu}$ and $\frac{BE}{ED}=\frac{\lambda}{1-\lambda}$.

Using Ratio Theorem,

First considering ΔOBD,

$$\overrightarrow{OE}=\frac{\lambda\overrightarrow{OD}+(1-\lambda)\overrightarrow{OB}}{\lambda+(1-\lambda)}$$
$$=\lambda\left(\frac{1}{2}\mathbf{a}\right)+(1-\lambda)\mathbf{b}$$
$$=\frac{1}{2}\lambda\mathbf{a}+(1-\lambda)\mathbf{b}$$

And considering ΔOAC,

$$\overrightarrow{AE}=\frac{\mu\overrightarrow{AC}+(1-\mu)\overrightarrow{AO}}{\mu+(1-\mu)}$$
$$=\mu\left(\frac{1}{2}\overrightarrow{AB}\right)+(1-\mu)(-\mathbf{a})$$
$$=\frac{1}{2}\mu(\mathbf{b}-\mathbf{a})+(\mu-1)\mathbf{a}$$
$$=\left(\frac{1}{2}\mu-1\right)\mathbf{a}+\frac{1}{2}\mu\mathbf{b}$$

$$\therefore \overrightarrow{OE}=\overrightarrow{OA}+\overrightarrow{AE}=\mathbf{a}+\left(\frac{1}{2}\mu-1\right)\mathbf{a}+\frac{1}{2}\mu\mathbf{b}$$
$$=\frac{1}{2}\mu\mathbf{a}+\frac{1}{2}\mu\mathbf{b}$$

(The shorter alternative is to consider a parallelogram with adjacent sides *OA* and *OB*)

Therefore,

$$\overrightarrow{OE}=\frac{1}{2}\lambda\mathbf{a}+(1-\lambda)\mathbf{b}=\frac{1}{2}\mu\mathbf{a}+\frac{1}{2}\mu\mathbf{b}.$$

Since **a** and **b** are non-zero and non-parallel, this implies that

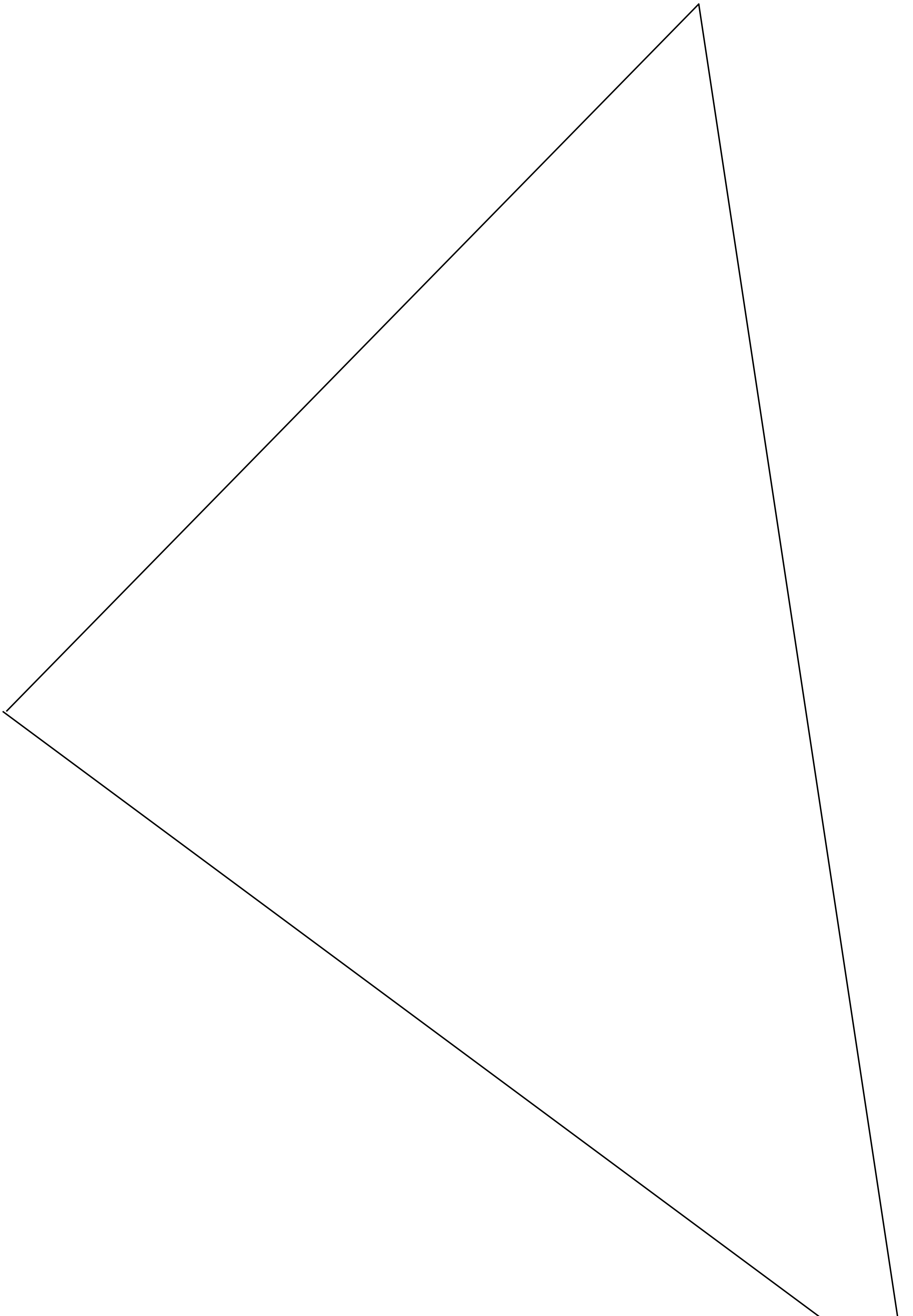
$$\begin{cases}\frac{1}{2}\lambda=\frac{1}{2}\mu\\ 1-\lambda=\frac{1}{2}\mu\end{cases}$$

Solving simultaneously, we get

$$\lambda=\mu=\frac{2}{3}$$

Applying the same idea for ΔOAF,

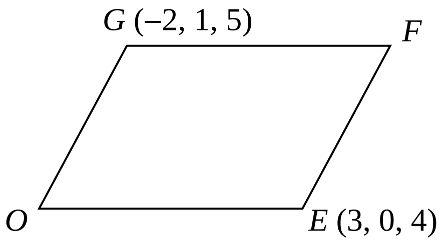
Hence, $\overrightarrow{OE}=\frac{2}{3}\overrightarrow{OC}$, $\overrightarrow{AE}=\frac{2}{3}\overrightarrow{AF}$ and $\overrightarrow{BE}=\frac{2}{3}\overrightarrow{BD}$. (shown)



H2 Mathematics (9758)
Chapter 5 Vectors
Discussion Questions

Level 1

1 **(a)** *OEFG* is a parallelogram as shown in the diagram.



(i) Find $\overrightarrow{OF}$ and $\overrightarrow{EG}$.

(ii) Find the size of the angle *GEO*.

(b) The points *P*, *Q* and *R* have coordinates $(4, 1, 1)$, $(-8, 5, -15)$ and $(7, 0, 5)$ respectively. Show that *P*, *Q*, and *R* are collinear.

(c) Given $\mathbf{a} = \mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{b} = \mathbf{j} - 3\mathbf{k}$, find a vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

2 **2010(9740)/I/1**

The position vectors **a** and **b** are given by

$$\mathbf{a} = 2p\mathbf{i} + 3p\mathbf{j} + 6p\mathbf{k} \text{ and } \mathbf{b} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k},$$

where $p > 0$. It is given that $|\mathbf{a}| = |\mathbf{b}|$.

(i) Find the exact value of *p*. [2]

(ii) Show that $(\mathbf{a}+\mathbf{b}) \cdot (\mathbf{a}-\mathbf{b}) = 0$. [3]

3 **2009 MJC Promo**

The points *A*, *B* and *C* have position vectors $\begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}$, $\begin{pmatrix} 3 \\ y \\ 7 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ 10 \\ 11 \end{pmatrix}$ with respect to the origin *O* respectively.

(i) Find the value of *y* such that *A*, *B* and *C* are collinear. [2]

(ii) Find the exact area of triangle *OAC*. [3]

(iii) The point *D* divides *OC* internally such that *OD* : *OC* = 2 : 5. Find the vector $\overrightarrow{AD}$. [3]

Level 2

4 The position vectors of the points *A*, *B* and *C* are $\overrightarrow{OA} = -8\mathbf{j} - \mathbf{k}$, $\overrightarrow{OB} = \mathbf{i} + 5\mathbf{k}$ and $\overrightarrow{OC} = p\mathbf{i} + (2p - 11)\mathbf{j} - 4\mathbf{k}$ respectively. Find the

(i) unit vector(s) parallel to the vector $\overrightarrow{AB}$,

(ii) position vector of the midpoint of *AB*,

(iii) value of p such that *A*, *B* and *C* are collinear,

(iv) position vector of *D* such that *ABCD* is a parallelogram and $p = 6$.

(v) position vector of *E* such that point E is on *AB* produced and $AB : AE = 2 : 5$.

5 **2009(9740)/II/2**

Relative to the origin *O*, two points *A* and *B* have position vectors given by $\mathbf{a} = 14\mathbf{i} + 14\mathbf{j} + 14\mathbf{k}$ and $\mathbf{b} = 11\mathbf{i} - 13\mathbf{j} + 2\mathbf{k}$ respectively.

(i) The point *P* divides the line *AB* in the ratio $2:1$. Find the coordinates of *P*. [2]

(ii) Show that *AB* and *OP* are perpendicular. [2]

(iii) The vector $\mathbf{c}$ is a unit vector in the direction of $\overrightarrow{OP}$. Write $\mathbf{c}$ as a column vector, and give the geometrical meaning of $|\mathbf{a} \cdot \mathbf{c}|$. [2]

(iv) Find $\mathbf{a} \times \mathbf{p}$, where $\mathbf{p}$ is the vector $\overrightarrow{OP}$, and give the geometrical meaning of $|\mathbf{a} \times \mathbf{p}|$. Hence write down the area of triangle *OAP*. [4]

6 **(a)** Find the unit vector in the direction of $-\mathbf{i} - 3\mathbf{j}$.

(b) The vector $\mathbf{v}$ has a magnitude of 5 and is parallel to $\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$.

Find the possible vector(s) $\mathbf{v}$.

(c) Find the vector $\mathbf{r}$ given that $\mathbf{r} + \mathbf{j} - 2\mathbf{k}$ is parallel to the *x*-axis and $\mathbf{r} - 2\mathbf{i}$ is parallel to $-2\mathbf{j} + 4\mathbf{k}$.

7 Relative to the origin *O*, two points *A* and *B* have position vectors given by $\mathbf{a} = 3\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = 5\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}$ respectively.

(i) Find the length of the projection of $\overrightarrow{OA}$ on $\overrightarrow{OB}$.

(ii) Hence, or otherwise, find the position vector of the point *C* on *OB* such that *AC* is perpendicular to *OB*.

8 $\mathbf{a}$ and $\mathbf{b}$ are vectors such that $|\mathbf{a}| = \sqrt{3}$, $|\mathbf{b}| = 1$, and the angle between them is $\frac{5\pi}{6}$.

(a) By considering $(2\mathbf{a}+\mathbf{b})\cdot(2\mathbf{a}+\mathbf{b})$, find the exact value of $|2\mathbf{a}+\mathbf{b}|$.

(b) Find

(i) the length of projection of **a** onto **b**,

(ii) the projection vector of **a** onto **b**.

9 **2014(9740)/I/3**

(i) Given that $\mathbf{a}\times\mathbf{b} = \mathbf{0}$, what can be deduced about the vectors **a** and **b**? [2]

(ii) Find a unit vector **n** such that $\mathbf{n}\times(\mathbf{i}+2\mathbf{j}-2\mathbf{k}) = \mathbf{0}$. [2]

(iii) Find the cosine of the acute angle between $\mathbf{i}+2\mathbf{j}-2\mathbf{k}$ and the *z*-axis. [1]

10 **2012(9740)/I/5**

Referred to the origin O, the points *A* and *B* have position vectors **a** and **b** such that

$$\mathbf{a} = \mathbf{i} - \mathbf{j} + \mathbf{k} \quad \text{and} \quad \mathbf{b} = \mathbf{i} + 2\mathbf{j}.$$

The point *C* has position vector **c** given by $\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b}$ where λ and μ are positive constants.

(i) Given that the area of triangle *OAC* is $\sqrt{126}$, find μ. [4]

(ii) Given instead that $\mu = 4$ and that $OC = 5\sqrt{3}$, find the possible coordinates of *C*. [4]

<u>Level 3</u>

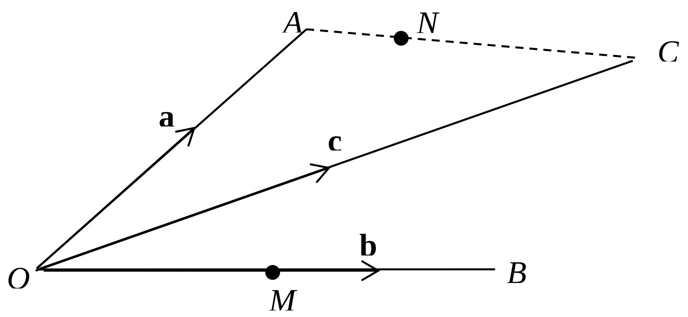
11 The points *A*, *B* and *C* have position vectors **a**, **b** and $\frac{3}{2}\mathbf{a}+\frac{1}{2}\mathbf{b}$ respectively. The point *P* on *AB* is such that $AP:PB = \lambda : 1-\lambda$ and the point *P* on *OC* is such that $OP:PC = \mu : 1-\mu$.

(i) Express $\overrightarrow{OP}$ in terms of λ, **a** and **b**.

(ii) By expressing $\overrightarrow{OP}$ in terms of μ, **a** and **b**, find the values of λ and μ. Hence show that P is the midpoint of OC.

(iii) It is given that the position vectors of the points *A* and *B* are $2\mathbf{j}+\mathbf{k}$ and $12\mathbf{i}-2\mathbf{j}-3\mathbf{k}$ respectively. The point *Q* lies on *OA* such that *PQ* is perpendicular to *OA*. Find the position vector of the point *Q*.

12 2013(9740)/I/6



The origin O and the points A, B and C lie in the same plane, where $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$ (see diagram).

(i) Explain why c can be expressed as $\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b}$, for constants λ and μ. [1]

The point N is on AC such that $AN : NC = 3 : 4$.

(ii) Write down the position vector of N in terms of **a** and **c**. [1]

(iii) It is given that the area of triangle ONC is equal to the area of triangle OMC, where M is the mid-point of OB. By finding the areas of these triangles in terms **a** and **b**, find λ in terms of μ in the case where λ and μ are both positive. [5]

13 2016(9740)/I/5

The vectors **u** and **v** are given by $\mathbf{u} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{v} = a\mathbf{i} + b\mathbf{k}$, where a and b are constants.

(i) Find $(\mathbf{u}+\mathbf{v})\times(\mathbf{u}-\mathbf{v})$ in terms of a and b. [2]

(ii) Given that the **i**- and **k**-components of the answer to part **(i)** are equal, express $(\mathbf{u}+\mathbf{v})\times(\mathbf{u}-\mathbf{v})$ in terms of a only. Hence find, in an exact form, the possible values of a for which $(\mathbf{u}+\mathbf{v})\times(\mathbf{u}-\mathbf{v})$ is a unit vector. [4]

(iii) Given instead that $(\mathbf{u}+\mathbf{v}).(\mathbf{u}-\mathbf{v}) = 0$, find the numerical value of $|\mathbf{v}|$. [2]

14 2018/9758 A Level/I/6

Vectors **a**, **b** and **c** are such that $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{a} \times 3\mathbf{b} = 2\mathbf{a} \times \mathbf{c}$.

(i) Show that $3\mathbf{b} - 2\mathbf{c} = \lambda\mathbf{a}$, where λ is a constant. [2]

(ii) It is now given that **a** and **c** are unit vectors, that the modulus of **b** is 4 and that the angle between **b** and **c** is 60°. Using a suitable scalar product, find exactly the two possible values of λ. [5]

Answer Key

1	**(a)(i)** $\overrightarrow{OF}=\begin{pmatrix}1\\1\\9\end{pmatrix}$, $\overrightarrow{EG}=\begin{pmatrix}-5\\1\\1\end{pmatrix}$ **(a)(ii)** 65.0° **(c)** $\begin{pmatrix}-2\\3\\1\end{pmatrix}$ or $\begin{pmatrix}2\\-3\\-1\end{pmatrix}$
2	**(i)** $p=\frac{3}{7}$
3	**(i)** $y=4$ **(ii)** $\frac{3}{2}\sqrt{195}$ **(iii)** $\begin{pmatrix}0\\3\\-3/5\end{pmatrix}$
4	**(i)** $\frac{\sqrt{101}}{101}\begin{pmatrix}1\\8\\6\end{pmatrix}$ or $-\frac{\sqrt{101}}{101}\begin{pmatrix}1\\8\\6\end{pmatrix}$ **(ii)** $\begin{pmatrix}1/2\\-4\\2\end{pmatrix}$ **(iii)** $p=-\frac{1}{2}$ **(iv)** $\begin{pmatrix}5\\-7\\-10\end{pmatrix}$ **(v)** $\begin{pmatrix}2.5\\12\\14\end{pmatrix}$
5	**(i)** $(12,-4,6)$ **(iii)** $\frac{1}{7}\begin{pmatrix}6\\-2\\3\end{pmatrix}$ **(iv)** $\begin{pmatrix}140\\84\\-224\end{pmatrix}$ **;** $98\sqrt{2}$ square units
6	**(a)** $\frac{\sqrt{10}}{10}\begin{pmatrix}-1\\-3\\0\end{pmatrix}$ **(b)** $\frac{5\sqrt{14}}{14}\begin{pmatrix}2\\3\\-1\end{pmatrix}$ or $-\frac{5\sqrt{14}}{14}\begin{pmatrix}2\\3\\-1\end{pmatrix}$ **(c)** $\mathbf{r}=\begin{pmatrix}2\\-1\\2\end{pmatrix}$
7	**(i)** $2\sqrt{2}$ **(ii)** $\frac{2}{5}\begin{pmatrix}5\\-4\\3\end{pmatrix}$
8	**(a)** $\sqrt{7}$ **(b)(i)** $\frac{3}{2}$ units **(b)(ii)** $-\frac{3}{2}\mathbf{b}$
9	**(ii)** $\frac{1}{3}\begin{pmatrix}1\\2\\-2\end{pmatrix}$ or $-\frac{1}{3}\begin{pmatrix}1\\2\\-2\end{pmatrix}$ **(iii)** $\frac{2}{3}$
10	**(i)** 6 **(ii)** $C(5,7,1)$ or $C\left(\frac{17}{3},\frac{19}{3},\frac{5}{3}\right)$
11	**(i)** $(1-\lambda)\mathbf{a}+\lambda\mathbf{b}$ **(ii)** $\mu\left(\frac{3}{2}\mathbf{a}+\frac{1}{2}\mathbf{b}\right)$ **(iii)** $\frac{2}{5}\begin{pmatrix}0\\2\\1\end{pmatrix}$
12	**(ii)** $\frac{3\mathbf{c}+4\mathbf{a}}{7}$ **(iii)** $\lambda=\frac{8}{7}\mu$
13	**(i)** $2\begin{pmatrix}b\\2b-2a\\-a\end{pmatrix}$ **(ii)** $a=\frac{\sqrt{2}}{12}$ or $a=-\frac{\sqrt{2}}{12}$ **(iii)** 3
14	**(ii)** $\therefore \lambda=-2\sqrt{31}$ or $\lambda=2\sqrt{31}$

H2 Mathematics (9758)
Chapter 5 Vectors
Extra Practice Questions

1 2014 IJC Promo/9 (Modified)

Relative to the origin *O*, the points *A* and *B* have position vectors $\mathbf{i}+3\mathbf{j}-2\mathbf{k}$ and $-6\mathbf{i}+4\mathbf{j}-7\mathbf{k}$ respectively.

(i) Find the size of angle *OAB*, giving your answer to the nearest 0.1°. [3]

(ii) Find a unit vector perpendicular to $\overrightarrow{OA}$ and $\overrightarrow{OB}$. [3]

2 2016/MI Prelim/I/Q3

Points *A* and *B* have coordinates $(3,\ 1,\ 2)$ and $(1,-5,\ 4)$. The point *C* lies on *AB* produced such that $AC:BC=3:2$. Find the position vector of point *C*. [2]

Determine the position vector of point *D* such that *OADC* is a parallelogram, and find the exact area of *OADC*. [4]

3 2017/CJC Prelim/II/Q2

Referred to the origin *O*, the points *A*, *B*, *P* and *Q* have position vectors **a**, **b**, **p** and **q** respectively, such that $|\mathbf{a}|=2$, **b** is a unit vector, and the angle between **a** and **b** is $\frac{\pi}{4}$.

(i) Give a geometrical interpretation of $|\mathbf{b}\cdot\mathbf{a}|$. [1]

(ii) Find $|\mathbf{a}\times\mathbf{b}|$, leaving your answer in exact form. [2]

It is also given that $\mathbf{p}=3\mathbf{a}+(\mu+2)\mathbf{b}$ and $\mathbf{q}=(\mu+3)\mathbf{a}+\mu\mathbf{b}$, where $\mu\in\mathbb{R}$.

(iii) Show that $\mathbf{p}\times\mathbf{q}=\left(\mu^2+2\mu+6\right)(\mathbf{b}\times\mathbf{a})$. [3]

(iv) Hence find the smallest area of the triangle *OPQ* as μ varies. [3]

4 **2014 TPJC Promo/4**

Referred to the origin *O*, the points A and B have position vectors **a** and **b** respectively. The point *C* lies on *AB* such that $AC:CB=1:3$.

(i) Write down the position vector of *C* in terms of **a** and **b**. [1]

It is given that $\mathbf{a}=\begin{pmatrix}1\\3\\0\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}3\\5\\k\end{pmatrix}$, where *k* is a positive constant, and $|\mathbf{b}|=\sqrt{5}|\mathbf{a}|$.

(ii) Show that $k=4$. [2]

(iii) Find the exact area of triangle *OAC*. [3]

(iv) Find the exact length of projection of $\overrightarrow{OC}$ onto the line *OB*. [3]

5 **2015 DHS Promo/2**

The position vectors of the points *A* and *B* relative to the origin *O* are $\begin{pmatrix}2\\-1\\1\end{pmatrix}$ and $\begin{pmatrix}-2\\5\\1\end{pmatrix}$ respectively. The point *P* lies between *A* and *B* such that $\overrightarrow{AP}=\lambda\overrightarrow{AB}$ where $0<\lambda<1$.

(i) Find the position vector of *P* in terms of λ. [1]

(ii) If $\overrightarrow{OP}$ is perpendicular to $\overrightarrow{AB}$, find the value of λ. [2]

Given that $\lambda=\frac{1}{3}$,

(iii) Find the area of triangle *OPA*. [2]

(iv) Write down the ratio of the area of triangle *OPB* to the area of triangle *OPA*. [1]

6 **2015 NYJC Promo/4**

Relative to the origin *O*, the position vectors of two points *A* and *B* are **a** and **b** respectively, where **a** and **b** are non-zero and non-parallel vectors. The vector **a** is a unit vector which is perpendicular to $2\mathbf{a}+5\mathbf{b}$. The angle between **a** and **b** is $\frac{2\pi}{3}$.

(i) Show that $|\mathbf{b}|=\frac{4}{5}$. [3]

(ii) The point *M* divides *AB* in the ratio $\lambda:1-\lambda$. The point *N* is such that *OMBN* is a parallelogram. By considering $\overrightarrow{ON}$ in terms of **a** and **b**, find the area of triangle *OAN* in terms of λ. [4]

7 2012/DHS/I/7

Referred to the origin O, the points A and B are such that $\overrightarrow{OA}=\mathbf{a}$ and $\overrightarrow{OB}=\mathbf{b}$. The point C lies on OB such that $\overrightarrow{OC}=p\overrightarrow{OB}$, where p is a constant. D is on AC such that $AD:DC=2:3$ and E is on AB such that $AE:EB=1:3$.

(i) Find $\overrightarrow{OD}$ and $\overrightarrow{OE}$ in terms of **a**, **b** and p. [2]

(ii) Given that O, D and E are collinear, find p. [3]

(iii) If OB = 5, show that the shortest distance from E to OB can be expressed as $k\left|\mathbf{a}\times\mathbf{b}\right|$, where k is a constant to be found. [3]

(iv) Give a geometrical interpretation of $\left|\mathbf{a}\bullet\hat{\mathbf{b}}\right|$. [1]

8 2008/HCI/I/12a

The points A, B, C have position vectors **a**, **b**, **c** respectively.

(i) Given that non-zero numbers λ, μ are such that $\lambda\mathbf{a}+\mu\mathbf{b}+\mathbf{c}=0$ and $\lambda+\mu+1=0$. Show that A, B, C are collinear. [2]

(ii) Show that the point P with position vector **p** given by $\mathbf{p}=4\mathbf{a}-3\mathbf{b}$ lies on BA produced, and find the ratio $PA : PB$. [3]

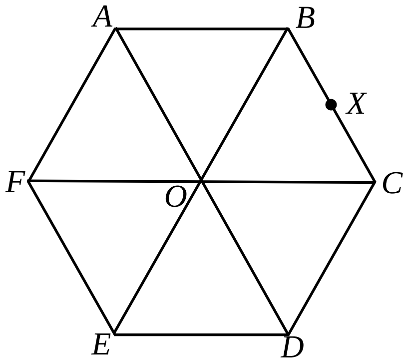
9 2019/TJC/Prelim 9758/2019/01/Q9

(a) Vectors **u** and **v** are such that $\mathbf{u}.\mathbf{v}=-1$ and $(\mathbf{u}\times\mathbf{v})+\mathbf{u}$ is perpendicular to $(\mathbf{u}\times\mathbf{v})+\mathbf{v}$.

Show that $\left|\mathbf{u}\times\mathbf{v}\right|=1$. [3]

Hence find the angle between **u** and **v**. [3]

(b) The figure shows a regular hexagon $ABCDEF$ with O at the centre of the hexagon. X is the midpoint of BC.



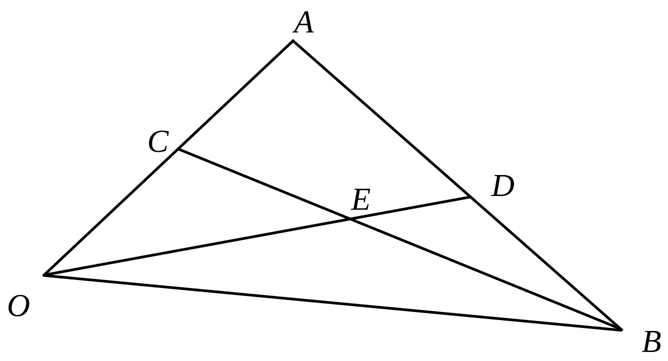
Given that $\overrightarrow{OA}=\mathbf{a}$ and $\overrightarrow{OB}=\mathbf{b}$, find $\overrightarrow{OF}$ and $\overrightarrow{OX}$ in terms of **a** and **b**. [2]

Line segments AC and FX intersect at the point Y. Determine the ratio $AY : YC$. [4]

10 2012/HCI/II/Q4

(a) Referred to the origin O, the position vectors of three points A, B and P are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{a}+5\mathbf{b}$ respectively. Given that $\mathbf{b}$ is a unit vector, the angle AOB is $60°$ and AB is perpendicular to OP, find $|\mathbf{a}|$. [4]

(b)



In the triangle OAB where O is the origin, the position vectors of the points A and B are $\mathbf{a}$ and $\mathbf{b}$ respectively. The point C is the midpoint of OA, the point E on BC is such that $CE : EB = 3 : 4$, and the line OE meets AB at D. Find the ratio $AD : AB$. [4]

Answer Key

No	Year	JC/CI	Answers
1	**2014**	**IJC**	(i) 100.7° (ii) $\frac{1}{\sqrt{1014}}\begin{pmatrix}-13\\19\\22\end{pmatrix}$
2	**2016**	**MI**	$\overrightarrow{OC}=\begin{pmatrix}-3\\-17\\8\end{pmatrix}$, $\overrightarrow{OD}=\begin{pmatrix}0\\-16\\10\end{pmatrix}$, $6\sqrt{138}$ units^2
3	**2017**	**CJC**	(ii) $\sqrt{2}$ (iv) $\frac{5\sqrt{2}}{2}$
4	**2014**	**TPJC**	(i) $\overrightarrow{OC}=\frac{1}{4}(3\mathbf{a}+\mathbf{b})$, (iii) $\frac{\sqrt{11}}{2}$ (iv) $\frac{13\sqrt{2}}{5}$
5	**2015**	**DHS**	(i) $\overrightarrow{OP}=\begin{pmatrix}2\\-1\\1\end{pmatrix}+\lambda\begin{pmatrix}-4\\6\\0\end{pmatrix}$; (ii) $\lambda=\frac{7}{26}$; (iii) $\frac{\sqrt{29}}{3}$ units^2 or 1.80 units^2; (iv) 2:1
6	**2015**	**NYJC**	(ii) $\frac{(1-\lambda)\sqrt{3}}{5}$
7	**2012**	**DHS**	(i) $\overrightarrow{OD}=\frac{3\mathbf{a}+2p\mathbf{b}}{5}$, $\overrightarrow{OE}=\frac{3\mathbf{a}+\mathbf{b}}{4}$ (ii) $p=\frac{1}{2}$ (iii) $k=\frac{3}{20}$
8	**2008**	**HCI**	(ii) $PA:PB=3:4$
9	**2019**	**TJC**	(a) $\theta=135°$ (b) 3:2
10	**2012**	**HCI**	(i) $\sqrt{6}-1$ (ii) $3:5$