NANYANG JUNIOR COLLEGE

JC1 END-OF-YEAR EXAMINATION

Higher 2

CANDIDATE NAME

CT CLASS | 2 | 4 | | |

MATHEMATICS 9758/01

Paper 1

2 October 2024

2.5 hours

Additional Materials: Printed Answer Booklet

List of Formulae and Results (MF27)

READ THESE INSTRUCTIONS FIRST

Answer **all** questions.
Write your answers on the Printed Answer Booklet. Follow the instructions on the front cover of the answer booklet.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are **not** allowed in a question, you must present the mathematical steps using mathematical notations and not calculator commands.
You must show all necessary working clearly.

The number of marks is given in brackets [ ] at the end of each question or part question.

The total number of marks for this paper is **80**.

1 A curve C has equation

$$y = \frac{a}{x^2} + be^{2x} + c,$$

where a, b and c are constants.

It is given that C passes through the point with coordinates $\left(1, 2e^2 - 1\right)$. When C undergoes a scaling of factor 2 parallel to the x-axis, the transformed curve passes through the point with coordinates $\left(-1, 8 + \frac{2}{e}\right)$. When C undergoes a translation by negative 1 unit in the direction of the y-axis, the transformed curve passes through the point with coordinates $\left(\sqrt{3}, 2e^{2\sqrt{3}} - 4\right)$. Find the values of a, b and c. [4]

2 **(a)** Show that $x^2 + 4x + 9$ is always positive for all real values of x. [2]

(b) Hence, without using a calculator, solve $\dfrac{\left(x^2 + 4x + 9\right)(x+2)^2}{x^2 - 6x + 7} \le 0$. [3]

3 The curve D has equation $y = \dfrac{2x - 5}{x^2 + 2x - 3}$.

(a) Sketch the graph of D, stating the equations of any asymptotes, the coordinates of the points where the curve crosses the axes and the stationary point(s). [3]

(b) By considering a suitable curve, hence, find the exact range of values of k, where $k > 0$, such that the equation $(x+3)^2 + \left(\dfrac{2x-5}{x^2 + 2x - 3}\right)^2 = k^2$ has at least one **positive** real root. [3]

4 During a clinical trial, the concentration U, of a specific blood agent is measured at one-hour intervals following the initial administration of the trial drug to a patient.

The following readings were obtained

$$U_3 = 88,\ U_4 = 76 \text{ and } U_5 = 70,$$

where U_t denotes the reading t hours after the drug was first administered.

It is believed that U satisfies the relationship

$$U_{t+1} = p + qU_t,\ t \ge 0,$$

where p and q are constants.

(a) Find the values of p and q. [2]

(b) Determine the initial concentration of the blood agent when the drug was initially administered. [3]

(c) Describe how the concentration behaves over a long period of time. [1]

5 The position vectors of points A, B and C, relative to the origin O are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively, where

$$\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}, \quad \mathbf{b} = 4\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}, \quad \mathbf{c} = 3\mathbf{i} - \mathbf{j} - \mathbf{k}.$$

(a) Find $\mathbf{a} \times \mathbf{b}$ and deduce the exact area of triangle OAB. [3]

(b) Hence, find the volume of the tetrahedron $OABC$.

[The volume of a tetrahedron is $\frac{1}{3} \times$ base area $\times$ perpendicular height.] [3]

(c) Find the perpendicular distance from B to line OA. [2]

6 **(a)** The sum, S_n, of the first n terms of a series is given by $S_n = n^2 + 2n$. Show that this is an arithmetic series. Find the values of the first term and the common difference. [4]

(b) In a geometric progression, the first term is 12 and the sixth term is $-\frac{3}{8}$. Let the sum of the first n terms of the progression and the sum to infinity be S_n and S respectively. Find the least value of n for which the difference between S_n and S is less than 0.001. [4]

7 Functions f and g are defined by

$$\mathrm{f}: x \mapsto |x + 20|, \quad x \in \mathbb{R},$$

$$\mathrm{g}: x \mapsto x + \frac{\alpha}{x}, \quad x \in \mathbb{R},\ x > 0.$$

(a) Explain why the composite function gf does not exist. [2]

(b) Determine the range of values of α such that g^{-1} exists. [2]

For the rest of the question, it is given that $\alpha = 4$ and the domain of g is further restricted to $x \geq \beta$.

(c) Determine the least value of β for which g^{-1} exists. Using this value of β, hence find $\mathrm{g}^{-1}(x)$ and state its domain. [4]

8 **(a)** Given that $y = \sqrt{\ln(x + \mathrm{e})}$ for $x > 1 - \mathrm{e}$, show that

$$2y\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^{-y^2}.$$

Hence find the first three terms of the Maclaurin expansion of y. Give the coefficients as exact fractions in their simplest form. [4]

(b) Using standard series from the List of Formulae (MF27), expand $\ln(x + \mathrm{e})$ as far as the term in x^2. Hence, verify the correctness of your expansion in **(a)**, showing your working clearly. [4]

(c) Use your expansion in part **(a)** to find an approximation to $\sqrt{\ln\left(\frac{1 + 10\mathrm{e}}{10}\right)}$, leaving your answer in terms of e. [2]

9 The plane Π_1 has equation $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, where λ and μ are parameters. The plane Π_2 has equation $x - 4y + 5z = 12$.

(a) Find the acute angle between Π_1 and Π_2. [3]

(b) Find a vector equation of the line of intersection between Π_1 and Π_2. [2]

(c) The line l passes through the point A with position vector $m\mathbf{i} + (2m+1)\mathbf{j} - 3\mathbf{k}$ and is parallel to $3n\mathbf{i} - 3\mathbf{j} + n\mathbf{k}$, where m and n are positive constants. Given that the perpendicular distance from A to the plane Π_1 is $\frac{15}{\sqrt{6}}$ and that the acute angle between l and Π_1 is $\sin^{-1}\frac{2}{\sqrt{6}}$, find the values of m and n. [6]

10 A tiny robot moves in a way that traces out the curve with parametric equations given by

$$x = 1 + t^2, \quad y = 2\sin^{-1} t$$

from the point where $t = -1$ to the point where $t = 1$.

(a) Sketch the curve traced out by the robot, indicating clearly the exact coordinates of the endpoints. [2]

(b) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t. [2]

The robot crosses the x-axis at the point P.

(c) Find the cartesian equation of the tangent to the curve at P. [2]

(d) Find the angle between the direction in which the robot is moving and the positive x-axis at the instant it reaches the point where $y = \frac{\pi}{3}$. [2]

A tiny magnet is placed at the point $(2, 2)$.

(e) Show that the distance s between the robot and the magnet is given by

$$s = \sqrt{\left(t^2 - 1\right)^2 + 4\left(\sin^{-1} t - 1\right)^2}. \quad [1]$$

(f) The magnet will attract the robot if it comes within 0.25 unit of its position. By using differentiation to find the minimum distance between the magnet and the robot, determine whether the robot will be attracted by the magnet.
[You need not show that the distance is a minimum.] [5]