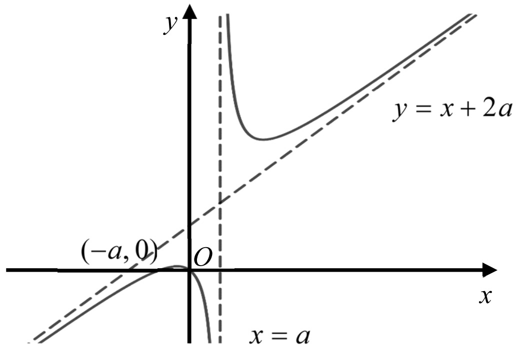
2023 H2 FM Y5 Common Test

1(a)	$\sum_{r=2}^{n}\left(\frac{1}{2^r}+\frac{1}{r^2-1}\right)=\sum_{r=2}^{n}\left[\frac{1}{2^r}+\frac{1}{2}\left(\frac{1}{r-1}-\frac{1}{r+1}\right)\right]$ $=\frac{\frac{1}{4}\left(1-\left(\frac{1}{2}\right)^{n-1}\right)}{1-\frac{1}{2}}+\frac{1}{2}\left(\begin{array}{l}\frac{1}{1}-\frac{1}{3}\\+\frac{1}{2}-\frac{1}{4}\\+\frac{1}{3}-\frac{1}{5}\\+\dots\\+\frac{1}{n-3}-\frac{1}{n-1}\\+\frac{1}{n-2}-\frac{1}{n}\\+\frac{1}{n-1}-\frac{1}{n+1}\end{array}\right)$ $=\frac{1}{2}\left(1-\left(\frac{1}{2}\right)^{n-1}\right)+\frac{1}{2}\left(1+\frac{1}{2}-\frac{1}{n}-\frac{1}{n+1}\right)$ $=\frac{5}{4}-\left(\frac{1}{2}\right)^{n}-\frac{1}{2n}-\frac{1}{2(n+1)}$	
(b)	$\sum_{r=2}^{\infty}\left(\frac{1}{2^r}+\frac{1}{r^2-1}\right)$ $=\lim_{n\to\infty}\left(\frac{5}{4}+\left(\frac{1}{2}\right)^{n}+\frac{1}{2n}-\frac{1}{2(n+1)}\right)$ $=\frac{5}{4}$	

2	 Intercepts: $(0,0)$, $(-a,0)$ Vertical Asymptotes: $x=a$ $\frac{x^2+ax}{x-a}=\frac{x^2+ax-2a^2+2a^2}{x-a}=x+2a+\frac{2a^2}{x-a}$ Oblique Asymptotes: $y=x+2a$ Turning points: $\frac{dy}{dx}=1-\frac{2a^2}{(x-a)^2}=0 \Rightarrow x=a\pm\sqrt{2}a$ $\left(\left(1-\sqrt{2}\right)a,\left(3-2\sqrt{2}\right)a\right)$ and $\left(\left(1+\sqrt{2}\right)a,\left(3+2\sqrt{2}\right)a\right)$	
	Set of values is $\mathbb{R}\setminus\left[\left(3-2\sqrt{2}\right)a,\left(3+2\sqrt{2}\right)a\right]$. Accept $y\le\left(3-2\sqrt{2}\right)a$ or $y\ge\left(3+2\sqrt{2}\right)a$	

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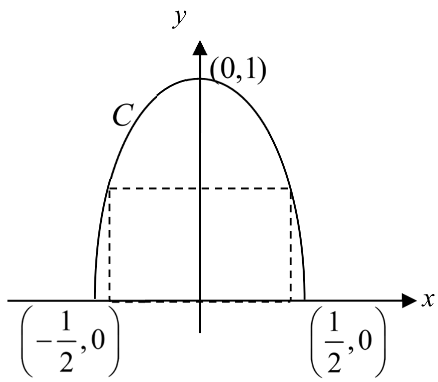


$4x^2 + y^2 = 1 \Rightarrow y = \sqrt{1-4x^2}$

Area of rectangle $A = 2xy = 2x\sqrt{1-4x^2}$

$\frac{\mathrm{d}A}{\mathrm{d}x} = 2x\frac{(-8x)}{2\sqrt{1-4x^2}} + 2\sqrt{1-4x^2} = \frac{2-16x^2}{\sqrt{1-4x^2}}$

When $\frac{\mathrm{d}A}{\mathrm{d}x} = 0$,

$2(1-4x^2) = 8x^2$

$x = \pm\frac{1}{2\sqrt{2}}$

Thus the x-coordinates for the bottom corners of the rectangle are $\frac{1}{2\sqrt{2}}$ and $-\frac{1}{2\sqrt{2}}$ but we only take the positive value of x for the area of the rectangle.

Using first derivative test:

$\frac{\mathrm{d}A}{\mathrm{d}x} = \frac{2-16x^2}{1-4x^2} = \frac{2(1+2\sqrt{2}x)(1-2\sqrt{2}x)}{1-4x^2}$

x	$\frac{1}{2\sqrt{2}}^{-}$	$\frac{1}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}^{+}$
$\frac{\mathrm{d}A}{\mathrm{d}x}$	$+$	0	$-$

Thus the area of the rectangle inscribed is maximum at

$2\frac{1}{2\sqrt{2}}\sqrt{1-4\left(\frac{1}{8}\right)} = \frac{1}{2}$ units^2

OR

Using second derivative test:

	$\frac{dA}{dx} = \frac{2-16x^2}{\sqrt{1-4x^2}}$ $\frac{d^2A}{dx^2} = \frac{\sqrt{1-4x^2}(-32x) - (2-16x^2)\frac{-8x}{2\sqrt{1-4x^2}}}{1-4x^2}$ $\frac{d^2A}{dx^2} = \frac{(1-4x^2)(-32x)+4x(2-16x^2)}{(1-4x^2)^{\frac{3}{2}}} = \frac{-8x(3+8x^2)}{(1-4x^2)^{\frac{3}{2}}}$ Since $2x$ is the length of the rectangle, $x > 0$ When x = $\frac{1}{2\sqrt{2}}$, $\frac{d^2A}{dx^2} = \frac{-8\left(\frac{1}{2\sqrt{2}}\right)\left(3+8\left(\frac{1}{8}\right)\right)}{\left(1-4\left(\frac{1}{8}\right)\right)^{\frac{3}{2}}} < 0$ Thus, area is a maximum when $x = \frac{1}{2\sqrt{2}}$ Thus the area of the rectangle inscribed is maximum at $2\frac{1}{2\sqrt{2}}\sqrt{1-4\left(\frac{1}{8}\right)} = \frac{1}{2}$ units^2	

4(a) **[5]**	$(9-x)^{\frac{5}{2}}(1+3x^2)^{\frac{5}{2}} = 9^{\frac{5}{2}}\left(1-\frac{x}{9}\right)^{\frac{5}{2}}(1+3x^2)^{\frac{5}{2}}$ $\left(1-\frac{x}{9}\right)^{\frac{5}{2}} = 1+\frac{5}{2}\left(-\frac{x}{9}\right)+\frac{\frac{5}{2}\left(\frac{3}{2}\right)}{2!}\left(-\frac{x}{9}\right)^2+\frac{\frac{5}{2}\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)}{3!}\left(-\frac{x}{9}\right)^3+\ldots$ $= 1-\frac{5x}{18}+\frac{5}{216}x^2-\frac{5}{11664}x^3+\ldots$ $(1+3x^2)^{\frac{5}{2}} = 1+\frac{5}{2}\left(3x^2\right)+\ldots = 1+\frac{15}{2}x^2$ $(9-x)^{\frac{5}{2}}(1+3x^2)^{\frac{5}{2}} = 9^{\frac{5}{2}}\left(1-\frac{x}{9}\right)^{\frac{5}{2}}(1+3x^2)^{\frac{5}{2}}$ $= 243\left(1-\frac{5x}{18}+\frac{5}{216}x^2-\frac{5}{11664}x^3+\ldots\right)\left(1+\frac{15}{2}x^2+\ldots\right.$ $= 243\left(1-\frac{5x}{18}+\frac{15}{2}x^2+\frac{5}{216}x^2-\frac{5}{11664}x^3-\frac{25}{12}x^3+\ldots\right.$ $= 243-\frac{135}{2}x+\frac{14625}{8}x^2-\frac{24305}{48}x^3+\ldots$	
(b) **[2]**	Expansion of $\left(1-\frac{x}{9}\right)^{\frac{5}{2}}$ is valid for $\left\|\frac{x}{9}\right\| < 1 \Rightarrow \|x\| < 9$. Expansion of $(1+3x^2)^{\frac{5}{2}}$ is valid for $\left\|3x^2\right\| < 1 \Rightarrow \|x\| < \frac{1}{\sqrt{3}}$. Hence, expansion of $(9-x)^{\frac{5}{2}}(1+3x^2)^{\frac{5}{2}}$ is valid for $\|x\| < \frac{1}{\sqrt{3}}$. This further implies $-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$. Range of validity for which expansion is valid is $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.	

5[8]

Use de Moivre's theorem to show that $\sin 5\theta = \cos^5\theta(t^5 - 10t^3 + 5t)$ where $t = \tan\theta$.

$$\begin{aligned}
\cos 5\theta + \mathrm{i}\sin 5\theta &= (\cos\theta + \mathrm{i}\sin\theta)^5 \\
&= \cos^5\theta + 5\mathrm{i}\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta \\
&\qquad - 10\mathrm{i}\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + \mathrm{i}\sin^5\theta \\
&= (\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta) \\
&\qquad + \mathrm{i}(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta)
\end{aligned}$$

Compare imaginary part,

$$\begin{aligned}
\sin 5\theta &= 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta \\
&= \cos^5\theta(5\tan\theta - 10\tan^3\theta + \tan^5\theta) \\
&= \cos^5\theta(t^5 - 10t^3 + 5t)
\end{aligned}$$

where $t = \tan\theta$. (shown)

Let $\theta = \dfrac{\pi}{5}$ and $x = \tan^2\theta = t^2$

$$\sin 5\theta = \cos^5\theta(t^5 - 10t^3 + 5t)$$

$$\Rightarrow \sin\pi = \cos^5\frac{\pi}{5}(t^5 - 10t^3 + 5t)$$

$$0 = \left(\cos^5\frac{\pi}{5}\right)t(t^4 - 10t^2 + 5)$$

Since $\cos^5\dfrac{\pi}{5} \neq 0$ and $\tan\dfrac{\pi}{5} \neq 0$,

$t^4 - 10t^2 + 5 = 0$

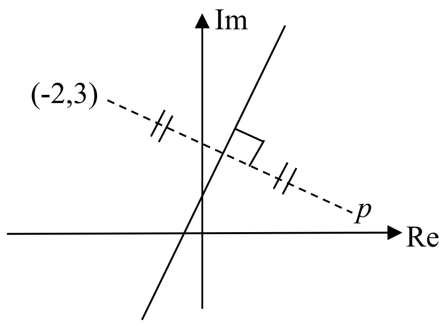
$x^2 - 10x + 5 = 0$

$\tan^2\dfrac{\pi}{5}$ is a root of $x^2 - 10x + 5 = 0$. (deduced)

$$\tan^2\frac{\pi}{5} = \frac{10 \pm \sqrt{80}}{2} = 5 \pm 2\sqrt{5}$$

Since $0 < \tan\dfrac{\pi}{5} < \tan\dfrac{\pi}{4} = 1$,

$\tan^2\dfrac{\pi}{5} < 1$ so $\tan^2\dfrac{\pi}{5} = 5 - 2\sqrt{5}$

6[2]	Perpendicular bisector of the line segment from the points represented by $-2+3\text{i}$ and p on Argand diagram. 	
(b) **[6]**	**(i)** To find line equation passing through $(1,8)$ and $(-2,-1)$, $\frac{y-8}{x-1}=\frac{8-(-1)}{1-(-2)} \Rightarrow y=3x+5$ Represent $-2+3\text{i}$ from part (a) as point $(-2,3)$ To equation of line passing through $(-2,3)$ and perpendicular to $y=3x+5$, $y=-\frac{1}{3}x+c$ $3=-\frac{1}{3}(-2)+c \Rightarrow c=\frac{7}{3}$ $y=-\frac{1}{3}x+\frac{7}{3}$ To find point of intersection between $y=3x+5$ and $y=-\frac{1}{3}x+\frac{7}{3}$, $3x+5=-\frac{1}{3}x+\frac{7}{3}$ $\frac{10}{3}x=-\frac{8}{3}$ $x=-\frac{4}{5}$ $y=3\left(-\frac{4}{5}\right)+5=\frac{13}{5}$ Point of intersection is $\left(-\frac{4}{5},\frac{13}{5}\right)$. Represent $p=(x,y)$,	

	$\left(-\frac{4}{5},\frac{13}{5}\right)=\left(\frac{x+(-2)}{2},\frac{y+3}{2}\right)$ $(x,y)=\left(\frac{2}{5},\frac{11}{5}\right)$ $\therefore p=\frac{2}{5}+\frac{11}{5}\mathrm{i}$ the least possible value of $\lvert z-p\rvert=\sqrt{\left(\frac{2}{5}+\frac{4}{5}\right)^2+\left(\frac{11}{5}-\frac{13}{5}\right)^2}=\sqrt{\left(\frac{6}{5}\right)^2+\left(-\frac{2}{5}\right)^2}=\frac{2\sqrt{10}}{5}$ units Alternatively, From previous part, the line equation passing through $(1,8)$ and $(-2,-1)$, $y=3x+5$ the least possible value of $\lvert z-p\rvert=\frac{\left\lvert\frac{11}{5}-3\left(\frac{2}{5}\right)-5\right\rvert}{\sqrt{1^2+3^2}}=\frac{4}{\sqrt{10}}=\frac{2\sqrt{10}}{5}$ units		

7(a)	Let P_n be the statement $u_n = 2n^2 - 4n + 6 - \left(\frac{1}{2}\right)^n$, for all integers $n \geq 2$. When $n = 1$, $u_1 = 2 - 4 + 6 - \frac{1}{2} = \frac{7}{2}$ (as given) Since LHS = RHS, P_1 is true. Assume P_k is true for some $k \geq 1$, i.e. assume $u_k = 2k^2 - 4k + 6 - \left(\frac{1}{2}\right)^k$. To prove that P_{k+1} is true, i.e. to prove $u_{k+1} = 2(k+1)^2 - 4(k+1) + 6 - \left(\frac{1}{2}\right)^{k+1}$ Now, LHS $= u_{k+1} = \frac{1}{2}u_k + (k+1)^2$ $= \frac{1}{2}\left[2k^2 - 4k + 6 - \left(\frac{1}{2}\right)^k\right] + (k+1)^2$ $= k^2 - 2k + 3 - \left(\frac{1}{2}\right)^{k+1} + (k+1)^2$ $= k^2 + 2k + 1 - 4k - 4 + 6 - \left(\frac{1}{2}\right)^{k+1} + (k+1)^2$ $= (k+1)^2 - 4(k+1) + 6 - \left(\frac{1}{2}\right)^{k+1} + (k+1)^2$ $= 2(k+1)^2 - 4(k+1) + 6 - \left(\frac{1}{2}\right)^{k+1} =$ RHS $\therefore P_k$ is true $\Rightarrow P_{k+1}$ is true. Since P1 is also true, by Mathematical induction P_n is true for all positive integers *n*.	
(b)	$9 > 8$ $3 > 2\sqrt{2}$ $1 + 2 > 2\sqrt{2}$ $1 > 2\sqrt{2} - 2$ (proven) Let P_n be the statement $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} ... + \frac{1}{\sqrt{n}} > 2\sqrt{n+1} - 2$, for all positive integers *n*. From previous part, P_1 is true. Assume P_k is true for some $k \geq 1$, i.e. assume: $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} ... + \frac{1}{\sqrt{k}} > 2\sqrt{k+1} - 2$. To prove that P_{k+1} is true,	

	i.e. to prove: $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}...+\frac{1}{\sqrt{k+1}}>2\sqrt{k+2}-2$. Now, LHS $=1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}...+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}} \; >\left(2\sqrt{k+1}-2\right)+\frac{1}{\sqrt{k+1}}$. We want to show that: $\left(2\sqrt{k+1}-2\right)+\frac{1}{\sqrt{k+1}}>2\sqrt{k+2}-2$ i.e. $2\sqrt{k+1}+\frac{1}{\sqrt{k+1}}-2\sqrt{k+2}>0$ $\Leftrightarrow 2k+3-2\sqrt{(k+1)(k+2)}>0$ $2k+3-2\sqrt{\left(k+\frac{3}{2}\right)^2-\frac{1}{4}}>2k+3-2\sqrt{\left(k+\frac{3}{2}\right)^2}=2k+3-2\left(k+\frac{3}{2}\right)=0$ $\therefore P_k$ is true. Since P_1 is true and P_k is true $\Rightarrow P_{k+1}$ is true, by mathematical induction, P_k is true for all positive integers.	

8(a) **(i)**	$u_n = 0.7u_{n-1} + 150,\ u_1 = 2000$	
(ii)	$u_n = 0.7u_{n-1} + 150$ $= 0.7(0.7u_{n-2} + 150) + 150$ $= 0.7^2 u_{n-2} + 150(0.7 + 1)$ $= 0.7^2(0.7u_{n-3} + 150) + 150(0.7 + 1)$ $= 0.7^3 u_{n-3} + 150(0.7^2 + 0.7 + 1)$ $= \dots$ $= 0.7^{n-1}u_1 + 150(0.7^{n-2} + 0.7^{n-3} + \dots + 0.7 + 1)$ $= 0.7^{n-1}u_1 + 150\left(\frac{1-0.7^{n-1}}{1-0.7}\right)$ $= 0.7^{n-1}(2000) + 500(1 - 0.7^{n-1})$ $= 500 + 1500(0.7^{n-1})$	
(iii)	Number of fishes at the start of February 2023 = ~~0.88(2000) + 150~~ 0.525 (2000) + 150 = ~~1910~~ 1200 Let $q = 1 - \frac{p}{100}$, x be the number of months where he operated under the new model, starting from February 2023. $u_n = (0.75)(q)u_{n-1} + 150$ $n \to \infty,\ u_{n-1} \to 375$ and $u_n \to 375$ $375 = (0.75)(q)(375) + 150$ $q = 0.8$ $p = 0.2$ Alternatively, $u_x = (0.75)(q)u_{x-1} + 150$ $= (0.75q)((0.75q)u_{x-2} + 150) + 150$ $= \dots$ $= (0.75q)^{n-1}u_1 + 150\left(\frac{1-(0.75q)^{x-1}}{1-0.75q}\right)$ $= \frac{150}{1-0.75q} + \left(\text{~~1910~~ 1200} - \frac{150}{1-0.75q}\right)(0.75q)^{x-1}$ As $x \to \infty,\ 0.75q^{x-1} \to 0,\ u_x \to \frac{150}{1-0.75q}$	

	Solving $\dfrac{150}{1-0.75q}=375$, $q=0.8$ $\therefore\ p=20$	
(b)	$P_{n+1}-P_n=2(P_n-P_{n-1})$ $P_{n+1}=3P_n-2P_{n-1}$ Characteristic Equation: $\lambda^2-3\lambda+2=0$ $(\lambda-2)(\lambda-1)=0$ $\lambda=2$ or $\lambda=1$ General solution: $P_n=A(2^n)+B(1^n)$ Given $P_0=3000,\ \ P_1=3500$, $3000=2A+B$ $3500=4A+B$ Solving, $A=250,\ B=2500$ $\therefore P_n=250(2^n)+2500$ Solving $250(2^n)+2500=60000$, Using GC, n = 7, Aug 2023	