STUDENT NAME: ______________________________

TEACHER INITIALS: ________

CANDIDATE SESSION NUMBER

0	2	5	0	1	2				

EXAMINATION CODE

8	8	1	6	-	7	2	0	1

ST JOSEPH'S INSTITUTION
JC2 PRELIMINARY EXAMINATION 2016

MATHEMATICS
HIGHER LEVEL
PAPER 1
Tuesday

5th July 2016
2 hours
1400 – 1600 hrs

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- **Section A**: Answer all questions showing working and answers in the spaces provided in the exam paper.
- **Section B**: Answer all questions using the foolscap paper provided.
- The use of calculators is **not** permitted in this paper.
- A clean copy of the **Mathematics HL Formulae Booklet** is required for this paper.
- Unless otherwise stated in the question all numerical answers are to be given exactly.
- The maximum mark for this examination paper is *[120 marks]*.
- This question paper consists of **14** printed pages including the Cover Sheet.

FOR MARKER USE ONLY:

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	Q12	Q13	Q14	Q15	TOTAL
															/120

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are advised to show all working.

SECTION A (60 marks)

1 [Maximum mark: 4]

The mean and standard deviation of a data set K is given by μ and σ. Denote by $(aK + b)$ the set derived by adding b after multiplying by a each element in K.

(a) Express in terms of μ the mean of the data set $(2K - 1)$. [1]

(b) Express in terms of σ the standard deviation of $(2K + 1)$. [1]

(c) Find σ if the variance of $(2K + 1)$ is 3. [2]

2 [Maximum mark: 7]

(a) Given that $\log_{11} 2 + 2\log_{11}(a-b) = \log_{11} a + \log_{11} b$, find the value of $\frac{a}{b}$. [4]

(b) Solve the equation: $3^{2x}(2^{3x}-4) - 2^{3x} + 4 = 0$. [3]

3 [Maximum mark: 6]

Let $\alpha \in \mathbb{C}$ be a fifth root of unity such that $\mathrm{Im}(\alpha) \neq 0$.

(i) Express α in the form $re^{i\theta}$. [1]

(ii) Enumerate the other four fifth roots of unity as a power of α. [1]

(iii) Evaluate $\sum_{k=1}^{49} \alpha^k$. [4]

4 [Maximum mark: 5]

Show that no complex number $z \in \mathbb{C}$ such that $|z| = 1$ satisfies $z^2 - 3z^* = i$, where $i^2 = -1$.

5 [Maximum mark: 6]

Given that $\{a_k\}_{k\in\mathbb{N}}$ is a geometric sequence with common ratio $r = \frac{1}{3}$ such that

$$\sum_{k=1}^{\infty} a_k = 10,$$

.

(a) show that [4]

$$\sum_{k=1}^{\infty} a_k^2 = \sum_{k=1}^{\infty} 5a_k.$$

(b) Hence or otherwise, evaluate [2]

$$\sum_{k=1}^{\infty} a_k(a_k + 5).$$

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6 [Maximum mark: 5]

If $y^{y+1} = \sqrt{x^5 + 1} + \tan x + \ln(\cos(x))$, find $\frac{dy}{dx}$ at $(0, 1)$.

7 [Maximum mark: 5]

(a) Suppose $f(0) = 12$ and f' is continuous such that $\int_0^4 f'(x)dx = 17$, find $f(4)$. [2]

(b) Suppose f is continuous such that $\int_0^4 f(x)dx = 10$, evaluate $\int_0^2 f(2x)\,dx$. [3]

8 [Maximum mark: 5]

Enumerate the terms containing x^0, x^1 and x^2 in the expansion of $(1 - 4x + 4x^2)^5$.

9 [Maximum mark: 5]

Let $P(x) = (b - x)(ax^2 + (a + 1)x + b)$, where $a, b \in \mathbb{R}$. Find the range of values of a given that 3 is the only real root of $P(x)$.

10 [Maximum mark: 6]

Suppose the solution of the equation $x + \tan x = \pi$ over the interval $\left(0, \frac{1}{2}\pi\right)$ is $x = a$. Find the other solutions of the equation over the interval $(0, 2\pi)$ in terms of a and π.

11 [Maximum mark: 6]

Determine the probability that a randomly selected integer between 100 and 500 inclusive is divisible by 3 or 5.

*Do **NOT** write solutions on this page.*

SECTION B (60 marks)

Answer all questions on the foolscap paper provided. Please start each question on a new page.

12 [Maximum mark: 15]

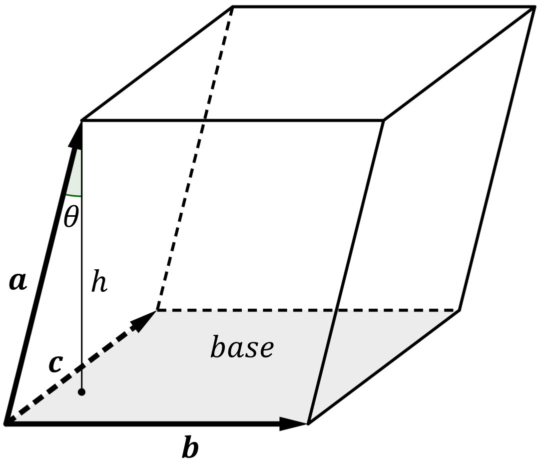
(a) (i) For vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ in three dimensions, prove that [5]

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}$$

(ii) Hence, show that for three coplanar vectors $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$: [2]

$$(\boldsymbol{v}_1 \times \boldsymbol{v}_2) \cdot \boldsymbol{v}_j = 0 \text{ if } j = 1 \text{ or } 2$$

(b) Show that the volume of the parallelepiped pictured below is given by $|\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})|$, where $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ are vectors representing the respective sides. [4]



(c) Find the equation of the plane consisting of all points equidistant to the points $(1, 2, -1)$ and $(5, 4, 5)$. [4]

13 [Maximum mark: 15]

(a) Prove that for $\binom{2n}{n} < 2^{2n-2}$ for all integers $n \geq 5$. [7]

(b) Find the value of s such that

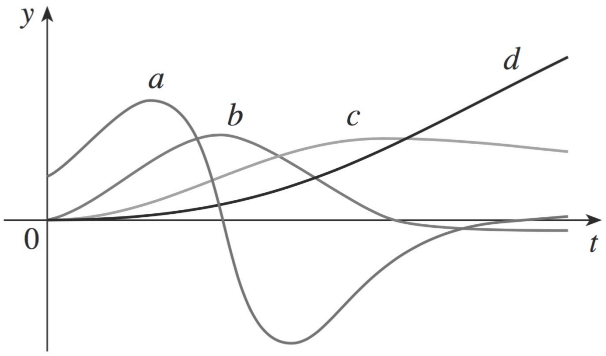
$$P(x) = -x^4 + sx^3 + (s^2 - 3)x^2 - (s + 2)x + 2$$

is divisible by $(x - 1)^2$. [5]

(c) Describe with reason the nature of the roots of $P(x)$ defined in (b) when $s = 0$. [3]

14 [Maximum mark: 20]

(a) The figure below shows the graphs of four functions. One is the position (displacement) function of a car, one is the velocity of the car, one is its acceleration, and one is its jerk (which is defined as the rate of change of acceleration). Identify each curve, and explain your choices. [5]



(b) The displacement of a particle from an origin is given by the equation $s = \frac{1-v}{2v-1}$ $(s \geq 0)$, where s is its distance from the origin given its velocity v.

(i) Find an expression for v in terms of s. [2]

(ii) Interpret what happens to the velocity of the car as it continues to go farther from its origin. [1]

(iii) Find an expression for the acceleration of the car in terms of s. [4]

(c) (i) Find the stationary point of $f(x) = x^{1/3} - x^{2/3}$ and classify it as maximum, minimum or point of inflexion. [6]

(ii) Describe with reason the tangent line at $x = 0$. [2]

15 [Maximum mark: 10]

(a) The graph of $y = \cos x$ is transformed to $y = 2\cos(3x - \pi) + 1$ via the following sequence of ordered transformations:

Reflection in the x-axis;
Vertical scaling by a factor of a, where $a > 0$;
Horizontal scaling by a factor of b, where $b > 0$;
Translation by $\binom{h}{k}$.

Find a, b, h and k. [3]

(b) (i) The function $f(x) = 2\cos(3x - \pi) + 1$ is one-one over the interval $[0, t]$. Find the largest possible value of t. [2]

(ii) Given the domain found in (i), find the inverse of f. [3]

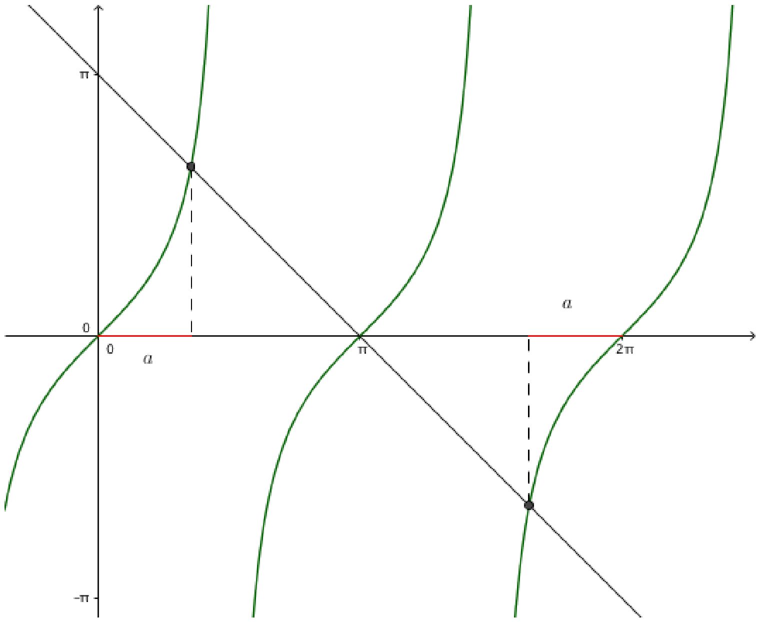
(iii) State the domain of the inverse found in (ii). [2]

End of Paper

JC2 HL Math Preliminary Examination 2016 Paper 1 (Markscheme)

Qn	Suggested Solutions	Marks
1	**Descriptive Statistics**	**[Maximum mark: 4]**
(a)	$E(2k+1) = 2\mu - 1.$	**A1**
(b)	$\sqrt{Var(2K+1)} = 2\sigma.$	**A1**
(c)	$3 = Var(2K+1) = 4\sigma^2$, and so $\sigma = \frac{1}{2}\sqrt{3}$.	**M1** **A1**
2	**Exponential/Logarithmic Equation**	**[Maximum mark: 7]**
(a)	$\log_{11} 2(a-b)^2 = \log_{11} ab \Longrightarrow 2a^2 - 5ab + 2b^2 = 0$ $(2a-b)(a-2b) = 0$ $\frac{a}{b} = \frac{1}{2}$ (rej. Because $a > b$) or $\frac{a}{b} = 2.$	**M1** **M1** **A1A1**
(b)	$(3^{2x} - 1)(2^{3x} - 4) = 0$ $x = 0$ or $x = \frac{2}{3}.$	**M1** **A1A1**
3	**De Moivre's**	**[Maximum mark: 6]**
(i)	Accept any of the following: $e^{2\pi i/5}$, $e^{4\pi i/5}$, $e^{6\pi i/5}$, $e^{8\pi i/5}$	**A1**
(ii)	Other 5th roots of unity: α^2, α^3, α^4 and $\alpha^5 = 1$.	**A1**
(iii)	$1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 = 0 \Longrightarrow \alpha + \alpha^2 + \alpha^3 + \alpha^4 = -1$ Thus, $\sum_{k=1}^{49} \alpha^k = \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \cdots + \alpha^{46} + \alpha^{47} + \alpha^{48} + \alpha^{49} = 0 + \cdots + 0 + \alpha^{46} + \alpha^{47} + \alpha^{48} + \alpha^{49}$ $= \alpha + \alpha^2 + \alpha^3 + \alpha^4 = -1.$	**A1** **M1A1** **A1**
4	**Complex Numbers**	**[Maximum mark: 5]**
	$z^2 - 3z^* = i \Longrightarrow z^3 = iz + 3$ Geometrically, since $\|z\| = 1$, then z^3 is in the unit circle, but $iz + 3$ is not. To see this, let $z = a + ib$ for some real numbers a and b. Recall that $a^2 + b^2 = 1$. Now, suppose $\|iz + 3\| = 1$, then it follows that $a^2 + (3-b)^2 = 1$ which implies $b = \frac{3}{2}$, which is a contradiction since this would mean $a^2 = -\frac{5}{4}$.	**M1** **A1** - $\|z^3\| = 1$ **M1** - $let\ z = a + ib$ **A1** - $b = \frac{3}{2}$ **R1**

Qn	**Suggested Solutions**	**Marks**
5	**Geometric Progression**	**[Maximum mark: 6]**
(a)	$\frac{a_1}{1-\frac{1}{3}} = 10 \Rightarrow a_1 = \frac{20}{3}$.	**M1A1**
	$\{a_k^2\}$ is GP with first term $a_1^2 = \frac{400}{9}$ and common ratio $r^2 = \frac{1}{9}$. Thus, $\sum_{k=1}^{\infty} a_k^2 = \frac{400/9}{1-1/9} = 50 = 5(10) = \sum_{k=1}^{\infty} 5a_k$.	**M1A1**
(b)	$\sum_{k=1}^{\infty} a_k(a_k+5) = \sum_{k=1}^{\infty} 2a_k^2 = 100$.	**M1A1**
6	**Implicit differentiation**	**[Maximum mark: 5]**
	$$(y+1)\ln y = \ln\left(\sqrt{x^5+1} + \tan x + \ln(\cos(x))\right)$$	**M1 – correct way of taking ln of both sides**
	$$\left(\ln y + \frac{y+1}{y}\right)\frac{dy}{dx} = \frac{\frac{5x^4}{2\sqrt{x^5+1}} + \sec^2 x + \frac{-\sin x}{\cos x}}{\sqrt{x^5+1} + \tan x + \ln(\cos(x))}$$	**A1A1**
	At $(0,1)$, we have	**M1 – attempt to sub in x=0, y=1**
	$$\left(0 + \frac{1+1}{1}\right)\frac{dy}{dx} = \frac{0+1+0}{1+0+0} \Rightarrow \frac{dy}{dx} = \frac{1}{2}$$	**A1 (N0)**
7	**Integration**	**[Maximum mark: 5]**
(a)	$$17 = \int_0^4 f'(x)dx = f(4) - f(0)$$	**M1**
	$$\Rightarrow f(4) = 12 + 17 = 29$$	**A1**
(b)	Let $x = 2y$. If $x = 0, y = 0$; if $x = 4, y = 2$.	**M1**
	$$10 = \int_0^4 f(x)dx = \int_0^2 f(2y)2dy$$	**M1**
	$$\Rightarrow \int_0^2 f(2y)dy = 5$$	**A1 (N0)**
8	**Binomial Theorem**	**[Maximum mark: 5]**
	$$(1-4x+4x^2)^5 = (1-2x)^{10}$$	**M1A1**
	$$= 1 - 10(2x) + 45(2x)^2 + \cdots$$	**M1**
	$$= 1 - 20x + 180x^2 + \cdots$$	**A1A1**
	---- or -----	**--------- or -------**
	$$(1-4x+4x^2)^5 = [(1-4x)+4x^2]^{10}$$	**M1**
	$$= (1-4x)^5 + \binom{5}{1}(1-4x)(4x^2) + \cdots$$	**A1**
	$$= 1 - \binom{5}{1}(4x) + \binom{5}{2}(4x)^2 + 20x^2 + \cdots$$	**M1** **A1**
	$$= 1 - 20x + 180x^2 + \cdots$$	**A1**

Qn	Suggested Solutions	Marks
9	**Polynomials**	**[Maximum mark: 5]**
	3 is the unique real root implies $b = 3$ and $(a+1)^2 - 12a < 0$ $\Rightarrow a^2 - 10a + 1 < 0.$ By Q.F., $a = \frac{10 \pm \sqrt{10^2 - 4(1)(1)}}{2(1)} = 5 \pm 2\sqrt{6}.$ Thus, $5 - 2\sqrt{6} < a < 5 + 2\sqrt{6}.$	**A1** **M1 – use of $\Delta < 0$** **A1** **M1 – attempt to use quadratic formula** **A1 (simplified)**
10	**Trigonometry Sketching**	**[Maximum mark: 6]**
		**G1G1**
	Sketch $y = \tan x$ and $y = \pi - x$. By symmetrical properties of $y = \tan x$, the other solutions are $x = \pi$ and $x = 2\pi - a$.	**M1 – attempt to sketch** **R1A1A1**
11	**Arithmetic Progression + Probability**	**[Maximum mark: 6]**
	Number of integers: 401 Divisible by 3: $102, 105, \ldots, 498 \Rightarrow n_3 = \frac{498-102}{3} + 1 = 133$ Divisible by 5: $n_5 = \frac{500-100}{5} + 1 = 81$ Divisible by 15: $n_{15} = \frac{495-105}{15} + 1 = 27$ Thus, $P = \frac{133+81-27}{401} = \frac{187}{401}.$	**A1** **A1** **A1** **A1** **M1 – P.I.E.** **A1**

Qn	**Suggested Solutions**	**Marks**
12	**Vector Product / Equation of a plane**	**[Maximum mark: 15]**
(a)	**(i)** Let $\boldsymbol{a} := \begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}, \boldsymbol{b} := \begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}, \boldsymbol{c} := \begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}$	**M1**
	$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix} \cdot \begin{pmatrix}b_2c_3 - b_3c_2\\b_3c_1 - b_1c_3\\b_1c_2 - b_2c_1\end{pmatrix}$	**M1**
	$= a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1$	**A1**
	$= (a_2b_3 - a_3b_2)c_1 + (a_3b_1 - a_1b_3)c_2 + (a_1b_2 - a_2b_1)c_3$	**M1**
	$= \begin{pmatrix}a_2b_3 - a_3b_2\\a_3b_1 - a_1b_3\\a_1b_2 - a_2b_1\end{pmatrix} \cdot \begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix} = (\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}$	**A1**
	(ii) $(\boldsymbol{v_1} \times \boldsymbol{v_2}) \cdot \boldsymbol{v_1} = \boldsymbol{v_1} \cdot (\boldsymbol{v_1} \times \boldsymbol{v_2}) = (\boldsymbol{v_1} \times \boldsymbol{v_1}) \cdot \boldsymbol{v_2} = 0$ $(\boldsymbol{v_1} \times \boldsymbol{v_2}) \cdot \boldsymbol{v_2} = \boldsymbol{v_1} \cdot (\boldsymbol{v_2} \times \boldsymbol{v_2}) = 0.$	**M1A1**
(b)	Area of base is $\lvert \boldsymbol{b} \times \boldsymbol{c} \rvert$	**A1**
	Height is $\lvert \boldsymbol{a} \cdot \hat{\boldsymbol{n}} \rvert$, where $\hat{\boldsymbol{n}} \parallel \boldsymbol{b} \times \boldsymbol{c}$	**A1**
	Take $\hat{\boldsymbol{n}} = \frac{\boldsymbol{b} \times \boldsymbol{c}}{\lvert \boldsymbol{b} \times \boldsymbol{c} \rvert}$, so that the volume is	**M1**
	$\lvert \boldsymbol{b} \times \boldsymbol{c} \rvert \left\lvert \boldsymbol{a} \cdot \frac{\boldsymbol{b} \times \boldsymbol{c}}{\lvert \boldsymbol{b} \times \boldsymbol{c} \rvert} \right\rvert = \lvert \boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) \rvert$	**A1 (A0 – no clear explanation)**
(c)	The plane equidistant to the points contains the midpoint $(3, 3, 2)$ and has normal vector parallel to $\begin{pmatrix}5\\4\\5\end{pmatrix} - \begin{pmatrix}1\\2\\-1\end{pmatrix} = \begin{pmatrix}4\\2\\6\end{pmatrix}$.	**A1 - midpoint** **A1 – normal vector**
	Thus, the equation of the plane is given by $4x + 2y + 6z = 12 + 6 + 12 = 30$ or $2x + y + 3z = 15.$	**M1A1**
	OR	
	$\sqrt{(x-1)^2 + (y-2)^2 + (z+1)^2} = \sqrt{(x-5)^2 + (y-4)^2 + (z-5)^2}$	**A1A1**
	$\Leftrightarrow -2x - 4y + 2z + 6 = -10x - 8y - 10z + 66$	**M1**
	Thus, the equation of the plane is given by $8x + 4y + 12z = 60$ or $2x + y + 3z = 15.$	**A1**

Qn	**Suggested Solutions**	**Marks**
13	**Polynomial + Induction**	**[Maximum mark: 15]**
(a)	For $n = 5$,	
	$\binom{2(5)}{5} = \frac{10!}{5!\,5!} = \frac{10 \times 9 \times 8 \times 7 \times 6}{5 \times 4 \times 3 \times 2} = 252 < 256$	**M1A1**
	$= 2^{2(5)-2}$	**A1**
	Assume true for $n = k$, i.e., $\binom{2k}{k} < 2^{2(k)-2}$.	
	For $n = k + 1$, we prove that $\binom{2k+2}{k+1} < 2^{2k}$.	**M1**
	$\binom{2k+2}{k+1} = \frac{(2k+2)!}{(k+1)!\,(k+1)!} = \frac{(2k+2)(2k+1)(2k)!}{(k+1)(k+1)k!\,k!}$	**M1**
	$= \frac{2(2k+1)}{k+1}\binom{2k}{k} < \frac{2(2k+1)}{k+1}2^{2k-2}$	**A1**
	$= \left(4 - \frac{2}{k+1}\right)2^{2k-2} < 4 \times 2^{2k-2} = 2^{2k}$	
	Therefore, by mathematical induction, the ascertion is proved.	**A1 (A0 – when not clear)**
(b)	By the factor theorem and repeated roots, $P(1) = P'(1) = 0$.	**M1**
	$P(1) = -1 + s + (s^2 - 3) - (s + 2) + 2 = 0$ $\Longrightarrow s^2 - 4 = 0 \Longrightarrow s = \pm 2$	**A1**
	$P'(x) = -4x^3 + 3sx^2 + 2(s^2 - 3)x - (s + 2)$	**M1**
	$P'(1) = -4 + 3s + 2s^2 - 6 - s - 2 = 0$	**A1**
	$\Longrightarrow s^2 + s - 6 = (s - 2)(s + 3) = 0 \Longrightarrow s = 2 \; or - 3$	**A1**
	Thus, $s = 2$.	
(c)	When $s = 0$, $P(x) = -x^4 - 3x^2 - 2x + 2$.	
	$\alpha^2 + \beta^2 + \delta^2 + \gamma^2 = 0^2 - 2\left(\frac{-3}{-1}\right) = -6 < 0$ and $\frac{d}{a} < 0$, then $P(x)$ has two complex roots and two real roots.	**M1M1** **A1**
14	**Derivatives + Kinematics**	**[Maximum mark: 20]**
(a)	By observation, $d' = c$, $c' = b$ and $b' = a$.	**M1M1A1**
	Thus, d is displacement, c is velocity, b is acceleration and a is jerk.	**A1A1**

Qn	Suggested Solutions	Marks
(b)	(i)	
	$s=\frac{1-v}{2v-1} \Longrightarrow 2vs-s=1-v \Longrightarrow v=\frac{1+s}{1+2s}$	**M1A1**
	(ii) As $s\to\infty$, $v\to\frac{1}{2}$.	**A1**
	(iii) $a=\frac{dv}{dt}$ and $v=\frac{ds}{dt}$. Also, $\frac{dv}{dt}=\frac{dv}{ds}\frac{ds}{dt}$.	**M1A1**
	$a=\frac{dv}{dt}=\frac{(1+2s)-2(1+s)}{(1+2s)^2}\frac{ds}{dt}=-\frac{1+s}{(1+2s)^3}$	**M1A1**
(c)	(i)	
	$f'(x)=\frac{1}{3}x^{-\frac{2}{3}}-\frac{2}{3}x^{-\frac{1}{3}}=\frac{1}{3}x^{-\frac{2}{3}}\left(1-2x^{\frac{1}{3}}\right)$	**M1A1**
	Thus, $f'(x)=0 \Longrightarrow x=\frac{1}{8}$.	**A1**
	$f''(x)=-\frac{2}{9}x^{-\frac{5}{3}}+\frac{2}{9}x^{-\frac{4}{3}}=-\frac{2}{9}x^{-\frac{5}{3}}\left(1-x^{\frac{1}{3}}\right)$	**M1**
	and so $f''\left(\frac{1}{8}\right)<0$, which implies $\left(\frac{1}{8},\frac{1}{4}\right)$ is a maximum.	**R1A1**
	(ii) At $x=0$, f' does not exist. But $f(0)=0$, which can only mean that at $x=0$, we have a vertical tangent line.	**R1A1**
15	**Transformation of functions + Inverse Trigo**	**[Maximum mark: 10]**
(a)	$y=2\cos(3x-\pi)+1=-2\cos(3x)+1$	**(M1)**
	Thus, $a=2$, $b=\frac{1}{3}$, $h=0$ and $k=1$	**A1A1**
(b)	(i) period is $\frac{2\pi}{3}$ and so t = $\frac{2\pi}{3}\div 2=\frac{\pi}{3}$.	**M1A1**
	(ii)	
	$y=2\cos(3x-\pi)+1 \Longrightarrow x=2\cos(3y-\pi)+1$	**M1**
	$\frac{x-1}{2}=\cos(3y-\pi) \Longrightarrow y=\frac{1}{3}\cos^{-1}\left(\frac{x-1}{2}\right)+\frac{\pi}{3}$	**M1**
	Thus, $f^{-1}(x)=\frac{1}{3}\cos^{-1}\left(\frac{x-1}{2}\right)+\frac{\pi}{3}$.	**A1**
	(iii) Since the range of f is $-1\le y\le 3$, then it follows that $D_{f^{-1}}=\{x\in\mathbb{R}\mid -1\le x\le 3\}$.	**(R1)A1**

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ST JOSEPH'S INSTITUTION

JC2 PRELIMINARY EXAMINATION 2016

MATHEMATICS 7th July 2015

HIGHER LEVEL 2 hrs

PAPER 2

Thursday 0800 – 1000 hrs

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- **Section A**: Answer all questions showing working and answers in the spaces provided in the exam paper.
- **Section B**: Answer all questions using the foolscap paper provided.
- The use of a scientific or graphical calculator is permitted throughout.
- Ti-Nspire calculators must be in Press-to-Test mode and cleared of all previous data.
- TI-84+ graphical calculators must only have permitted Apps and be ram-cleared.
- A clean copy of the **Mathematics HL Formulae Booklet** is required for this paper.
- Unless otherwise stated in the question all numerical answers must be given exactly or to three significant figures.
- The maximum mark for this examination paper is *[120 marks]*.
- This question paper consists of 15 printed pages including the Cover Sheet.

FOR MARKER USE ONLY:

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	Q12	Q13	Q14	Q15	TOTAL
															/120

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graph display calculator should be supported by suitable working, for example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

SECTION A (60 marks)

Answer ***all*** *questions in the spaces provided. Working may be continued below the lines if necessary. Foolscap paper may be used for any additional working.*

1 [Maximum mark: 5]

By using an appropriate substitution , find $\int \frac{\tan(\ln y)}{y}\,dy, y > 0.$

2 [Maximum Mark: 4]

Only two international airlines fly daily into an airport. UN Air has 70 flights a day and IS Air has 65 flights a day. Passengers flying with UN Air have an 18% probability of losing their luggage and passengers flying with IS Air have a 23% probability of losing their luggage. You overhear someone in the airport complain about her luggage being lost.

Find the probability that she travelled with IS Air.

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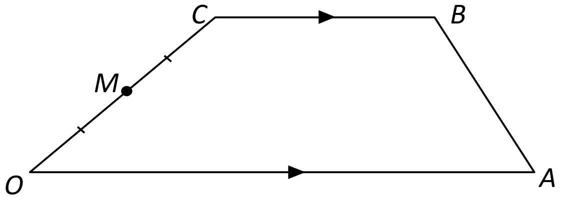
3 [Maximum Mark: 6]



$OABC$ is a trapezium such that CB is parallel to OA and $CB : OA = k : 1$, where k is a constant and $0 < k < 1$. It is given that $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OC} = \mathbf{c}$, and M is the midpoint of OC.

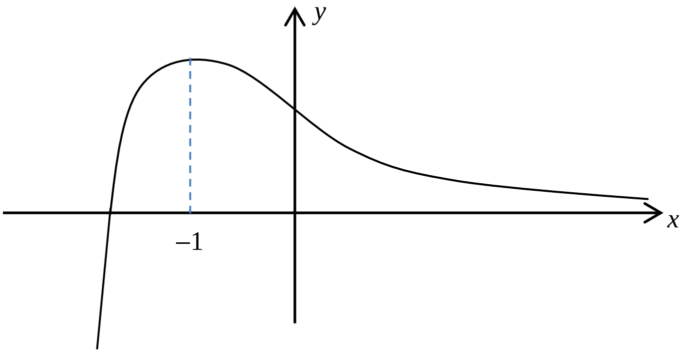
Find $\overrightarrow{OB}$ in terms of k, $\mathbf{a}$ and $\mathbf{c}$ and show that the area of triangle AMB can be written as $\lambda|\mathbf{a}\times\mathbf{c}|$, where λ is a constant to be found in terms of k.

4 [Maximum Mark: 6]

A curve *C* has the equation $p^2x^2 + y^2 = p^2$ where $p > 1$. The curve *C* is made up of C_1 when $y > 0$, and C_2 when $y \leq 0$.

(i) Sketch *C*, stating the coordinates of any points of intersection with the axes in terms of *p*. [2]

(ii) The graph of $y = \mathrm{f}(x) = p\left(x+2\right)\mathrm{e}^{-x}$ is given below. It has a maximum point at $x = -1$, a point of inflexion at $x = 0$ and a horizontal asymptote $y = 0$. It is also given that the gradient of $\mathrm{f}(x)$ is less than the gradient of graph C_2 for $-1 < x < 0$.



Sketch the graph of $y = \mathrm{f}'(x)$ on the same diagram as *C*. [3]

Hence, state the number of roots of the equation $p^2x^2 + \left[\mathrm{f}'(x)\right]^2 = p^2$. [1]

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5 [Maximum Mark: 9]

Do not use a graphic calculator in answering this question.

It is given that $\sin x > \dfrac{2x}{\pi}$ for $0 < x < \dfrac{\pi}{2}$.

(i) Explain why $\displaystyle\int_0^{\frac{\pi}{2}} \mathrm{e}^{-\sin x}\,\mathrm{d}x < \int_0^{\frac{\pi}{2}} \mathrm{e}^{-\frac{2x}{\pi}}\,\mathrm{d}x$. [2]

(ii) By making the substitution $u = \pi - x$, show that

$$\int_{\frac{\pi}{2}}^{\pi} \mathrm{e}^{-\sin x}\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \mathrm{e}^{-\sin u}\,\mathrm{d}u .$$ [2]

(iii) Hence show that $\displaystyle\int_0^{\pi} \mathrm{e}^{-\sin x}\,\mathrm{d}x < \frac{\pi}{\mathrm{e}}(\mathrm{e}-1)$. [5]

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6 [Maximum Mark: 7]

If $z=\omega$ is the solution of the equation $z^3=1$ which has the smallest positive argument, show that $1+\omega+\omega^2=0$. [2]

Solve the system of simultaneous equations

$$x+y+z=\ \ 3$$

$$x+\omega y+\omega^2 z=-3$$

$$x+\omega^2 y+\omega z=-3$$

giving your answer in numerical form (that is, **not** in terms of ω). [5]

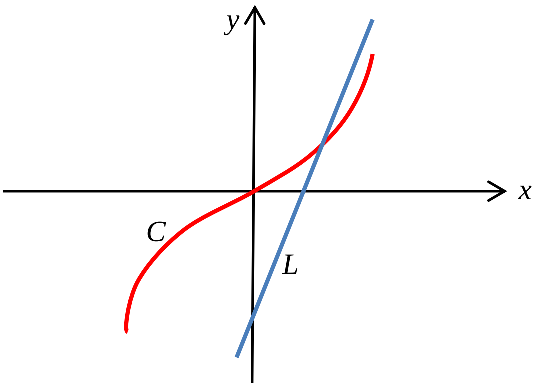
7 [Maximum Mark: 4]

A group of 12 people consists of 6 married couples.

The group is going on a flight and is assigned to sit in three distinct rows of four seats each. Find the number of ways in which the 12 people can be arranged if each row has at least 1 woman.

8   [Maximum mark: 7]

The diagram shows the curve C with equation $y = 2\sin^{-1} x$ and the line L with equation $y = \frac{8\pi}{3} x - \pi$. C and L intersect at the point where $x = \frac{1}{2}$.



The region S is defined by $y \le 2\sin^{-1} x$, $y \ge \frac{8\pi}{3} x - \pi$ and $x \ge 0$.

Find the exact volume of the solid obtained when S is rotated through 2π radians about the y-axis.

9 [Maximum mark: 5]

Jack and Jill play a game, by throwing a die in turn. If the die shows a 1, 2, 3 or 4, the player who threw the die wins the game. If the die shows a 5 or 6, the other player has the next throw. Jack plays first and the game continues until there is a winner.

(a) Write down the probability that Jack wins on his first throw. [1]
(b) Calculate the probability that Jill wins on her first throw. [1]
(c) Calculate the probability that Jack wins the game. [3]

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10 [Maximum mark: 7]

Solve the equation $z^5 + 32 = 0$, expressing your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$. [3]

z_1, z_2 and z_3 are three of the roots of $z^5 + 32 = 0$ such that $0 < \arg z_1 < \arg z_2 < \arg z_3 \leq \pi$.

The points *A* and *B* represent the roots z_1 and z_3 respectively in an Argand diagram. The line segment BA' is obtained by rotating the line segment *BA* through $\frac{\pi}{2}$ clockwise about the point *B*. Find the real part of the complex number represented by point A', giving your answer in exact trigonometric form. [4]

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*Do **NOT** write solutions on this page.*

SECTION B (60 marks)

*Answer **all** questions using the foolscap papers provided. Please start each question on a new page.*

11 [Maximum Mark: 15]

The function f is defined by

$$\mathrm{f} : x \mapsto \frac{1}{2-x^2},\ x \in \mathbb{R},\ x \leq 0,\ x \neq -\sqrt{2}$$

(i) Find the inverse function f^{-1} including its domain. [3]

(ii) Sketch the graphs of f and f^{-1} on the same diagram, giving the exact equation of any asymptote(s) and showing clearly the relationship between the two graphs. Hence find the value(s) of x for which $\mathrm{f} \circ \mathrm{f}\left(x\right) = x$. [6]

The function g is defined by

$$\mathrm{g} : x \mapsto \frac{1}{\sqrt{x}},\ x \in \mathbb{R},\ x > 0.$$

(iii) Find $\left(\mathrm{g} \circ \mathrm{f}\right)\left(x\right)$ and its domain. [2]

(iv) Find the exact value of $\int_{-\sqrt{2}}^{0} \left(\mathrm{g} \circ \mathrm{f}\right)\left(x\right)\ \mathrm{d}x$. [2]

(v) Find the derivative of $\frac{h(x)}{g(x)}$ at $x = 2$, given that h is a function in x such that $h(2) = 0$ and $h'(2) = 1.$ [2]

TURN OVER

*Do **NOT** write solutions on this page.*

12 [Maximum Mark: 10]

The life in the office buildings of a small city have occasional breakdowns. The breakdowns at any given time are independent of one another and can be modelled using a Poisson distribution with mean 0.2 per day.

(a) Determine the probability that there will be exactly four breakdowns during the month of June (June has 30 days). [2]

(b) Determine the probability that there are more than 3 breakdowns during the month of June. [2]

(c) Find the probability that the first breakdown in June occurs on 3rd of June. [2]

(d) It costs 1850 euros to service the lifts when they have breakdowns.
Find the expected cost of servicing lifts for the month of June. [1]

(e) Determine the minimum value of n if the probability that there will be no breakdowns in at most 2 out of the first n days in June is less than 0.01. [3]

13 [Maximum mark: 12]

Planes p_1 and p_2 have the following equations:

$$p_1: \quad 3x-2y+6z=2$$
$$p_2: \quad \mathbf{r}=\left(1+2s+2t\right)\mathbf{i}+\left(-2-3t\right)\mathbf{j}+\left(-s-2t\right)\mathbf{k}\,, \quad s,t\in\mathbb{R}$$

(i) Show that p_1 and p_2 are parallel and distinct planes. Hence find the shortest distance between these two planes. [6]

(ii) The line l has equation $\mathbf{r}=(5+\beta\lambda)\mathbf{i}+(-5+3\lambda)\mathbf{j}+(\alpha+\lambda)\mathbf{k}$, where $\lambda\in\mathbb{R}$ and α, β are real constants, $\alpha\in\mathbb{Z}$.

(a) Find conditions on α and β such that l only intersects one plane but not the other. [3]

(b) The angle between l and p_1 is 22°. Find the two possible values of β. [3]

*Do **NOT** write solutions on this page.*

14 [Maximum Mark: 11]

When $v>0$, the motion of a particle can be described by the equation $\frac{\mathrm{d}v}{\mathrm{d}x}=-\frac{1+v^2}{50}$ where x metres is the displacement from the origin, O.

Given that $\frac{\mathrm{d}x}{\mathrm{d}v}=\frac{1}{\frac{\mathrm{d}v}{\mathrm{d}x}}$, write down $\frac{\mathrm{d}x}{\mathrm{d}v}$ in terms of v.

Given that $v=10$ when $x=0$, find x in terms of v.

Hence show that $v=\frac{10-\tan\frac{x}{50}}{1+10\tan\frac{x}{50}}$.

15 [Maximum mark: 12]

(a) Show that $\cos(A+B)+\cos(A-B)=2\cos A\cos B$. [2]

(b) Let $T_n(x)=\cos(n\arccos x)$ where x is a real number, $x\in[-1,1]$ and $n\in\mathbb{Z}^+$.

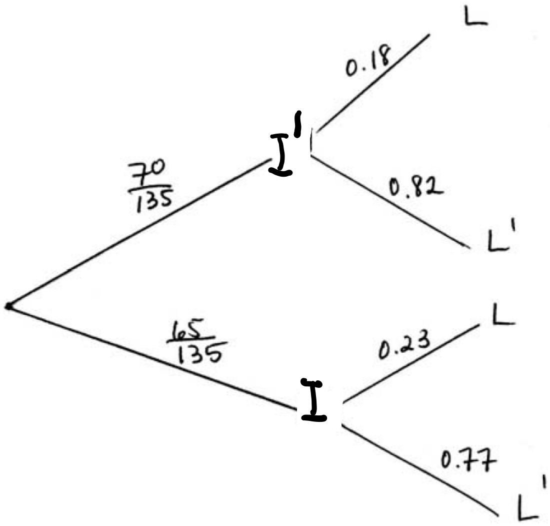
(i) Find $T_1(x)$. [1]

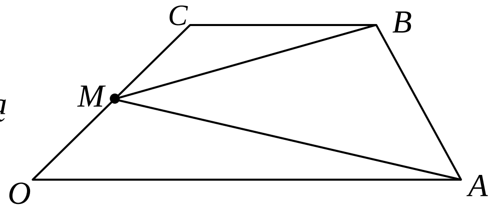
(ii) Show that $T_2(x)=2x^2-1$. [2]

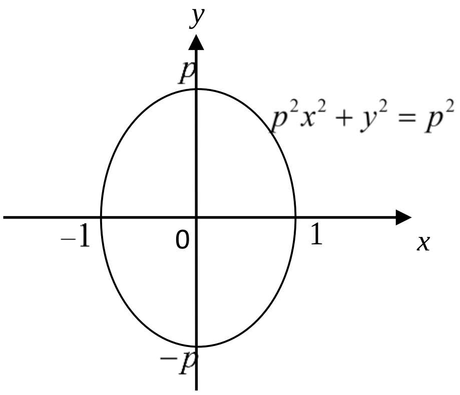
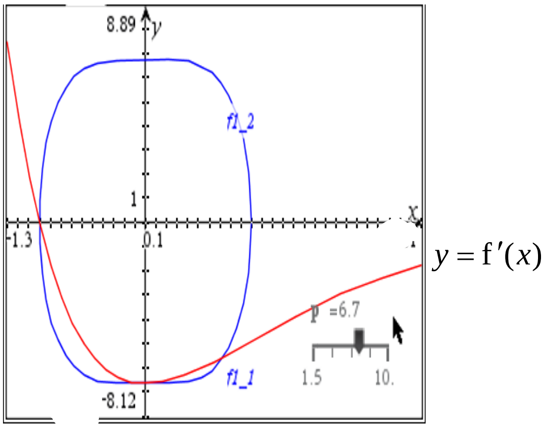
(c) Use the result in part (a) to show that $T_{n+1}(x)+T_{n-1}(x)=2xT_n(x)$. [4]

(d) Show that $\sum_{n=2}^{10}\left(T_{n+1}\left(\frac{1}{2}\right)+T_{n-1}\left(\frac{1}{2}\right)\right)=\sum_{n=2}^{10}\left(\cos\frac{n\pi}{3}\right)=k$, where k is a real number to be determined. [3]

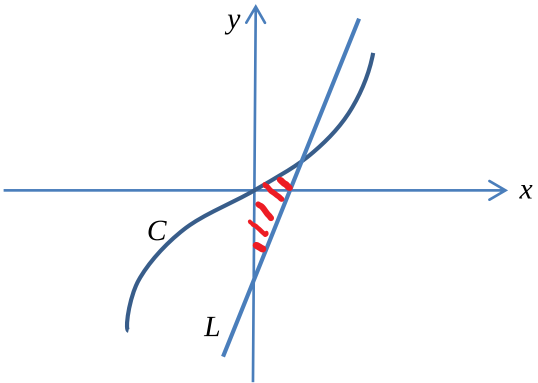
End of paper

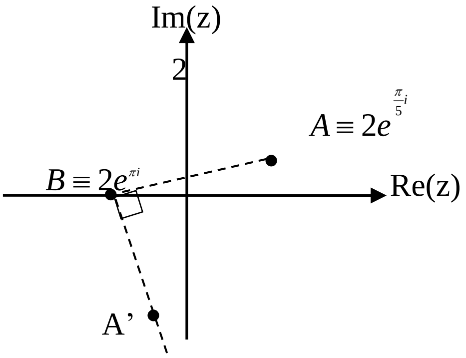
Qn	Solution	Mark
1	Let $u = \ln y \Rightarrow \frac{du}{dy} = \frac{1}{y}$ $\int \frac{\tan(\ln y)}{y} dy$ $= \int \tan(u) du$ $= \int \frac{\sin u}{\cos u} du$ $= -\ln\lvert\cos u\rvert + c$ $= -\ln\lvert\cos(\ln y)\rvert + c$	A1 M1A1 M1 A1 TOT=5
2		M1
	Let I be the event that a passenger travlled with IS Air and L be the event that a pasenger lost her luggage. $P(I \mid L) = \frac{P(I \cap L)}{P(L)}$ $= \frac{0.23 \times \frac{65}{135}}{0.18 \times \frac{70}{135} + 0.23 \times \frac{65}{135}}$ $= \frac{299}{551}$ $(or\ 0.543)$	A1 A1 A1 TOT=4

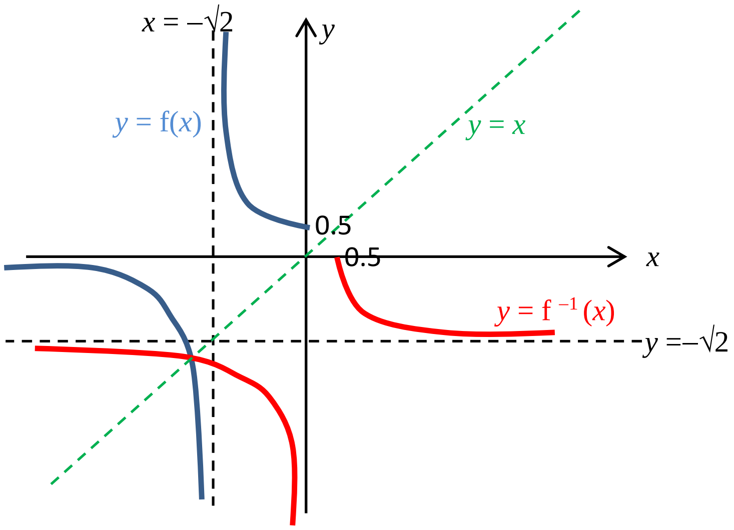
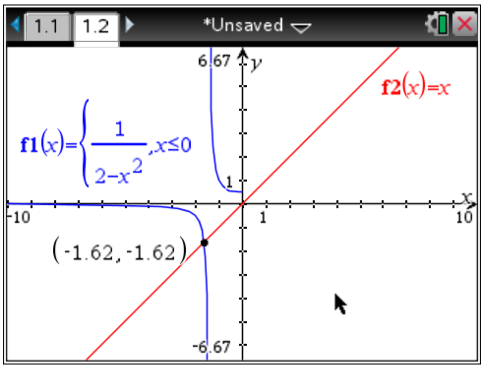
3	$\overrightarrow{CB}=k\underset{\sim}{a}$ $\overrightarrow{OB}=\overrightarrow{OC}+\overrightarrow{CB}=\underset{\sim}{c}+k\underset{\sim}{a}$ $\overrightarrow{OM}=\frac{1}{2}\underset{\sim}{c}$ 	**A1**
	Method 1	
	Area of trapezium OCBA $=\frac{1}{2}\left(\lvert CB\rvert+\lvert OA\rvert\right)\lvert OC\rvert\sin\theta$ $=\frac{1}{2}\left(k\lvert\underset{\sim}{a}\rvert+\lvert\underset{\sim}{a}\rvert\right)\lvert\underset{\sim}{c}\rvert\sin\theta$ $=\frac{1}{2}(k+1)\lvert\underset{\sim}{a}\rvert\lvert\underset{\sim}{c}\rvert\sin\theta$ $=\frac{1}{2}(k+1)\lvert\underset{\sim}{a}\times\underset{\sim}{c}\rvert$	
	Area of triangle *AMB* = Area of trapezium *OCBA* - Area of triangle *AMO* - Area of triangle *CMB*	M1
	$=\frac{1}{2}(k+1)\lvert(\underset{\sim}{a}\times\underset{\sim}{c})\rvert-\frac{1}{2}\left\lvert\left(\frac{1}{2}\underset{\sim}{c}\times\underset{\sim}{a}\right)\right\rvert-\frac{1}{2}\left\lvert\left(k\underset{\sim}{a}\times\frac{1}{2}\underset{\sim}{c}\right)\right\rvert$	A1A1A1
	$=\frac{1}{2}(k+1)\lvert(\underset{\sim}{a}\times\underset{\sim}{c})\rvert-\frac{1}{4}\lvert(\underset{\sim}{a}\times\underset{\sim}{c})\rvert-\frac{1}{4}k\lvert(\underset{\sim}{a}\times\underset{\sim}{c})\rvert$	A1
	$=\frac{1}{4}(1+k)\lvert(\underset{\sim}{a}\times\underset{\sim}{c})\rvert$	
	Method 2	
	$\overrightarrow{MB}=\overrightarrow{OB}-\overrightarrow{OM}=\underset{\sim}{c}+k\underset{\sim}{a}-\frac{1}{2}\underset{\sim}{c},\ \overrightarrow{MA}=\overrightarrow{OA}-\overrightarrow{OM}=\underset{\sim}{a}-\frac{1}{2}\underset{\sim}{c}$	
	Area of triangle *AMB* $=\frac{1}{2}\lvert\overrightarrow{MB}\times\overrightarrow{MA}\rvert=\frac{1}{2}\left\lvert\left(\underset{\sim}{c}+k\underset{\sim}{a}-\frac{1}{2}\underset{\sim}{c}\right)\times\left(\underset{\sim}{a}-\frac{1}{2}\underset{\sim}{c}\right)\right\rvert$	M1A1A1
	$=\frac{1}{2}\left\lvert\left(k\underset{\sim}{a}+\frac{1}{2}\underset{\sim}{c}\right)\times\left(\underset{\sim}{a}-\frac{1}{2}\underset{\sim}{c}\right)\right\rvert$	M1
	$=\frac{1}{2}\left\lvert k(\underset{\sim}{a}\times\underset{\sim}{a})-\frac{1}{2}k(\underset{\sim}{a}\times\underset{\sim}{c})+\frac{1}{2}(\underset{\sim}{c}\times\underset{\sim}{a})-\frac{1}{4}(\underset{\sim}{c}\times\underset{\sim}{c})\right\rvert$	
	$=\frac{1}{2}\left\lvert-\frac{1}{2}k(\underset{\sim}{a}\times\underset{\sim}{c})-\frac{1}{2}(\underset{\sim}{a}\times\underset{\sim}{c})\right\rvert\quad(\because\underset{\sim}{a}\times\underset{\sim}{a}=\underset{\sim}{c}\times\underset{\sim}{c}=\underset{\sim}{0},\underset{\sim}{c}\times\underset{\sim}{a}=-\underset{\sim}{a}\times\underset{\sim}{c})$	A1
	$=\frac{1}{4}(1+k)\lvert(\underset{\sim}{a}\times\underset{\sim}{c})\rvert$	TOT=6

4(i)	$p^2x^2+y^2=p^2$ where $p>1$ 	G1(shape) G1 (x, y intercepts)
(ii)		G1 (shape) G1 (x intercept at x=-1) G1 (min point at x=0)
	No. of roots = no. of intersection points between both graphs = 3	A1 TOT=6
5(i)	Given that $\sin x > \frac{2x}{\pi} \Rightarrow -\sin x < -\frac{2x}{\pi}$	R1
	$\Rightarrow \quad \mathrm{e}^{-\sin x} < \mathrm{e}^{-\frac{2x}{\pi}}$ since $y=\mathrm{e}^x$ is increasing	R1
	$\Rightarrow \quad \int_0^{\frac{\pi}{2}} e^{-\sin x}dx < \int_0^{\frac{\pi}{2}} e^{-\frac{2x}{\pi}}dx$	
(ii)	Let $u=\pi-x \Rightarrow \frac{du}{dx}=-1$	
	$\int_{\frac{\pi}{2}}^{\pi} e^{-\sin x}dx = \int_{\frac{\pi}{2}}^{0} e^{-\sin(\pi-u)}du$	A1
	$= \int_0^{\frac{\pi}{2}} e^{-\sin u}du \quad (\text{since } \sin(\pi-u) = -\sin u)$	A1

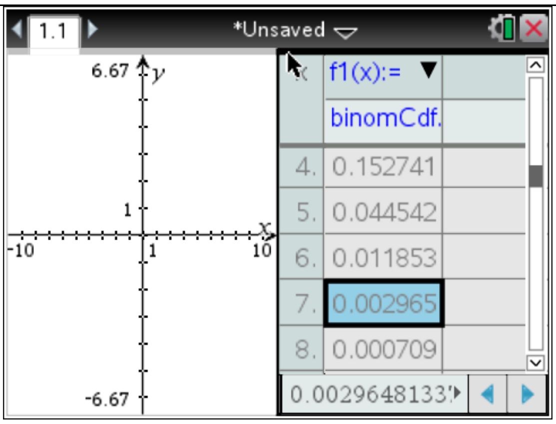
(iii)	$\int_0^{\pi} e^{-\sin x}dx=\int_0^{\frac{\pi}{2}} e^{-\sin x}dx+\int_{\frac{\pi}{2}}^{\pi} e^{-\sin x}dx$	M1
	$=2\int_0^{\frac{\pi}{2}} e^{-\sin x}dx$ $\left(\text{from the result in (ii)}\right)$	A1
	$<2\int_0^{\frac{\pi}{2}} e^{-\frac{2x}{\pi}}dx\left(\text{from the result in (i)}\right)$	A1
	$=2\left[\frac{-\pi}{2}e^{-\frac{2x}{\pi}}\right]_0^{\frac{\pi}{2}}$	M1A1
	$=\frac{\pi(e-1)}{e}$	TOT=9
6	Since $z=\omega$ is a solution of $z^3=1$, $\omega^3=1\Rightarrow(\omega-1)\left(1+\omega+\omega^2\right)=0$.	M1
	Since ω is the solution with the smallest positive argument, $\omega\neq1$. Hence, $1+\omega+\omega^2=0$	R1
	$x+y+z=3\quad-(1)$ $x+\omega y+\omega^2 z=-3\quad-(2)$ $x+\omega^2 y+\omega z=-3\quad-(3)$ $(1)+(2)+(3)$,	M1
	$3x+\left(1+\omega+\omega^2\right)y+\left(1+\omega+\omega^2\right)z=-3$ $\Rightarrow 3x=-3\ \left(\because 1+\omega+\omega^2=0\right)$	A1
	$\Rightarrow x=-1$ $y+z=4\quad-(4)$ $\omega y+\omega^2 z=-2\quad-(5)$ $\omega^2 y+\omega z=-2\quad-(6)$ $\omega\times(5)-(6)$, $\omega^3 z-\omega z=-2\omega+2$	
	$\Rightarrow z(1-\omega)=2(1-\omega)\ \left(\because \omega^3=1\right)$	M1 A1
	$\Rightarrow z=2\ \left(\because \omega\neq1\right)$	A1
	$\Rightarrow y=2$	TOT=7
7	**Method 1**	
	W _ _ _ W _ _ _ W _ _ _ W _ _ _	M1
	Required no. of ways $=\left({}^6C_3\right)(3!)(9!)=435456000$	A3
		TOT=4

	Method 2 No. of ways that 12 people are seated in three distinct rows of four seats without restriction $=12!$ No. of ways that 12 people are seated with 0 female in one of the rows $=\left({}^{3}C_{1}\right)\left({}^{6}C_{4}\right)(8!)(4!)$ Hence required no. of ways $=12!-\left({}^{3}C_{1}\right)\left({}^{6}C_{4}\right)(8!)(4!)$ $=435456000$	
8	C: $y=2\sin^{-1}x$ L: $y=\frac{8\pi}{3}x-\pi$ C & L intersect at $\left(\frac{1}{2},\frac{\pi}{3}\right)$ And y-intercept of L is -π.  Volume obtained when S is rotated 2π radians about the y-axis	
	$=\frac{1}{3}\pi\left(\frac{1}{2}\right)^{2}\left(\frac{\pi}{3}+\pi\right)-\pi\int_{0}^{\frac{\pi}{3}}\left[\sin\left(\frac{y}{2}\right)\right]^{2}dy$	M1A1 M1A1
	$=\frac{\pi}{12}\left(\frac{4\pi}{3}\right)-\pi\int_{0}^{\frac{\pi}{3}}\frac{1-\cos y}{2}\,dy$	
	$=\frac{\pi^{2}}{9}-\frac{\pi}{2}\;y-\sin y\,{}_{0}^{\frac{\pi}{3}}$	M1A1
	$=\frac{\pi^{2}}{9}-\frac{\pi}{2}\left[\frac{\pi}{3}-\sin\frac{\pi}{3}\right]$	
	$=\frac{\pi^{2}}{9}-\frac{\pi^{2}}{6}+\frac{\pi}{2}\left[\frac{\sqrt{3}}{2}\right]$	
	$=\frac{\pi\sqrt{3}}{4}-\frac{\pi^{2}}{18}$	A1
		TOT=7

9(a)	P(Jack wins on his first throw) $=\frac{2}{3}$	A1
(b)	P(Jill wins on her first throw) $=\frac{1}{3}\times\frac{2}{3}=\frac{2}{9}$	A1
(c)	$\mathrm{P}\left(\text{Jack wins the game}\right)=\frac{2}{3}+\frac{1}{3}\times\frac{1}{3}\times\frac{2}{3}+\frac{1}{3}\times\frac{1}{3}\times\frac{1}{3}\times\frac{1}{3}\times\frac{2}{3}+\dots$ $=\frac{2}{3}\left[1+\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^4+\dots\right]$ $=\frac{2}{3}\left(\frac{1}{1-\frac{1}{9}}\right)$ $=\frac{3}{4}$	M1 A1 A1 TOT=5
10	$z^5+32=0$ $z^5=-32=32e^{(\pi+2k\pi)i}$ $z=2e^{\left(\frac{\pi}{5}+\frac{2k\pi}{5}\right)i},\quad k=0,\pm1,\pm2$ $=2e^{-\frac{3\pi}{5}i},2e^{-\frac{\pi}{5}i},2e^{\frac{\pi}{5}i},2e^{\frac{3\pi}{5}i},2e^{\pi i}$	M1 A1A1
	 Let the complex number represented by A’ be *z*. BA rotates 90° about B to get BA’: $z-2e^{i\pi}=(-i)\left(2e^{i\frac{\pi}{5}}-2e^{i\pi}\right)$ $z=(-2)-i\left[2e^{\frac{1}{5}\pi i}-(-2)\right]$ since $e^{i\pi}=-1$ $z=-2-i\left[2\cos\frac{\pi}{5}+2i\sin\frac{\pi}{5}+2)\right]$ Real part = $-2-2i^2\sin\frac{\pi}{5}=-2+2\sin\frac{\pi}{5}$	M1A1 A1 A1 TOT=7

11(i)	Let $y=\mathrm{f}(x) \Rightarrow y=\frac{1}{2-x^2}$ $\Rightarrow x^2=2-\frac{1}{y}$ $\Rightarrow x=-\sqrt{2-\frac{1}{y}}$ since $x\le 0$ Therefore $\mathrm{f}^{-1}(x)=-\sqrt{2-\frac{1}{x}}$, $x\in(-\infty,0)\cup\left[\frac{1}{2},\infty\right)$	M1 A1 A1
(ii)	 **Method 1:** $(\mathrm{f}\circ\mathrm{f})(x)=x$ $\Rightarrow \mathrm{f}(x)=\mathrm{f}^{-1}(x)$ $\Rightarrow \mathrm{f}(x)=x$ By GDC, $x=-1.62$ (3 sf) 	G2 (f) G2 (f^{-1}) M1A1

	Method 2: $(\mathrm{f}\circ \mathrm{f})(x)=x$ $\Rightarrow \mathrm{f}(x)=\mathrm{f}^{-1}(x)$ $\Rightarrow \mathrm{f}(x)=x$ $\Rightarrow \frac{1}{2-x^2}=x$ $\Rightarrow x^3-2x+1=0$ $\Rightarrow (x-1)\left(x^2+x-1\right)=0$ $\Rightarrow x=1,\ x=\frac{-1\pm\sqrt{1+4}}{2}=\frac{-1\pm\sqrt{5}}{2}$ Since x is negative, $x=\frac{-1-\sqrt{5}}{2}$.	M1A1
(iii)	$(\mathrm{g}\circ \mathrm{f})(x)=g\left(\frac{1}{2-x^2}\right)=\sqrt{2-x^2}$, $D_{gf=}$ $\left(-\sqrt{2},0\right]$	A1A1
(iv)	$\int_{-\sqrt{2}}^{0}(\mathrm{g}\circ \mathrm{f})(x)\ \mathrm{d}x = \frac{1}{4}\times$ Area of a circle with radius $\sqrt{2} = \frac{1}{4}\left(\pi\left(\sqrt{2}\right)^2\right)=\frac{\pi}{2}$	M1A1
(v)	$\frac{d}{dx}\left(\frac{h(x)}{g(x)}\right)=\frac{g(x)h'(x)-h(x)g'(x)}{\left(g(x)\right)^2}$ $\frac{d}{dx}\left(\frac{h(x)}{g(x)}\right)\Bigg\vert_{x=2}=\frac{g(2)h'(2)-h(2)g'(2)}{\left(g(2)\right)^2}=\frac{h'(2)}{g(2)}=\sqrt{2}$	M1A1 TOT=15
12(a)	Let X be the number of breakdowns in 30 days. $X \sim P_o(0.2\times 30)$ *i.e.* $X \sim P_o(6)$ $P(X=4)=0.13385=0.134\ (3sf)$	M1 A1
(b)	$P(X>3)=P(X\geq 4)=0.849$	M1A1
(c)	Let Y be the number of breakdowns in a day. $Y \sim P_o(0.2)$ $\left[P(Y=0)\right]^2 P(Y\geq 1)=0.122$	M1A1
(d)	Expected cost = $6\times 1850=11100$	A1
(e)	Let W be the no. of days, out of n, whereby there is no breakdowns. $W \sim B\left(n,e^{-0.2}\right)$ $P(W\leq 2)<0.01$ when $n=6$, $P(W\leq 2)=0.0118>0.01$ when $n=7$, $P(W\leq 2)=0.002965<0.01$ By GDC, least n is 7.	M1 M1 A1 TOT=10

	 f1(x):= binomCdf. 4. 0.152741 5. 0.044542 6. 0.011853 7. 0.002965 8. 0.000709 0.0029648133	
13(i)	**Proving parallel & distinct planes** Normal vectors of p_1 and p_2 are respectively $\mathbf{n}_1=\begin{pmatrix}3\\-2\\6\end{pmatrix}$ $\mathbf{n}_2=\begin{pmatrix}2\\0\\-1\end{pmatrix}\times\begin{pmatrix}2\\-3\\-2\end{pmatrix}$ $=\begin{pmatrix}-3\\2\\-6\end{pmatrix}$ Since $\mathbf{n}_1=-\mathbf{n}_2$, the normal vectors are parallel and hence the planes are parallel as well. Furthermore, $3(1)-2(-2)+6(0)=7\neq 2$, so the point $(1,-2,0)$ is on p_2 but not on p_1. Hence the two planes are distinct.	M1 A1 A1
	Finding shortest distance **Method 1:** Express equations of both planes in scalar product form $\mathbf{r.n}=d$ where $\mathbf{n}$ is a unit normal vector and d is the shortest distance between the origin and the plane. p_1: $\mathbf{r}.\begin{pmatrix}3\\-2\\6\end{pmatrix}=2$ $\Rightarrow$ $\mathbf{r}.\frac{1}{\sqrt{9+4+36}}\begin{pmatrix}3\\-2\\6\end{pmatrix}=\frac{2}{\sqrt{9+4+36}}$ $\Rightarrow$ $\mathbf{r.n}=\frac{2}{7}$ where $\mathbf{n}=\frac{1}{7}\begin{pmatrix}3\\-2\\6\end{pmatrix}$ p_2: $\mathbf{r}.\begin{pmatrix}-3\\2\\-6\end{pmatrix}=\begin{pmatrix}1\\-2\\0\end{pmatrix}.\begin{pmatrix}-3\\2\\-6\end{pmatrix}=-7$ $\Rightarrow$ $\mathbf{r}.\frac{-1}{\sqrt{9+4+36}}\begin{pmatrix}-3\\2\\-6\end{pmatrix}=\frac{(-1)(-7)}{\sqrt{9+4+36}}$ $\Rightarrow$ $\mathbf{r.n}=1$ where $\mathbf{n}=\frac{1}{7}\begin{pmatrix}3\\-2\\6\end{pmatrix}$ Since both planes are on the same side of the origin, shortest distance between both planes $=1-\frac{2}{7}$ $=\frac{5}{7}$	A1 M1A1

	Method 2:	
	Since $\frac{3}{2}(0)-(-1)+3(0)=1$, the point $A(0,-1,0)$ is on p_1.	
	The point $B(1,-2,0)$ is on p_2.	
	$\overrightarrow{AB}=\begin{pmatrix}1\\-2\\0\end{pmatrix}-\begin{pmatrix}0\\-1\\0\end{pmatrix}=\begin{pmatrix}1\\-1\\0\end{pmatrix}$	A1
	Shortest distance between planes $=\left\lvert\overrightarrow{AB}\cdot\hat{\mathbf{n}}_1\right\rvert$	M1A1
	$=\frac{1}{\sqrt{9+4+36}}\left\lvert\begin{pmatrix}1\\-1\\0\end{pmatrix}.\begin{pmatrix}3\\-2\\6\end{pmatrix}\right\rvert$	
	$=\frac{\lvert 3+2+0\rvert}{7}$	
	$=\frac{5}{7}$	
(ii)(a)	l: $\mathbf{r}=\begin{pmatrix}5\\-5\\\alpha\end{pmatrix}+\lambda\begin{pmatrix}\beta\\3\\1\end{pmatrix},\ \lambda\in\mathbb{R}$	
	Since the planes are parallel, for l to intersect one but not the other, we need l to be contained in exactly one plane at one time.	
	p_1: $\mathbf{r}.\begin{pmatrix}3\\-2\\6\end{pmatrix}=\begin{pmatrix}0\\-1\\0\end{pmatrix}.\begin{pmatrix}3\\-2\\6\end{pmatrix}=2$	
	p_2: $\mathbf{r}.\begin{pmatrix}3\\-2\\6\end{pmatrix}=\begin{pmatrix}1\\-2\\0\end{pmatrix}.\begin{pmatrix}3\\-2\\6\end{pmatrix}=7$	
	l parallel to p_1 and p_2 $\Rightarrow\begin{pmatrix}\beta\\3\\1\end{pmatrix}.\begin{pmatrix}3\\-2\\6\end{pmatrix}=0$	M1
	$\Rightarrow\beta=0$	A1
	l contained in p_1 $\Rightarrow\begin{pmatrix}5\\-5\\\alpha\end{pmatrix}.\begin{pmatrix}3\\-2\\6\end{pmatrix}=2$	
	$\Rightarrow\alpha=\frac{2-15-10}{6}$	
	$\Rightarrow\alpha=-\frac{23}{6}$ (reject $\because\alpha\in\mathbb{Z}$)	
	l contained in p_2 $\Rightarrow\begin{pmatrix}5\\-5\\\alpha\end{pmatrix}.\begin{pmatrix}3\\-2\\6\end{pmatrix}=7$	
	$\Rightarrow\alpha=\frac{7-15-10}{6}$	A1
	$\Rightarrow\alpha=-3$	
	$\therefore\ \beta=0,\ \alpha=-3.$	

13(b)	$\sin 22° = \left\| \dfrac{\begin{pmatrix}\beta\\3\\1\end{pmatrix}\cdot\begin{pmatrix}3\\-2\\6\end{pmatrix}}{\sqrt{\beta^2+9+1}\times\sqrt{9+4+36}} \right\|$ $0.37461 = \dfrac{3\|\beta\|}{7\sqrt{\beta^2+10}}$ Using GDC, $\beta = -5.69$ or 5.69 (3 s.f.)	M1 A1 A1 TOT=12
14	$\dfrac{dv}{dx} = -\dfrac{1+v^2}{50}$ $x = \int -\dfrac{50}{1+v^2}\ \mathrm{d}v$ $x = -50\tan^{-1}(v) + c$ When $x = 0, v = 10,$ $\Rightarrow 0 = -50\tan^{-1}(10) + c$ $\Rightarrow c = 50\tan^{-1}(10)$ $x = 50\left[\tan^{-1}(10) - \tan^{-1}(v)\right]$ $\Rightarrow \tan^{-1}(v) = \tan^{-1}(10) - \dfrac{x}{50}$ $\Rightarrow v = \tan\left(\tan^{-1}(10) - \dfrac{x}{50}\right)$ Applying additional formula, $\Rightarrow v = \dfrac{10 - \tan\dfrac{x}{50}}{1 + 10\tan\dfrac{x}{50}}$ (shown)	A1 M1 A1 M1 A1 A1 A1 M1A1 M1 A1 TOT=11

15(a)	$\cos(A+B)=\cos A\cos B-\sin A\sin B-(1)$ $\cos(A-B)=\cos A\cos B+\sin A\sin B-(2)$ (1)+(2), $\cos\left(A+B\right)+\cos\left(A-B\right)=2\cos A\cos B.$ (shown)	M1A1
(b)(i)	$T_1\left(x\right)=\cos\left(\cos^{-1}x\right)=x$	A1
(ii)	$T_2\left(x\right)=\cos\left(2\arccos x\right)=2\cos^2\left(\cos^{-1}x\right)-1=2x^2-1$ (shown)	M1A1
(c)	$T_{n+1}\left(x\right)+T_{n-1}\left(x\right)$ $=\cos\left(\left(n+1\right)\arccos x\right)+\cos\left(\left(n-1\right)\arccos x\right)$ $=2\cos A\cos B,$ where $A=n\arccos x, B=\arccos x$ $=2\cos\left(\arccos x\right)\cos\left(n\arccos x\right)$ $=2xT_n\left(x\right)$ (shown)	A1 M1 A1 A1
(d)	$x=\frac{1}{2}\Rightarrow\cos^{-1}(x)=\frac{\pi}{3}$ $\therefore T_n\left(\frac{1}{2}\right)=\cos\left(n\cos^{-1}\left(\frac{1}{2}\right)\right)=\cos\frac{n\pi}{3}$ Since $T_{n+1}\left(\frac{1}{2}\right)+T_{n-1}\left(\frac{1}{2}\right)=2\left(\frac{1}{2}\right)T_n\left(\frac{1}{2}\right)=\cos\frac{n\pi}{3},$ $\therefore\sum_{n=2}^{10}\left(T_{n+1}\left(\frac{1}{2}\right)+T_{n-1}\left(\frac{1}{2}\right)\right)=\sum_{n=2}^{10}\left(\cos\frac{n\pi}{3}\right)=-2$ (by GDC)	A1 M1 A1 TOT=12

STUDENT NAME: ______________________________

TEACHER INITIALS: _________

CANDIDATE SESSION NUMBER

0	2	5	0	1	2				

EXAMINATION CODE

8	8	1	7	-	7	2	0	1

ST JOSEPH'S INSTITUTION
JC2 PRELIMINARY EXAMINATION 2017

MATHEMATICS

HIGHER LEVEL

PAPER 1

Friday

30th June 2017

2 hours

1400 – 1600 hrs

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- **Section A**: Answer all questions showing working and answers in the spaces provided in the exam paper.
- **Section B**: Answer all questions using the foolscap paper provided.
- The use of calculators is **not** permitted in this paper.
- A clean copy of the **Mathematics HL Formulae Booklet** is required for this paper.
- Unless otherwise stated in the question all numerical answers are to be given exactly.
- The maximum mark for this examination paper is *[100 marks]*.
- This question paper consists of **11** printed pages including the Cover Sheet.

FOR MARKER USE ONLY:

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	TOTAL
											/100

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are advised to show all working.

SECTION A (50 marks)

1 [**Maximum mark: 5**]

Find the value of a and of b such that the following system of linear equations yields infinitely many solutions.

$$\begin{cases} 4z + 2y + x = 5 \\ 2z \qquad\; - x = b \\ az + y \;\; + x \; = -10 \end{cases}$$

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2 [Maximum mark: 5]

Using mathematical induction, prove the following statement for $n \in \mathbb{Z}^+$:

$$(\cos x)(\cos 2x)(\cos 4x)(\cos 8x) \cdots (\cos 2^{n-1}x) = \frac{\sin 2^n x}{2^n \sin x}$$

for all $x \in \mathbb{R}$ such that $\sin x \neq 0$.

3 [Maximum mark: 9]

Determine the real coefficients a and b of the cubic polynomial $2x^3 + ax^2 + bx + 50$ given that $2 - i$ is one of its roots where $i^2 = -1$. **[7]**

Hence or otherwise, find the zeros of $f(x) = 50x^3 + bx^2 + ax + 2$. **[2]**

4 [Maximum mark: 5]

The discrete random variable X has the following probability distribution function:

$$P(X = k) = ar^{k-1}, \qquad k = 1, 2, 3, \ldots$$

Find a and r given that $P(X = 8) = \left(\frac{3}{4}\right)^7 - \left(\frac{3}{4}\right)^8$.

5 [Maximum mark: 6]

By using an appropriate trigonometric identity, show that **[2]**

$$\int \sin^2(3x-1)\,dx = \frac{1}{2}x - \frac{1}{12}\sin(6x-2) + C$$

Hence, evaluate the following integral: **[4]**

$$\int 2x\sin^2(3x-1)\,dx$$

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6 [Maximum mark: 6]

Find the gradient of the line tangent to the curve

$$y = \log_y(x^2 + 2)$$

at the point $(5, 3)$. Leave your final answer in the form $\frac{a}{b + c \ln 3}$, where $a, b, c \in \mathbb{Z}$.

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7 [Maximum mark: 6]

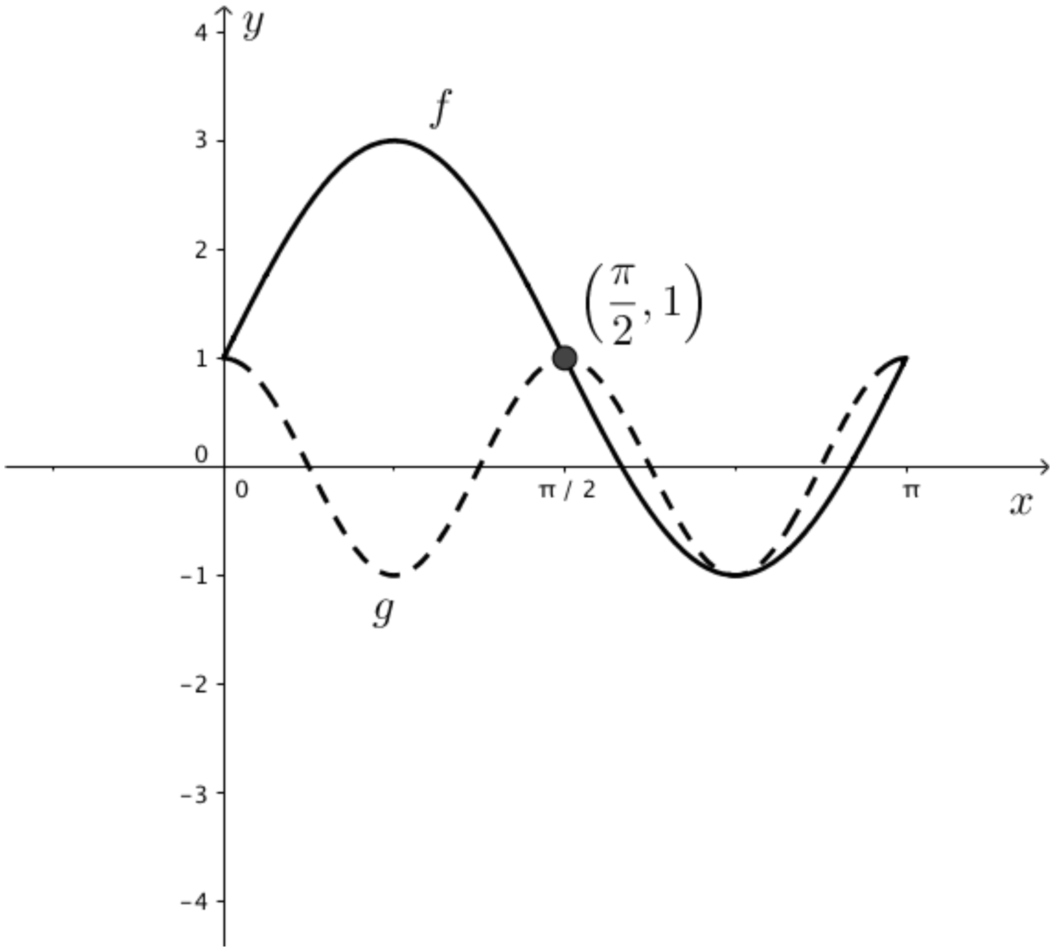
Solve the following system of complex equations:

$$\begin{cases} \arg z = \dfrac{\pi}{4} \\ 3(z - z^*) = z^2 \end{cases}$$

Leave your final answer in exponential form.

8 [Maximum mark: 8]

The graph of $y = f(x)$, which is shown below over the interval $[0, \pi]$, is obtained by transforming the graph of $y = \cos x$.



Find an appropriate expression for f in terms of cosine. **[3]**

Using the same set of axes, sketch the graph of $y = \frac{f(x)}{g(x)}$, clearly indicating the endpoints, maximum/ minimum points and asymptotes, if any. **[5]**

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*Do **NOT** write solutions on this page.*

SECTION B (50 marks)

Answer all questions on the foolscap paper provided. Please start each question on a new page.

9 [Maximum mark: 22]

Let $f(x)=\sqrt{\frac{x+1}{x}}, x \in D_f$ and $g(x)=\frac{x^2}{x^2+1}, x \in D_g$.

(a) Determine the maximal domains D_f and D_g. **[4]**

(b) Using the graph of $y=\frac{1}{x}$, find the range of f. **[2]**

(c) Show algebraically that f is one-one on D_f. **[3]**

(d) Hence, find an expression for f^{-1}, including its domain $D_{f^{-1}}$. **[4]**

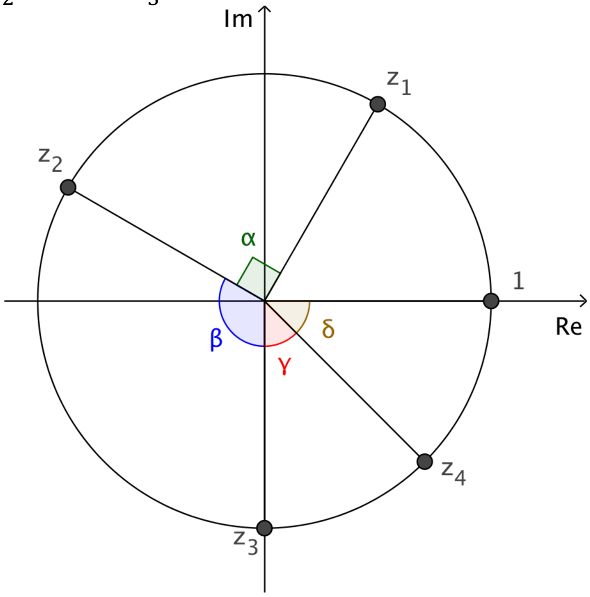
(e) On the other hand, justify that g is not one-one on D_g. **[2]**

(f) Find $g \circ f$ and show that $D_{g \circ f}=D_f$. **[4]**

(g) Sketch the graph of $g \circ f$, clearly indicating any intercepts and asymptotes. **[3]**

10 [Maximum mark: 13]

Let $z_k \in \mathbb{C}$ as in the Argand diagram below such that $|z_k|=1$ for $k=1,2,3,4$. Also, let α, β, γ and δ be positive angles such that $\alpha=\frac{\pi}{2}$ and $\beta=\frac{2\pi}{3}$.



(a) Find $\frac{z_2}{z_1}$, leaving your final answer in Cartesian form. **[2]**

(b) Determine z_1 given that $z_3=-i$, leaving your final answer in exponential form. **[2]**

(c) Find the smallest positive integer m such that ${z_2}^m=1$. **[2]**

(d) Let $w \in \mathbb{C}$. Find the possible values of $\arg w$ given that $w^2=i$. **[2]**

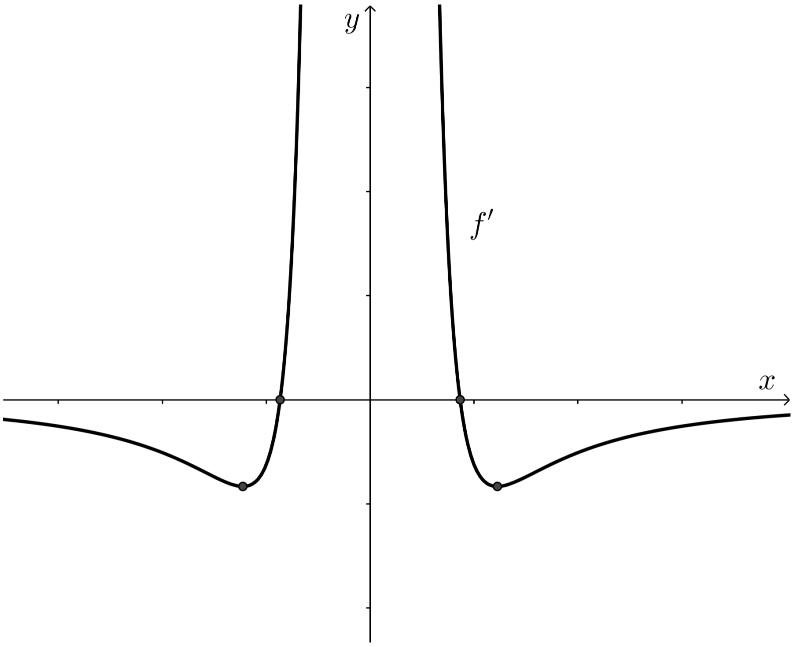
(e) Suppose $z_4^2=i \times z_3^2$. Prove that $\gamma=\delta$. **[3]**

(f) Find the probability that a randomly drawn point on the Argand diagram lies on the sector formed by γ **given** that the point is found inside the circle. **[2]**

*Do **NOT** write solutions on this page*

11 **[Maximum mark: 15]**

(a) The graph of $y = f'(x)$ appears below, where $x = 0$ is a vertical asymptote. It is known that $f'(\sqrt{3}) = f'(-\sqrt{3}) = 0$ and the minimum value of f' occurs when $x = \pm\sqrt{6}$.



i. Identify the x-coordinate of the maximum and minimum points of $y = f(x)$. **[3]**

ii. Also, identify the x-coordinate of all the points of inflexion of $y = f(x)$. **[2]**

iii. Sketch the graph of $y = f(x)$ as accurately as possible. **[4]**

(b) Consider the cubic polynomial $y = (ax + b)(x^2 - 3x + 1)$, where $a, b \in \mathbb{R}$.

Dividing y' by $(x - 1)$ yields a remainder of -1, while dividing y'' by the same divisor yields a remainder of -6.

Determine the value of a and of b. **[6]**

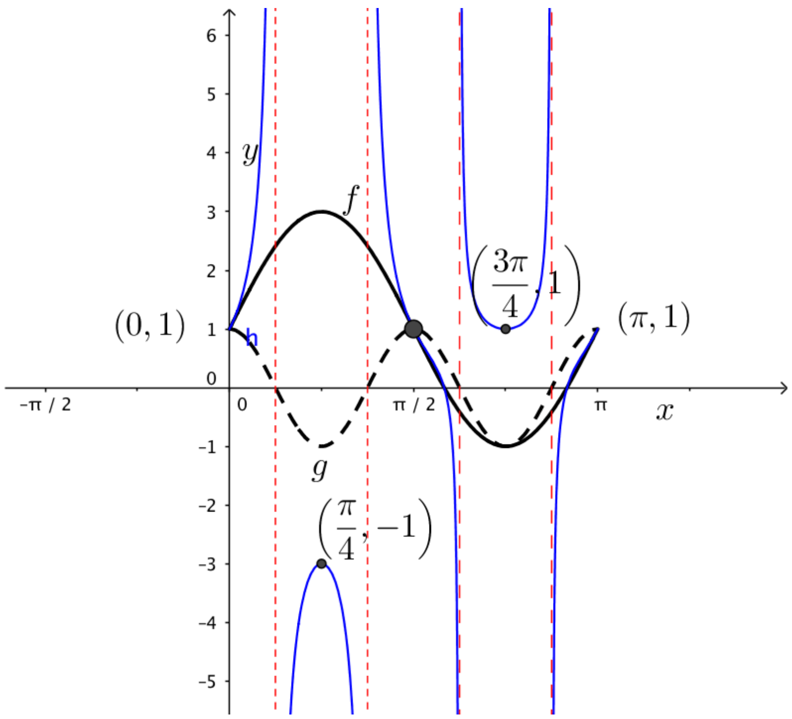
End of Paper

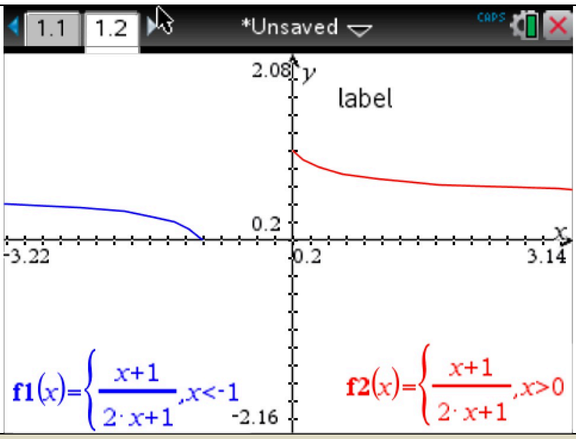
JC2 HL Math Preliminary Examination 2017 Paper 1 (Markscheme)

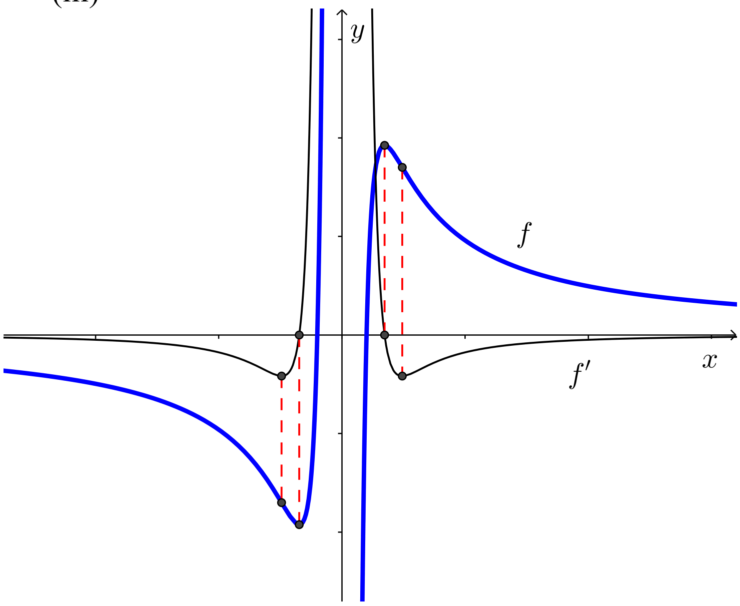
Qn	Suggested Solutions	Marks
1	**System of Linear Equations – Vectors/RREF**	**[Maximum: 5]**
	RREF Method:	
	$\left.\begin{matrix}1&2&4\\-1&0&2\\1&1&a\end{matrix}\right\vert\begin{matrix}5\\b\\-10\end{matrix} \Rightarrow \left.\begin{matrix}0&2&6\\-1&0&2\\0&1&a+2\end{matrix}\right\vert\begin{matrix}b+5\\b\\b-10\end{matrix}$	**M1A1**
	$\Rightarrow \left.\begin{matrix}-1&0&2\\0&1&a+2\\0&0&2-2a\end{matrix}\right\vert\begin{matrix}b\\b-10\\25-b\end{matrix}$	**M1**
	Thus, $a = 1$ and $b = 25$.	**A1A1**
		-OR-
	Linear Combination Method:	
	Eq1−2Eq3: $(4-2a)z - x = 25$.	**M1A1**
	The line of intersection intersects $2z - x = b$ at infinitely many points if and only if $a = 1$ and $b = 25$	**(M1)A1A1**
	Vector Method:	**-OR-**
	$\overrightarrow{n_1}\times\overrightarrow{n_3} = \begin{pmatrix}1\\2\\4\end{pmatrix}\times\begin{pmatrix}1\\1\\a\end{pmatrix} = \begin{pmatrix}2a-4\\4-a\\-1\end{pmatrix} \perp \overrightarrow{n_2} = \begin{pmatrix}-1\\0\\2\end{pmatrix}$	**A1**
	Thus, $4 - 2a - 2 = 0 \Rightarrow a = 1$.	**A1**
	Let $z = 0$, so that Eq 1 and 3 become $2y + x = 5$ and $y + x = -10$ which gives us $x = -25$ and $y = 15$.	**(M1)**
	Substituting $z = 0$ and $x = -25$ in Eq 2 gives us $b = -25$.	**M1** **A1**

Qn	Suggested Solutions	Marks
2	**Mathematical Induction/Compound Angle Formula**	**[Maximum: 5]**
	$n = 1$: $\frac{\sin 2x}{2\sin x} = \frac{2\sin x\cos x}{2\sin x} = \cos x$.	**A1**
	Suppose $\frac{\sin 2^k x}{2^k \sin x} = (\cos x)(\cos 2x)(\cos 4x)\cdots(\cos 2^{k-1}x)$	**M1**
	Thus, $(\cos x)\cdots(\cos 2^k x) = \frac{\sin 2^k x}{2^k \sin x}(\cos 2^k x) = \frac{\frac{1}{2}\sin 2^{k+1}x}{2^k \sin x}$	**M1A1**
	which is equal to $\frac{\sin 2^{k+1}x}{2^{k+1}\sin x}$.	
	Thus, by mathematical induction, the statement is true for all positive integer n.	**A1**
3	**Vieta's Formula/Complex Roots**	**[Maximum: 9]**
	$2 + i$ is also a root.	**A1**
	Let r be the 3rd root. Thus, $\frac{50}{2} = -(2+i)(2-i)r \Rightarrow 25 = -5r \Rightarrow r = -5$	**M1** **A1**
	Therefore, $-\frac{a}{2} = (2+i) + (2-i) - 5 = -1 \Rightarrow a = 2$	**M1** **A1**
	and $\frac{b}{2} = -5(2+i) - 5(2-i) + (2+i)(2-i) = -15$ $\Rightarrow b = -30$	**M1A1**
	Since, $0 = x^3 f\left(\frac{1}{x}\right) = 2x^3 + 2x^2 - 30x + 50$, then the zeros of f are $-\frac{1}{5}, \frac{1}{2-i} = \frac{2+i}{5}$ and $\frac{1}{2+i} = \frac{2-i}{5}$.	**R1** **A1 – all**

Qn	Suggested Solutions	Marks
4	**Discrete Probability/Geometric Progression**	**[Maximum: 5]**
	$1=\sum_{k=1}^{\infty}P(X=k)=\sum_{k=1}^{\infty}ar^{k-1}=\frac{a}{1-r}\Rightarrow a=1-r$	**R1A1**
	$r=\frac{ar^7}{ar^6}=\frac{\left(\frac{3}{4}\right)^7-\left(\frac{3}{4}\right)^8}{\left(\frac{3}{4}\right)^6-\left(\frac{3}{4}\right)^7}=\frac{\frac{3}{4}-\left(\frac{3}{4}\right)^2}{1-\frac{3}{4}}\Rightarrow r=\frac{3}{4}$	**M1** **A1**
	Thus, $a=\frac{1}{4}$.	**A1**
5	**Integration by Parts**	**[Maximum: 6]**
	Note that $\cos 2\theta=1-2\sin^2\theta$.	**A1**
	Thus, $\int\sin^2(3x-1)\,dx=\frac{1}{2}\int(1-\cos(6x-2))dx$	**M1**
	$=\frac{1}{2}x-\frac{1}{12}\sin(6x-2)+C$	**AG**
	Using by parts, let $u=2x$ and $dv=\sin^2(3x-1)$. Thus, $du=2dx$ and $v=\frac{1}{2}x-\frac{1}{12}\sin(6x-2)$.	**M1**
	Therefore,	
	$I=x^2-\frac{1}{6}x\sin(6x-2)-\int\left(x-\frac{1}{6}\sin(6x-2)\right)dx$	**A1**
	$=x^2-\frac{1}{6}x\sin(6x-2)-\frac{1}{2}x^2-\frac{1}{36}\cos(6x-2)+C$	**M1A1**
6	**Implicit Differentiation/Logarithms**	**[Maximum: 6]**
	Note that $y=\frac{\ln(x^2+2)}{\ln y}$.	**A1**
	$\frac{dy}{dx}=\frac{\frac{2x(\ln y)}{x^2+2}-\frac{\ln(x^2+2)}{y}\frac{dy}{dx}}{(\ln y)^2}$	**M1** **A1 -** $\frac{2x(\ln y)}{x^2+2}$ **A1 -** $\frac{\ln(x^2+2)}{y}\frac{dy}{dx}$
	At (5,3)	
	$\frac{dy}{dx}(\ln 3)^2=\frac{10}{27}\ln 3-\frac{1}{3}\ln 27\frac{dy}{dx}$	**M1**
	$\frac{dy}{dx}=\frac{\frac{10}{27}\ln 3}{(\ln 3)^2+\ln 3}=\frac{10}{27+27\ln 3}$	**A1**

Qn	**Suggested Solutions**	**Marks**
7	**System of Complex Equations**	**[Maximum: 6]**
	$\arg(z)=\frac{\pi}{4}\Rightarrow z=x+ix, x>0.$	**A1**
	$\Rightarrow 3(2ix)=2ix^2\Rightarrow x^2-3x=0\Rightarrow x=0,3$	**M1A1A1**
	Reject $x=0$. Thus, $z=3+3i=3\sqrt{2}e^{\frac{i\pi}{4}}$.	**A1 – rej.** **A1**
8	**Transformation of Graph**	**[Maximum: 8]**
	$f(x)=2\sin(2x)+1=2\cos\left(2x-\frac{\pi}{2}\right)+1$	**M1A1A1**
		**G1 – shape: all 5 branches** **G1 – 4 asymptotes** **G1 – Endpoints** **G1 – 2 roots** **G1 – Max and Min**
9	**Functions**	**[Maximum: 22]**
(a)	$D_f:\frac{x+1}{x}\geq 0\Rightarrow x\leq -1 \text{ or } x>0=(-\infty,-1]\cup(0,\infty)$	**M1A1A1**
	$D_g=\mathbb{R}$	**A1**
(b)	$f(x)=\sqrt{1+\frac{1}{x}}$	**M1**
	Thus, $R_f=[0,1)\cup(1,\infty)$	**A1A1**

Qn	Suggested Solutions	Marks
(c)	$f(a)=f(b)\Rightarrow\sqrt{1+\frac{1}{a}}=\sqrt{1+\frac{1}{b}}\Rightarrow a=b$	M1A1
(d)	$y=\sqrt{1+\frac{1}{x}}\Rightarrow x=\sqrt{1+\frac{1}{y}}\Rightarrow y=\frac{1}{x^2-1}$. Thus, $f^{-1}(x)=\frac{1}{x^2-1}, x\in[0,1)\cup(1,\infty)$.	M1A1 A1A1
(e)	$g(1)=g(-1)=\frac{1}{2}$, thus, g is not 1-1.	M1A1
(f)	$$g(f(x))=\frac{\frac{x+1}{x}}{\frac{(x+1)}{x}+1}=\frac{x+1}{2x+1}$$ Thus, $D_{g\circ f}=\left\{\mathbb{R}-\left\{-\frac{1}{2}\right\}\right\}\cap\{(-\infty,-1]\cup(0,\infty)\}=D_f$.	M1 A1 M1A1
(g)		G1 – shape G1 – intercepts (-1, 0) & hollow point (0, 1) G1 – asymptote
10	**Roots of Unity**	**[Maximum: 13]**
(a)	$$\frac{z_2}{z_1}=i$$	(M1)A1
(b)	$$z_1\times i\times\omega=-i\Rightarrow e^{i\left(\theta+\frac{7\pi}{6}\right)}=e^{i\frac{3\pi}{2}}$$ Thus, $\theta=\frac{3\pi}{2}-\frac{7\pi}{6}=\frac{\pi}{3}$ and so $z_1=e^{i\frac{\pi}{3}}$.	M1 A1
(c)	$$z_2^m=e^{i\frac{5m\pi}{6}}=1\Rightarrow m=12$$	M1A1

Qn	**Suggested Solutions**	**Marks**
(d)	$z^2 = i \Rightarrow e^{i\theta} = e^{i\frac{\pi}{2}} \Rightarrow \arg z = \frac{\pi}{4} or \frac{5\pi}{4}$.	**A1A1**
(e)	From the given assumption, it follows that $\arg z_4 = -\frac{\pi}{4} \Rightarrow \delta = \frac{\pi}{4}$. Thus $\gamma = \frac{\pi}{4}$ as well, proving that $\gamma = \delta$.	**M1A1** **A1**
(f)	$Prob = \frac{45}{360} = \frac{1}{8}$	**(M1)A1**
11	**Techniques of Differentiation/Remainder Theorem**	**[Maximum: 15]**
(a)	(i) Minimum at $x = -\sqrt{3}$ and max at $x = \sqrt{3}$ (ii) Inflexion at $x = \pm\sqrt{6}$. (iii)  	**(M1)A1A1** **A1A1** **G1– shape** **G1– asymptotes** **G1– max/min** **G1– inflexion**
(b)	$y' = 3ax^2 + (2b - 6a)x + (a - 3b)$ $y'' = 6ax + (2b - 6a)$ $y'(1) = 3a + 2b - 6a + a - 3b = -2a - b = -1$ $y''(1) = 6a + 2b - 6a = 2b = -6$ $b = -3$ and $a = 2$	**A1** **A1** **M1** **M1** **A1A1**

STUDENT NAME: ______________________

TEACHER INITIALS: ________

CANDIDATE SESSION NUMBER

0	2	5	0	1	2				

EXAMINATION CODE

8	8	1	7	-	7	2	0	2

ST JOSEPH'S INSTITUTION
JC2 PRELIMINARY EXAMINATION 2017

MATHEMATICS
HIGHER LEVEL
PAPER 2
Thursday

6th July 2017
2 hours
0800 – 1000 hrs

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- **Section A**: Answer all questions showing working and answers in the spaces provided in the exam paper.
- **Section B**: Answer all questions using the foolscap paper provided.
- The use of a scientific or graphical calculator is permitted in this paper.
- TI-Nspire calculators must be in Press-to-Test mode and cleared of all previous data.
- TI-84+ graphical calculators must only have permitted Apps and be ram-cleared.
- It is the responsibility of the student to ensure their calculator is examination ready.
- A clean copy of the **Mathematics HL Formulae Booklet** is required for this paper.
- Unless otherwise stated in the question all numerical answers are to be given exactly.
- The maximum mark for this examination paper is *[100 marks]*.
- This question paper consists of **13** printed pages including the Cover Sheet.

FOR MARKER USE ONLY:

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	Q12	Q13	TOTAL
													/100

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphical display calculator should be supported by suitable working; for example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are advised to show all working.

SECTION A (50 marks)

*Answer **all** questions in the spaces provided. Working may be continued below the lines if necessary. Foolscap paper may be used for any additional working.*

1 [Maximum mark: 3]

The random variable X is the number of successes in 200 independent trials of an experiment in which the probability of success at any one trial is p.
Given that $\mathrm{E}\left(X^2\right)=10.6008$, find the value of p.

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2 [Maximum mark: 10]

The graph of $y = \mathrm{f}(x)$ undergoes, in succession, the following transformations:

A: A scaling parallel to the *x*-axis by factor $\frac{1}{3}$.

B: A translation of -4 units in the direction of *y*-axis.

The resulting graph has equation $y = \dfrac{7-6x}{3x-2}$.

(i) Find an expression for $\mathrm{f}(x)$. [3]

(ii) By considering $\mathrm{f} \circ \mathrm{f}$, find an expression for the composite function $\mathrm{f}^{101}(x)$. [3]

(iii) State, with a reason, if f is self-inverse. [2]

(iv) Find the value of $\mathrm{f}^{-1}(-1)$. [2]

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TURN OVER

3 [Maximum mark: 7]

(i) Prove that $\dfrac{\sin(A-B)}{\cos A\cos B}=\tan A-\tan B.$ [3]

(ii) **Hence** show that

$$\frac{\sin x}{\cos x\cos 2x}+\frac{\sin x}{\cos 2x\cos 3x}+......+\frac{\sin x}{\cos nx\cos(n+1)x}=\frac{\sin nx}{\cos x\cos(n+1)x}.$$ [4]

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4 [Maximum mark: 10]

(i) Differentiate $\ln\left(\sec\alpha + \tan\alpha\right)$ with respect to α, leaving your final answer in simplified form. [2]

(ii) By using the substitution $x = 1 + \sec\theta$, find the exact value of $\int_{1+\sqrt{2}}^{3} \frac{x+1}{\sqrt{x^2 - 2x}}\,\mathrm{d}x$. [8]

5 [Maximum mark: 5]

Alfred, Bernard and Caleb have k marbles altogether. When Alfred gives Bernard 30 marbles and Bernard gives Caleb 12 marbles, the number of marbles Alfred, Bernard and Caleb each has respectively is in the ratio 1:2:3.

Find the least value of k, assuming that each of them has some marbles initially.

6 [Maximum mark: 3]

A vehicle rental company has 7 cars and 4 vans available for rental per day.
It is known that the request for cars has a mean of 4 per day; and independently, the request for vans has a mean of 2 per day.

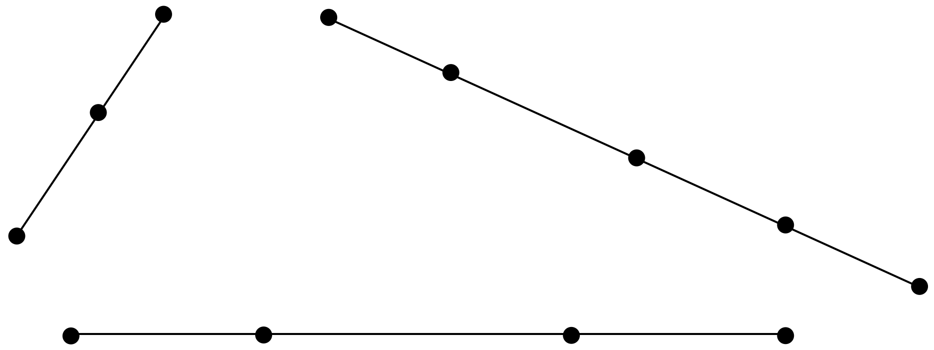
Find the probability that some requests for a vehicle have to be refused on a particular day.

7 [Maximum mark: 7]

Solve the equation $z^6 + z^5 + z^4 + z^3 + z^2 + z + 1 = 0$, giving the six roots in trigonometric form. Hence deduce the exact value of $\cos\left(\frac{2\pi}{7}\right) + \cos\left(\frac{4\pi}{7}\right) + \cos\left(\frac{6\pi}{7}\right)$.

8 [Maximum mark: 5]

The diagram shows three straight lines with 12 distinct points.



(i) Find the number of line segments that can be formed by joining any two points from different lines. [2]

(ii) Find the number of different triangles that can be formed from these points. [3]

*Do **NOT** write solutions on this page.*

SECTION B (50 marks)

Answer all questions on the foolscap paper provided. Please start each question on a new page.

9 [Maximum mark: 8]

(a) The first, second and third terms of an arithmetic progression are α, β and α^2 respectively where $\alpha < 0$. The first, second and third terms of a geometric progression are α, α^2 and β respectively. Find the value of α. [3]

(b) Let $S_n = 1 + 2x + 3x^2 + ... + nx^{n-1}$ where $0 < x < 1$ and n is a positive integer.

Show that $\int S_n \ \mathrm{d}x = \dfrac{x\left(1-x^n\right)}{1-x} + c$, where c is a constant.

Hence, deduce that $S_n = \dfrac{1-(n+1)x^n + nx^{n+1}}{(1-x)^2}$.

Given that for $0 < x < 1$, $nx^n \to 0$ as $n \to \infty$, state, in terms of x, $\lim\limits_{n\to\infty} S_n$. [5]

10 [Maximum mark: 10]

(a) Show that, for any positive integer n,

$$3\left(5^{n+1}+1\right) > 6\left(5^n+1\right).$$

Hence prove by induction that, for $n \geq 2$,

$$3^{n-1}\left(5^n+1\right) > 6^n.$$ [7]

(b) It is given that $\mathrm{g}(x) = \dfrac{\lambda x^2}{x^2+\lambda}$, where λ is a non-zero positive constant.

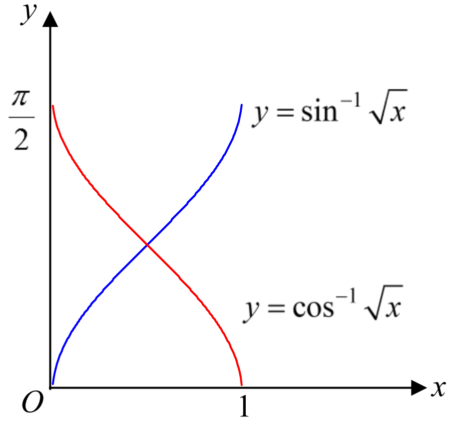
(i) Determine if g is an even function. [1]

(ii) Sketch $y = \mathrm{g}(x)$. [2]

*Do **NOT** write solutions on this page.*

11 [Maximum mark: 9]

(a)

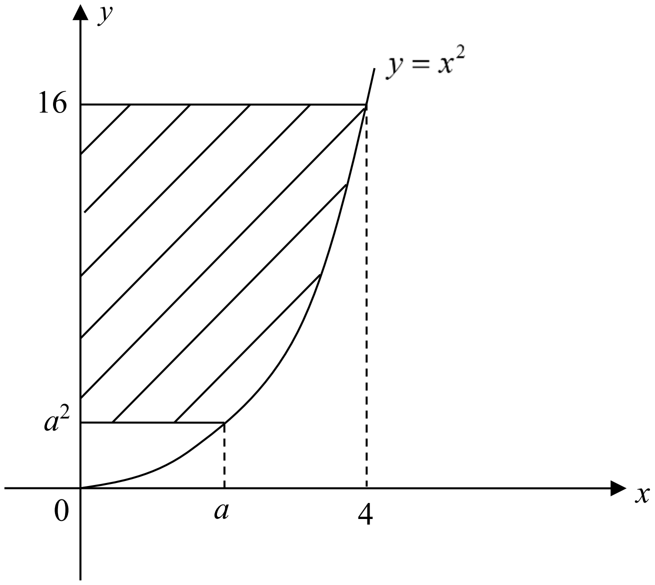


The diagram shows part of the curves of $y=\sin^{-1}\sqrt{x}$ and $y=\cos^{-1}\sqrt{x}$. The two curves intersect at point P.

(i) State the coordinates of P in exact form. [1]

(ii) The region A is bounded by the 2 curves and the x-axis.
Without the use of GDC, find the area of A. [4]

(b)



The diagram above shows a region bounded by the curve $y=x^2$, the lines $y=a^2$, $y=16$ and the y-axis. When this region is rotated 360° about the x-axis, the volume formed is $\frac{3968\pi}{5}$ units^3. Find the value of a. [4]

*Do **NOT** write solutions on this page.*

12 [Maximum mark: 15]

(a) By considering the scalar product $\mathbf{a}\cdot\mathbf{b}$, where $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$ and $\mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$, prove that

$$(a_1b_1+a_2b_2+a_3b_3)^2 \le (a_1{}^2+a_2{}^2+a_3{}^2)(b_1{}^2+b_2{}^2+b_3{}^2).$$ [3]

(b) The angle between the vectors **a** and **b** is 60° and $|\mathbf{a}|=2$, $|\mathbf{b}|=1$.

Find $|2\mathbf{a}-\mathbf{b}|$. [4]

(c) Three non-zero vectors **p**, **q** and **r** are such that $\mathbf{p}\times 3\mathbf{q}=2\mathbf{p}\times\mathbf{r}$.

Find a linear relationship between **p**, **q** and **r**. [3]

(d) Plane π has equation $3x+2y+5z=45$.

Obtain a vector equation of π in the form $\mathbf{r}=\mathbf{t}+\lambda\mathbf{u}+\mu\mathbf{v}$, $\lambda,\mu\in\mathbb{R}$,

given that **t** and **u** are of the form $p\mathbf{i}+p\mathbf{j}$ and $2\mathbf{i}+q\mathbf{j}$ respectively, where p and q are constants to be determined, and **u** is perpendicular to **v**. [5]

13 [Maximum mark: 8]

Jane goes into a candy shop and decides to buy 3 candies. She has a choice of 10 strawberry flavoured and 15 chocolate flavoured candies to choose from.

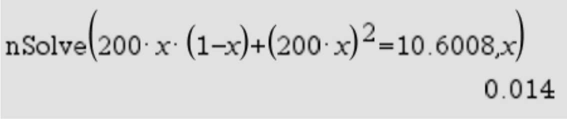
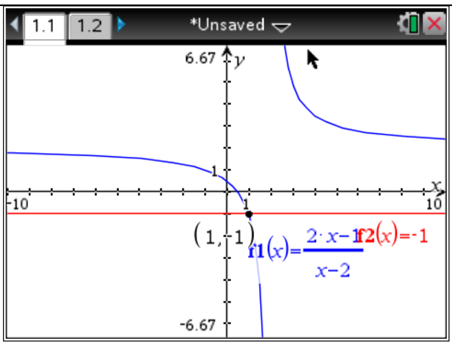
Let X be the number of strawberry candies she selected.

(i) Show that $\mathrm{P}(X=2)=\dfrac{27}{92}$. [3]

(ii) Find $E(X)$ and $Var(X)$. [5]

End of paper

JC2 HL Math Preliminary Examination 2017 Paper 2 (Mark Scheme)

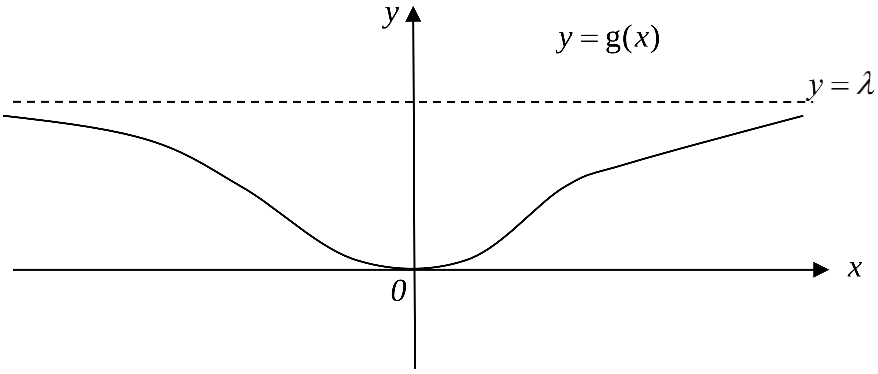
Qn	Suggested Solutions	Marks
1	$X \sim \mathrm{B}(200,\ p)$ $\mathrm{E}\left(X^2\right)=\mathrm{Var}(X)+\left[\mathrm{E}(X)\right]^2=10.6008$ $\Rightarrow 200p(1-p)+\left[200p\right]^2=10.6008$ Bt GDC, $p=0.014$.  nSolve$\left(200\cdot x\cdot(1-x)+(200\cdot x)^2=10.6008,x\right)$ 0.014	M1 A1 A1
2(i)	$\dfrac{7-6x}{(3x-2)} \xrightarrow{B'} \dfrac{7-6x}{(3x-2)}+4=\dfrac{6x-1}{3x-2}$ $\dfrac{6x-1}{3x-2} \xrightarrow{A'} \dfrac{6\left(\frac{1}{3}x\right)-1}{3\left(\frac{1}{3}x\right)-2}=\dfrac{2x-1}{x-2}$ Therefore, $\mathrm{f}(x)=\dfrac{2x-1}{x-2}$.	A1 A1 A1
(ii)	$\mathrm{f}\circ\mathrm{f}(x)=\mathrm{f}\left(\dfrac{2x-1}{x-2}\right)=\dfrac{2\left(\frac{2x-1}{x-2}\right)-1}{\frac{2x-1}{x-2}-2}=\dfrac{3x}{3}=x$ $\mathrm{f}^{101}(x)=\mathrm{f}(x)=\dfrac{2x-1}{x-2}$.	M1A1 A1
(iii)	Since f is 1-1 $\Rightarrow \mathrm{f}^{-1}(x)$ exists and $\mathrm{f}\circ\mathrm{f}(x)=x \Rightarrow \mathrm{f}(x)=\mathrm{f}^{-1}(x)$ i.e. f is self-inverse.	R2
(iv)	 Since $\mathrm{f}(1)=-1 \Rightarrow \mathrm{f}^{-1}(-1)=1$.	M1A1

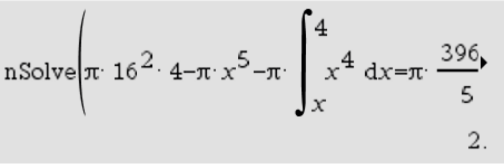
3(i)	$\frac{\sin(A-B)}{\cos A\cos B}=\frac{\sin A\cos B-\cos A\sin B}{\cos A\cos B}$ $=\frac{\sin A\cos B}{\cos A\cos B}-\frac{\cos A\sin B}{\cos A\cos B}$ $=\frac{\sin A}{\cos A}-\frac{\sin B}{\cos B}$ $=\tan A-\tan B$ (shown)	**M1** **M1** **A1**
(ii)	$\frac{\sin x}{\cos x\cos 2x}+\frac{\sin x}{\cos 2x\cos 3x}+......+\frac{\sin x}{\cos nx\cos(n+1)x}$ $=\frac{\sin(2x-x)}{\cos 2x\cos x}+\frac{\sin(3x-2x)}{\cos 3x\cos 2x}+.......+\frac{\sin[(n+1)x-nx]}{\cos[(n+1)x]\cos x}$ $=(\tan 2x-\tan x)+(\tan 3x-\tan 2x)+...........+[\tan(n+1)x-\tan x]$ $=\tan(n+1)x-\tan x$ $=\frac{\sin[(n+1)x-x]}{\cos x\cos(n+1)x}$ $=\frac{\sin nx}{\cos x\cos(n+1)x}$ (shown)	**M1** **A1** **A1** **A1**
4(i)	$\frac{d}{d\alpha}\left(\ln\left(\sec\alpha+\tan\alpha\right)\right)=\frac{\sec\alpha\tan\alpha+\sec^2\alpha}{\sec\alpha+\tan\alpha}=\sec\alpha$	**M1A1**
(ii)	$x=1+\sec\theta\ \therefore\frac{\mathrm{d}x}{\mathrm{d}\theta}=\sec\theta\tan\theta$ When $x=3,\theta=\frac{\pi}{3}$ When $x=1+\sqrt{2},\theta=\frac{\pi}{4}$ $\int_{1+\sqrt{2}}^{3}\frac{x+1}{\sqrt{x^2-2x}}\mathrm{d}x$ $=\int_{1+\sqrt{2}}^{3}\frac{x+1}{\sqrt{(x-1)^2-1}}\mathrm{d}x$ $=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{2+\sec\theta}{\sqrt{(\sec^2\theta-1)}}\sec\theta\tan\theta\mathrm{d}\theta$ $=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{2+\sec\theta}{\tan\theta}\sec\theta\tan\theta\mathrm{d}\theta$ $=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\left(2\sec\theta+\sec^2\theta\right)\mathrm{d}\theta$ $=\left[2\ln\left(\sec\theta+\tan\theta\right)+\tan\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$ $=2\ln\left(\frac{\sqrt{3}+2}{\sqrt{2}+1}\right)+\sqrt{3}-1$	**M1** **A1 (limits)** **A1(** $\frac{\mathrm{d}x}{\mathrm{d}\theta}=\sec\theta\tan\theta$ **)** **A1(** $\frac{2+\sec\theta}{\sqrt{(\sec^2\theta-1)}}$ **)** **M1A1A1** **A1**

5	Let $A,\ B,\ C$ be the no. of marbles Alfred, Bernard and Caleb has respectively. $A+B+C=k$ Given that $A-30:B+18:C+12=1:2:3,$ $\Rightarrow\begin{cases} 2(A-30)=B+18 \\ 3(A-30)=C+12 \\ 3(B+18)=2(C+12) \end{cases}$ $\Rightarrow\begin{cases} 2A-B=78\ \ ---(1) \\ 3A-C=102\ ---(2) \\ 3B-2C=-30\ ---(3) \end{cases}$ By GDC, $A=34+\frac{1}{3}C$ $B=-10+\frac{2}{3}C$ Since $A,\ B,\ C\ \in\mathbb{Z}^{+}, B=-10+\frac{2}{3}C>0\Rightarrow C>15$ least $C=18,\ B=2,\ A=40$ Hence, least $k=18+2+40=60.$	**A2 -** **any 2 equations out of 3** **A1** **M1** **A1**
	<u>Alternatively,</u> Let $A,\ B,\ C$ be the no. of marbles Alfred, Bernard and Caleb has respectively. Given that $A-30:B+18:C+12=1:2:3,$ $\Rightarrow\begin{cases} 2(A-30)=B+18 \\ 3(A-30)=C+12 \\ 3(B+18)=2(C+12) \end{cases}$ $\Rightarrow\begin{cases} 2A-B=78\ \ ---(1) \\ 3A-C=102\ ---(2) \\ 3B-2C=-30\ ---(3) \end{cases}$ $A+B+C=k$ $A+B+C-k=0---(4)$ By GC, $A=30+\frac{1}{6}k$ $B=-18+\frac{1}{3}k$ $C=-12+\frac{2}{3}k$ Since $A,\ B,\ C\ \in\mathbb{Z}^{+}, B=-18+\frac{1}{3}k>0\Rightarrow k>54$ Since k must be a multiple of 6, least $k=60.$	**A2 -** **any 2 equations out of 3** **A1** **M1** **A1**

6	“Some requests for a vehicle have to be refused on a particular day” is equivalent to “Either demand for a car or a van is not met”. $\text{P}\left(\text{Either demand for a car or a van is not met. }\right)$ $=\text{P}(X>7 \text{ or } Y>4)$ $\quad\left[\text{Note that } \text{P}(A\cup B)=\text{P}(A' \ \cap B').\right]$ $=1-\text{P}(X\le 7 \text{ and } Y\le 4)$ $=1-\text{P}(X\le 7)P(Y\le 4)$ $=0.101$ [**Note that** $\text{P}(X\le 7 \text{ and } Y\le 4)=\text{P}(X\le 7)\times\text{P}(Y\le 4)$ is due to the independence property of Poisson random variable.]	**M1** **M1** **A1**
7	$z^6+z^5+z^4+z^3+z^2+z+1=0$ $\Rightarrow \dfrac{z^7-1}{z-1}=0, \quad z\ne 1$ $\Rightarrow z^7=1$ $\Rightarrow z=e^{\frac{2\pi k}{7}i}, \ k=-3,-2,-1,1,2,3$ $\Rightarrow z=\cos\left(\dfrac{2\pi}{7}\right)\pm i\sin\left(\dfrac{2\pi}{7}\right),$ $\cos\left(\dfrac{4\pi}{7}\right)\pm i\sin\left(\dfrac{4\pi}{7}\right),$ $\cos\left(\dfrac{6\pi}{7}\right)\pm i\sin\left(\dfrac{6\pi}{7}\right).$	**M1** **A1** **A1A1**
	$z^6+z^5+z^4+z^3+z^2+z+1=0$ Sum of roots = –1 $\Rightarrow \cos\left(\dfrac{2\pi}{7}\right)+i\sin\left(\dfrac{2\pi}{7}\right)+\cos\left(\dfrac{4\pi}{7}\right)+i\sin\left(\dfrac{4\pi}{7}\right)+\cos\left(\dfrac{6\pi}{7}\right)+i\sin\left(\dfrac{6\pi}{7}\right)+$ $\cos\left(\dfrac{2\pi}{7}\right)-i\sin\left(\dfrac{2\pi}{7}\right)+\cos\left(\dfrac{4\pi}{7}\right)-i\sin\left(\dfrac{4\pi}{7}\right)+\cos\left(\dfrac{6\pi}{7}\right)-i\sin\left(\dfrac{6\pi}{7}\right) \quad =-1$ $2\left[\cos\left(\dfrac{2\pi}{7}\right)+\cos\left(\dfrac{4\pi}{7}\right)+\cos\left(\dfrac{6\pi}{7}\right)\right]=-1$ $\Rightarrow \cos\left(\dfrac{2\pi}{7}\right)+\cos\left(\dfrac{4\pi}{7}\right)+\cos\left(\dfrac{6\pi}{7}\right)=-0.5$	**M1** **A1** **A1**
8(i)	Method 1: ${}^5C_1\,{}^4C_1+{}^5C_1\,{}^3C_1+{}^4C_1\,{}^3C_1=47$ Method 2: ${}^{12}C_2-{}^3C_2-{}^4C_2-{}^5C_2$ $=47$	**M1A1**

(ii)	Method 1: ${}^3C_1\,{}^4C_1\,{}^5C_1 + {}^3C_2\,{}^4C_1\,{}^5C_0 + {}^3C_2\,{}^4C_0\,{}^5C_1 + {}^3C_0\,{}^4C_1\,{}^5C_2$ $+{}^3C_0\,{}^4C_2\,{}^5C_1 + {}^3C_1\,{}^4C_2\,{}^5C_0 + {}^3C_1\,{}^4C_0\,{}^5C_2$ $= 205$ Method 2: ${}^{12}C_3 - {}^3C_3 - {}^4C_3 - {}^5C_3$ $= 205$	**M1A1** **A1** **M1A1** **A1**
9(a)	$\beta - \alpha = \alpha^2 - \beta \quad (1)$ $\dfrac{\alpha^2}{\alpha} = \dfrac{\beta}{\alpha^2} \therefore \beta = \alpha^3 \quad (2)$ Sub (2) in (1), $\alpha^3 - \alpha = \alpha^2 - \alpha^3$ $2\alpha^3 - \alpha^2 - \alpha = 0$ Since $\alpha \neq 0$, $2\alpha^2 - \alpha - 1 = 0$ $\alpha = 1, \ -\dfrac{1}{2}$ Since $\alpha < 0$, $\alpha = -\dfrac{1}{2}$.	$\left.\begin{matrix} \\ \\ \end{matrix}\right\}$ **A1** **M1** **A1**
(b)	$\int S_n \mathrm{d}x = \int \left(1 + 2x + 3x^2 + \ldots + nx^{n-1}\right)\mathrm{d}x$ $= x + x^2 + x^3 + \ldots + x^n + c$ $= \dfrac{x(1-x^n)}{1-x} + c$ (shown) $S_n = \dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{x(1-x^n)}{1-x} + c\right) = \dfrac{(1-x)(1-(n+1)x^n) + x(1-x^n)}{(1-x)^2}$ $= \dfrac{1-(n+1)x^n + nx^{n+1}}{(1-x)^2}$ (deduced) Since for $0 < x < 1$, $nx^n \to 0$ as $n \to \infty$, $\lim\limits_{n\to\infty} S_n = \dfrac{1}{\left(1-x\right)^2}$	**M1** integrating **A1** geometric series **M1** quotient rule **A1** **A1**
10(a)	$3(5^{n+1} + 1) – 6(5^n + 1)$ $= 3.5.5^n + 3 – 6.5^n – 6$ $= 9.5^n – 3$ $\geq 42 \quad (\because 5^n \geq 5 \text{ for } n \geq 1)$ > 0 $\therefore \underline{\underline{3(5^{n+1} + 1) > 6(5^n + 1)}}$	**M1** **A1** **R1**

	Let P_n be the statement "$3^{n-1}\left(5^n+1\right)>6^n$", $n\in\mathbb{Z}^+,\ n\geq 2$. When $n = 2$, LHS = $3(5^2 + 1) = 78$. RHS = $6^2 = 36$ LHS > RHS $\therefore$ P_2 is true. Assume that P_k is true for some $k\in\mathbb{Z}^+,\ k\geq 2$, i.e. $3^{k-1}\left(5^k+1\right)>6^k$. To prove: P_{k+1} is true, i.e. $3^k\left(5^{k+1}+1\right)>6^{k+1}$. When $n = k+1$, $3^k\left(5^{k+1}+1\right)=3^{k-1}.3\left(5^{k+1}+1\right)$ $> 3^{k-1}.6(5^k + 1)$ $> 6.6^k$ $= 6^{k+1}$ $\therefore$ P_k is true $\Rightarrow$ P_{k+1} is true. Since P_2 is true, and P_k is true $\Rightarrow$ P_{k+1} is true, by Maths Induction, P_n is true for all $n\in\mathbb{Z}^+,\ n\geq 2$.	**M1 (template)** **A1** **M1** **A1**
(b)(i)	$\mathrm{g}(x)=\dfrac{\lambda x^2}{x^2+\lambda}$ Since $\mathrm{g}(-x)=\dfrac{\lambda x^2}{x^2+\lambda}=\mathrm{g}(x)$, g is even.	**R1**
(ii)		**G1 (asymptote)** **G1(shape)**
11(a) (i)	$\sin^{-1}\sqrt{x}=\cos^{-1}\sqrt{x}$ $\Rightarrow \sin^{-1}\sqrt{x}=\cos^{-1}\sqrt{x}=\dfrac{\pi}{4}$ $\Rightarrow \sqrt{x}=\dfrac{1}{\sqrt{2}}$ $\Rightarrow x=\dfrac{1}{2}$ $\therefore P=\left(\dfrac{1}{2},\dfrac{\pi}{4}\right)$	**A1**

(ii)	Area of A = $\int_0^{\frac{\pi}{4}}\cos^2 y\,\mathrm{d}y-\int_0^{\frac{\pi}{4}}\sin^2 y\,\mathrm{d}y$ $= \int_0^{\frac{\pi}{4}}\cos^2 y-\sin^2 y\,\mathrm{d}y$ $= \int_0^{\frac{\pi}{4}}\cos 2y\,\mathrm{d}y$ $= \left[\frac{\sin 2y}{2}\right]_0^{\frac{\pi}{4}}$ $= \frac{1}{2}$ units^2	**M1** **M1** **A1** **A1**
(b)	$\pi\left(16^2\right)(4)-\pi\left(a^2\right)(a)-\int_a^4 x^4dx=\frac{3968\pi}{5}$ By GDC, nsolve,  $\text{nSolve}\left(\pi\cdot 16^2\cdot 4-\pi\cdot x^5-\pi\cdot\int_x^4 x^4\,\mathrm{d}x=\pi\cdot\frac{396}{5}\right.$ 2. $a=2.$	**M1** **A1** $\left[\pi\left(a^2\right)(a)\right]$ **A1** $\left[\int_a^4 x^4dx\right]$ **A1**
12(a)	$\mathbf{a}\cdot\mathbf{b} =\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\theta$ $\Rightarrow\lvert\mathbf{a}\cdot\mathbf{b}\rvert\le\ \lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\quad\because\lvert\cos\theta\rvert\le 1$ $\Rightarrow\lvert\mathbf{a}\cdot\mathbf{b}\rvert^2\le\ \lvert\mathbf{a}\rvert^2\lvert\mathbf{b}\rvert^2$ If $\mathbf{a}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix},\quad\mathbf{b}=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix},$ $\Rightarrow(a_1b_1+a_2b_2+a_3b_3)^2\le(a_1{}^2+a_2{}^2+a_3{}^2)(b_1{}^2+b_2{}^2+b_3{}^2)$ **<u>Alternative Method</u>** $\mathbf{a}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix},\quad\mathbf{b}=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$ $\mathbf{a}\cdot\mathbf{b}=a_1b_1+a_2b_2+a_3b_3$ $\lvert a\rvert\lvert b\rvert\cos\theta=a_1b_1+a_2b_2+a_3b_3$ $\sqrt{a_1{}^2+a_2{}^2+a_3{}^2}\sqrt{b_1{}^2+b_2{}^2+b_3{}^2}\cos\theta=a_1b_1+a_2b_2+a_3b_3$ $(a_1{}^2+a_2{}^2+a_3{}^2)(b_1{}^2+b_2{}^2+b_3{}^2)\cos^2\theta=(a_1b_1+a_2b_2+a_3b_3)^2$ $\cos^2\theta=\frac{(a_1b_1+a_2b_2+a_3b_3)^2}{(a_1{}^2+a_2{}^2+a_3{}^2)(b_1{}^2+b_2{}^2+b_3{}^2)}\le 1$ $\therefore(a_1b_1+a_2b_2+a_3b_3)^2\le(a_1{}^2+a_2{}^2+a_3{}^2)(b_1{}^2+b_2{}^2+b_3{}^2)$	**M1** **M1** **A1**

(b)	$\lvert 2\mathbf{a}-\mathbf{b}\rvert^2=(2\mathbf{a}-\mathbf{b}).(2\mathbf{a}-\mathbf{b})$ $=4\lvert\mathbf{a}\rvert^2-4\mathbf{a.b}+\lvert\mathbf{b}\rvert^2$ $=4(4)-4(2)(1)\cos 60^0+1=13$ $\Rightarrow\lvert 2\mathbf{a}-\mathbf{b}\rvert=\sqrt{13}$ (shown)	**M1** **A1** **A1** **A1**
(c)	$\mathbf{p}\times 3\mathbf{q}=2\mathbf{p}\times\mathbf{r}$ $\Rightarrow\mathbf{p}\times 3\mathbf{q}-2\mathbf{p}\times\mathbf{r}=\mathbf{0}$ $\Rightarrow\mathbf{p}\times(3\mathbf{q}-2\mathbf{r})=0$ $\Rightarrow\mathbf{p}\mathbin{//}3\mathbf{q}-2\mathbf{r}$ $\Rightarrow\mathbf{p}=k(3\mathbf{q}-2\mathbf{r}),\ k\in\mathbb{R}$	**M1** **A1** **A1**
(d)	The equation of the plane π is $3x+2y+5z=45.$ $(p,p,0)$ lies in $\pi\Rightarrow 3p+2p+0=45\Rightarrow p=9$ $\begin{pmatrix}2\\q\\0\end{pmatrix}$ is perpendicular to $\begin{pmatrix}3\\2\\5\end{pmatrix}\Rightarrow\begin{pmatrix}2\\q\\0\end{pmatrix}\cdot\begin{pmatrix}3\\2\\5\end{pmatrix}=0$ $6+2q=0\Rightarrow q=-3$ Since $\underset{\sim}{v}$ is perpendicular to both $\underset{\sim}{u}$ and $\underset{\sim}{n}$, $\underset{\sim}{u}\times\underset{\sim}{n}=\begin{pmatrix}3\\2\\5\end{pmatrix}\times\begin{pmatrix}2\\-3\\0\end{pmatrix}=\begin{pmatrix}15\\10\\-13\end{pmatrix}$ $\underset{\sim}{r}=\begin{pmatrix}9\\9\\0\end{pmatrix}+\lambda\begin{pmatrix}2\\-3\\0\end{pmatrix}+\mu\begin{pmatrix}15\\10\\-13\end{pmatrix},\lambda,\mu\in\mathbb{R}$ **Alternative method to find $\underset{\sim}{v}$** Let $\underset{\sim}{v}=\begin{pmatrix}x\\y\\z\end{pmatrix}$ $\begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}3\\2\\5\end{pmatrix}=0$ and $\begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}2\\-3\\0\end{pmatrix}=0$ $3x+2y+5z=0$ and $2x-3y=0$ $x=-\frac{15}{13}z,\ y=-\frac{10}{13}z,\ z=z$ Let $z=13$ (any non-zero number will work) $\mathbf{v}=\begin{pmatrix}-15\\-10\\13\end{pmatrix}$	**A1** **A1** **M1A1** **A1**

<table>
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<td>13(i)</td>
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$$P\left(X=2\right)=\frac{{}^{10}C_2\times{}^{15}C_1}{{}^{25}C_3}=\frac{27}{92}\quad\text{(shown)}$$

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<td>M1
A1A1</td>
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<tr>
<td>(ii)</td>
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x	0	1	2	3
P(X=x)	$\frac{{}^{10}C_0\times{}^{15}C_3}{{}^{25}C_3}=\frac{91}{460}$	$\frac{{}^{10}C_1\times{}^{15}C_2}{{}^{25}C_3}=\frac{21}{46}$	$\frac{27}{92}$	$\frac{6}{115}$

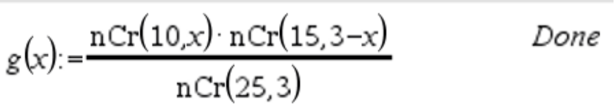
<u>Method 1</u>

$$E\left(X\right)=\frac{6}{5}$$

$$Var\left(X\right)=\mathrm{E}(X^2)-[\mathrm{E}(X)]^2=\frac{21}{10}-\left(\frac{6}{5}\right)^2=\frac{33}{50}$$

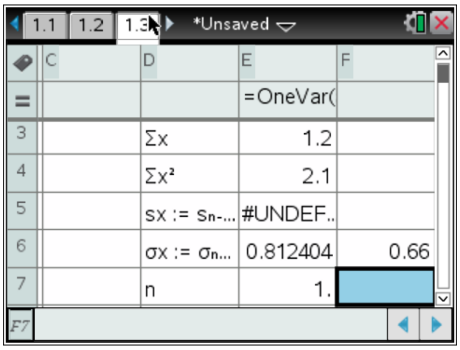
<u>Method 2</u>

By GDC,



$$g(x):=\frac{\mathrm{nCr}(10,x)\cdot \mathrm{nCr}(15,3-x)}{\mathrm{nCr}(25,3)}$$

Done



1.1 1.2 1.3 *Unsaved

	C	D	E	F
=			=OneVar(	
3		Σx	1.2	
4		Σx²	2.1	
5		sx := sₙ₋...	#UNDEF..	
6		σx := σₙ...	0.812404	0.66
7		n	1.	

F7

$$E\left(X\right)=1.2$$

$$Var\left(X\right)=\sigma^2=0.66$$

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STUDENT NAME: ______________________________

TEACHER INITIALS: ________

CANDIDATE SESSION NUMBER

0	2	5	0	1	2				

EXAMINATION CODE

8	8	1	8	-	7	2	0	1

ST. JOSEPH'S INSTITUTION
YEAR 6 PRELIMINARY EXAMINATION 2018

MATHEMATICS

HIGHER LEVEL

PAPER 1

Friday

29th June 2018

2 hours

1400 – 1600 hrs

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- **Section A**: Answer all questions showing working and answers in the spaces provided in the exam paper.
- **Section B**: Answer all questions using the foolscap paper provided.
- The use of calculators is **not** permitted in this paper.
- A clean copy of the **Mathematics HL Formulae Booklet** is required for this paper.
- Unless otherwise stated in the question all numerical answers are to be given exactly.
- The maximum mark for this examination paper is *[100 marks]*.
- This question paper consists of **10** printed pages including the Cover Sheet.

FOR MARKER USE ONLY:

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	TOTAL
										/100

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are advised to show all working.

SECTION A (50 marks)

1 [Maximum mark: 4]

Find the derivative y' from first principles given

$$y = \frac{3}{x^2}, x \neq 0.$$

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2 **[Maximum mark: 7]**

For some positive integer n,

$$P(x) = (1 + x - 2x^2)^n = 1 + nx + 3x^2 + \cdots$$

(a) Determine the maximum power of x in the expansion by first solving for n. [5]

(b) Find the sum of the roots of $P(x)$ given the value of n found above. [2]

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3 **[Maximum mark: 9]**

(a) Integrate by parts: [5]

$$\int (t+\pi)\cos\left(\frac{t}{2}\right)dt$$

(b) Hence, by introducing an appropriate substitution, evaluate [4]

$$\int_0^{\pi} w\sin\left(\frac{w}{2}\right)dw$$

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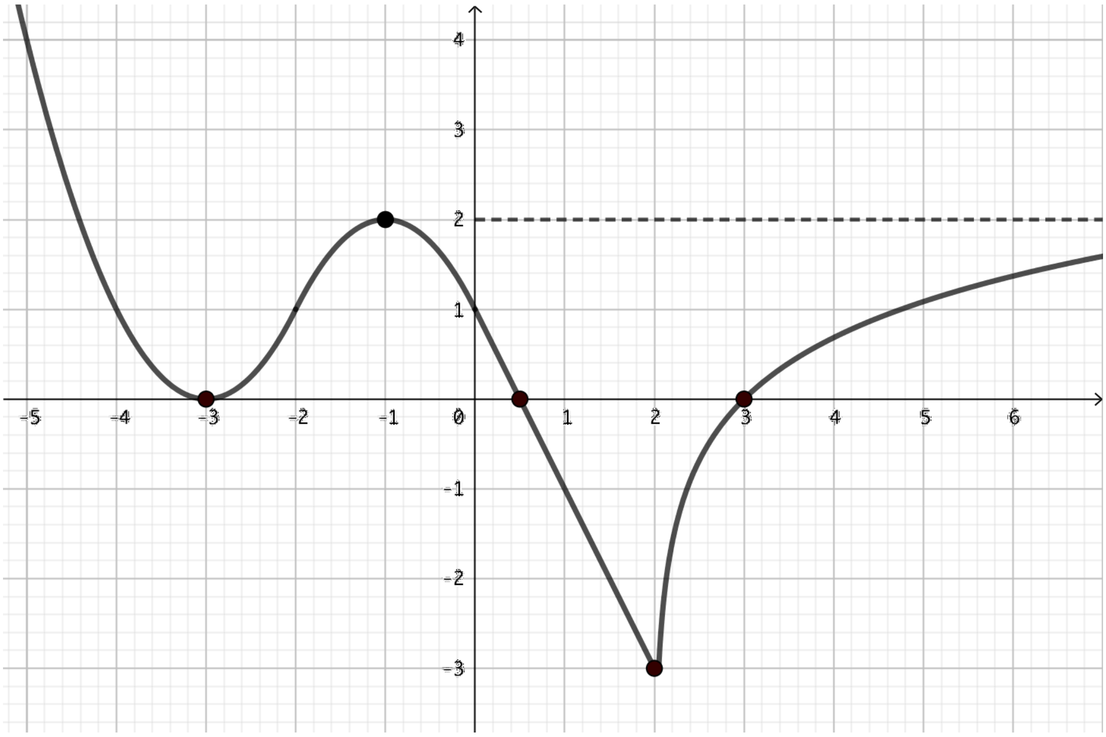
4 **[Maximum mark: 6]**

The graph of $y = f'(x)$ is shown below where $\lim_{x\to\infty} f'(x) = 2$.



(a) On the same set of axes above, sketch $y = f(x)$, indicating clearly the maximum/ minimum points and points of inflexion of the graph. [5]

(b) Evaluate $\lim_{x\to\infty} \frac{f(x)}{x}$. [1]

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5 **[Maximum mark: 6]**

Let $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$.

(a) Justify the following identity whenever $\gcd(a, b) = 1$, $a, b \in \mathbb{N}^+$: [2]

$$\sum_{i,j\in\mathbb{N}} \frac{1}{a^i \cdot b^j} = \left(\sum_{i\,\in\mathbb{N}} \frac{1}{a^i}\right)\left(\sum_{j\in\mathbb{N}} \frac{1}{b^j}\right)$$

where $\gcd(a, b)$ refers to the greatest common divisor of a and b.

(b) Using (a), find the value of the infinite sum [4]

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{12}+\frac{1}{15}+\frac{1}{16}+\cdots = \sum_{i,j,k\in\mathbb{N}} \frac{1}{2^i \cdot 3^j \cdot 5^k}$$

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6 **[Maximum mark: 10]**

(a) The vertices of a triangle are given by $A(-1,0,4)$, $C(1,1,0)$ and $D(2,3,1)$. Another point B is located on the line that connects A and C such that $3\overrightarrow{AC} = 2\overrightarrow{CB}$.

i. Find the vector $\overrightarrow{DB}$. [4]

ii. Find the Cartesian equation of the plane passing through A, B, C and D. [4]

(b) Calculate the shortest distance between the plane $3x - 2y + z = 4$ and the point $A(-1,0,4)$. [2]

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7 **[Maximum mark: 8]**

Let $w \in \mathbb{C}$ be an nth root of unity for some positive integer n such that $w^2 - w + 1 = 0$.

(a) Plot the two possible values of w on the Argand diagram, and hence prove that $n = 6$. [5]

(b) Evaluate $(1 - w)(1 - w^*)(1 - w^2)(1 - (w^*)^2)$, simplifying your final answer. [3]

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*Do **NOT** write solutions on this page.*

SECTION B (50 marks)

*Answer all questions on the foolscap paper provided. Please start each question on a **new** page.*

8 [Maximum mark: 17]

(a) The curve C has equation $x^2y + xy^2 + 54 = 0$.

Find the coordinates of the point on C at which the gradient is –1, showing that there is only one such point. [7]

(b)

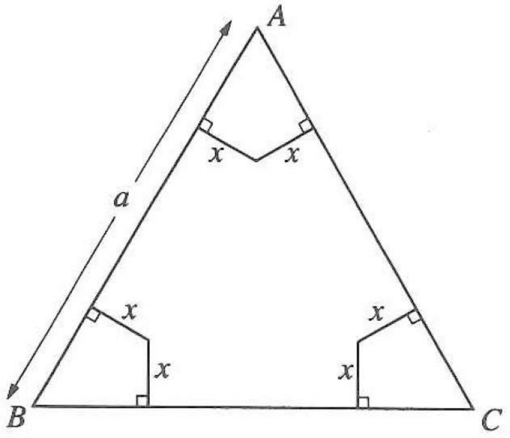


Fig. 1

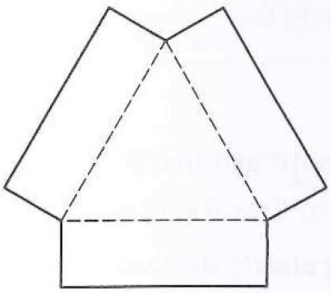


Fig. 2

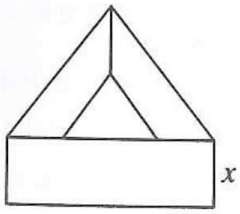


Fig. 3

Fig. 1 shows a piece of card, *ABC*, in the form of an equilateral triangle with sides of length *a*. A kite shape is cut from each corner, to give the shape shown in Fig. 2. The remaining card shown in Fig. 2 is folded along the dotted lines, to form the open triangular prism of height *x* shown in Fig. 3.

(i) Show that the volume V of the prism is given by $V = \frac{\sqrt{3}}{4}x\left(a - 2x\sqrt{3}\right)^2$. [3]

(ii) Find, in terms of *a*, the maximum value of *V*, proving that it is a maximum. [7]

Do ***NOT*** *write solutions on this page.*

9 [Maximum mark: 17]

(a) Solve the equation $\sin^{-1} x - \cos^{-1} x = \sin^{-1}(3x-2)$ where all the angles are principal values. [6]

(b) (i) Prove that $\cos(A+B) - \cos(A-B) = -2\sin A \sin B$. [2]

(ii) **Hence** prove, by mathematical induction, that

$$\sin 3x + \sin 5x + \sin 7x + ... + \sin\left[(2n+1)x\right] = \frac{\cos 2x - \cos\left[2(n+1)x\right]}{2\sin x}$$

for all $n \in \mathbb{Z}^+, \sin x \neq 0$. [9]

10 [Maximum mark: 16]

(a) Given that $y = \dfrac{x^2+x+1}{x-1}$, $x \in \mathbb{R}$, $x \neq 1$, find the range of values that y can take. [6]

(b) The roots of the equation $x^3 - px - q = 0$, where p and q are non-zero real constants, are given by $\alpha, \beta,$ and γ.

(i) Find the sum $\alpha^5 + \beta^5 + \gamma^5$ in terms of p and q. [6]

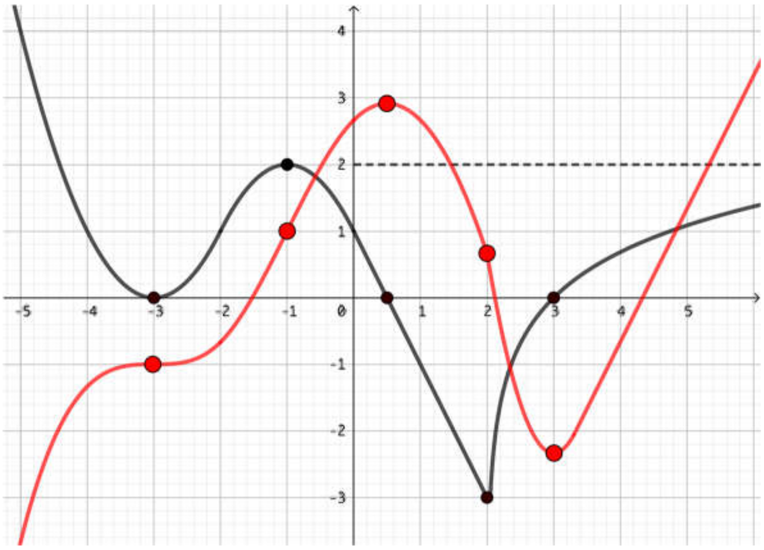
(ii) State the range of values of p for which the given equation has complex roots. [1]

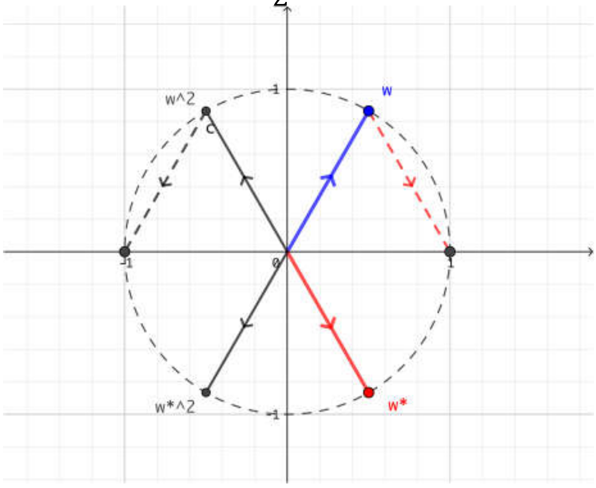
(iii) Using the range of values of p from part (ii), and further given that $\alpha^5 + \beta^5 + \gamma^5 = 5$, determine the sign of the real root of the equation, explaining your answer clearly. [3]

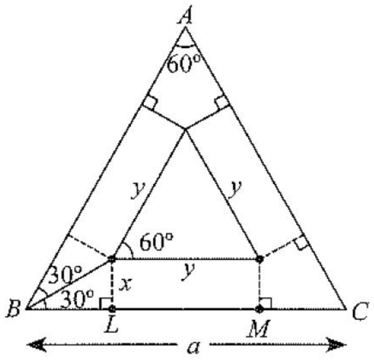
End of Paper

Year 6 HL Math Preliminary Examination 2018 Paper 1 (Markscheme)

Qn	Suggested solution	Markscheme
1	*Differentiation using first principles*	**[Max mark: 4]**
	$y' = \lim_{h\to 0}\frac{\frac{3}{(x+h)^2}-\frac{3}{x^2}}{h} = \lim_{h\to 0}\frac{3x^2-3(x+h)^2}{hx^2(x+h)^2}$ $= \lim_{h\to 0}\frac{-6x-3h}{x^2(x+h)^2} = -\frac{6}{x^3}$	M1A1 M1 – evaluation of $h \to 0$ A1
2	*Binomial Theorem*	**[Max mark: 7]**
(a)	$(1+x-2x^2)^n = (1-x)^n(1+2x)^n$ $= \left(1-\binom{n}{1}x+\binom{n}{2}x^2+\cdots\right)\left(1+\binom{n}{1}2x+\binom{n}{2}4x^2+\cdots\right)$ $4\binom{n}{2}-2\binom{n}{1}\binom{n}{1}+\binom{n}{2}=3$ $n^2-5n-6=(n-6)(n+1)=0$ Thus, $n=6$, and so the maximum power of x is 12.	M1 – any valid approach even if working is not correct, e.g. expanding twice A1 - quadratic M1A1 A1
(b)	Sum of roots is $6\left(1+\left(-\frac{1}{2}\right)\right)=3$	M1 – any valid approach even if working is wrong A1
3	*Integration by Parts + via Substitution*	**[Max mark: 9]**
(a)	$u=t+\pi$ and $dv=\cos\left(\frac{t}{2}\right)dt$ $dt=du$ and $v=2\sin\left(\frac{t}{2}\right)$ $\int(\boldsymbol{t}+\boldsymbol{\pi})\,\mathbf{cos}\left(\frac{\boldsymbol{t}}{\mathbf{2}}\right)\boldsymbol{dt} = 2(t+\pi)\sin\left(\frac{t}{2}\right)-\int 2\sin\left(\frac{t}{2}\right)dt$ $= 2(t+\pi)\sin\left(\frac{t}{2}\right)+4\cos\left(\frac{t}{2}\right)+C$	(M1) (A1) A1 M1A1 – A1A0 if $+C$ is missing
(b)	Let $w=t+\pi$ $\int_0^{\pi}\boldsymbol{w}\,\mathbf{sin}\left(\frac{\boldsymbol{w}}{\mathbf{2}}\right)\boldsymbol{dw} = \int_{-\pi}^{0}(t+\pi)\sin\left(\frac{t+\pi}{2}\right)dt$ $= \int_{-\pi}^{0}(t+\pi)\cos\left(\frac{t}{2}\right)dt = 4$	M1 A1 – correct bounds M1 - cos A1

Qn	Suggested solution	Markscheme
4	***Gradient Graph***	**[Max mark: 6]**
(a)	Inflexion at $x=-3,-1,2$ Max at $x=0.5$ Min at $x=3$  Note: as $x\to+\infty$, $f(x)$ becomes a straight line with gradient 2. Also $f''(2)$ does not exist.	A1 A1 A1 G1 – correct shape (excluding asymptotic behavior) G1
(b)	$\lim_{x\to\infty}\frac{f(x)}{x}=2$	A1
5	***GP***	**[Max mark: 6]**
(a)	Any logical explanation will do. Use of examples is also acceptable. $\sum_{i,j\in\mathbb{N}}\frac{1}{a^ib^j}=1+\frac{1}{a}+\frac{1}{b}+\frac{1}{a^2}+\frac{1}{a^2b}+\frac{1}{ab^2}+\frac{1}{b^2}+\cdots+\frac{1}{a^mb^n}+\cdots$ $=\left(1+\frac{1}{a}+\frac{1}{a^2}+\cdots\right)\left(1+\frac{1}{b}+\frac{1}{b^2}+\cdots\right)$	R1R1
(b)	$\left(\sum_{i\in\mathbb{N}}\frac{\mathbf{1}}{2^i}\right)\left(\sum_{j\in\mathbb{N}}\frac{\mathbf{1}}{3^j}\right)\left(\sum_{j\in\mathbb{N}}\frac{\mathbf{1}}{5^j}\right)=\left(\frac{1}{1-\frac{1}{2}}\right)\left(\frac{1}{1-\frac{1}{3}}\right)\left(\frac{1}{1-\frac{1}{5}}\right)$ $=2\times\frac{3}{2}\times\frac{5}{4}=\frac{15}{4}$	M1A1 M1A1
6	***Ratio Theorem and Distance between a Plane and a Point***	**[Max mark: 10]**
(a)(i)	$\overrightarrow{\boldsymbol{DA}}=\begin{pmatrix}-3\\-3\\3\end{pmatrix},\quad\overrightarrow{\boldsymbol{DC}}=\begin{pmatrix}-1\\-2\\-1\end{pmatrix}$ $\begin{pmatrix}-1\\-2\\-1\end{pmatrix}=\frac{1}{5}\left(3\begin{pmatrix}-3\\-3\\3\end{pmatrix}+2\overrightarrow{\boldsymbol{DB}}\right)$ $\overrightarrow{\boldsymbol{DB}}=\begin{pmatrix}2\\-0.5\\-7\end{pmatrix}$	A1A1 – any relevant vector M1 – correct use of Ratio Theorem or any other method A1

Qn	Suggested solution	Markscheme
(a)(ii)	$\begin{pmatrix}-3\\-3\\3\end{pmatrix}\times\begin{pmatrix}-1\\-2\\-1\end{pmatrix}=\begin{pmatrix}9\\-6\\3\end{pmatrix} \mid\mid \begin{pmatrix}3\\-2\\1\end{pmatrix}$ $3x-2y+z=\begin{pmatrix}3\\-2\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\0\end{pmatrix}=1$	M1A1 M1A1
(b)	$distance=\frac{\lvert 4-1\rvert}{\sqrt{3^2+2^2+1^2}}=\frac{3}{\sqrt{14}}$	M1 - any valid method A1
7	***Roots of unity***	**[Max mark: 8]**
(a)	$w=\frac{1\pm\sqrt{3}i}{2}=e^{\pm i\frac{\pi}{3}}$  $w^6=\left(e^{\pm i\frac{\pi}{3}}\right)^6=e^{\pm i\,2\pi}=1$	M1A1 G1 R1A1
(b)	$(1-w)(1-w^*)(1-w^2)(1-(w^*)^2)$ $=(1-(w+w^*)+ww^*)(1-(w^2+w^{*2})+w^2w^{*2})$ $=(1-1+1)(1-(-1)+1)$ $=3$ Note: Students can use the Cartesian form of w which may result to unnecessary calculation of w^2.	M1 – using whatever w found in (a) M1A1

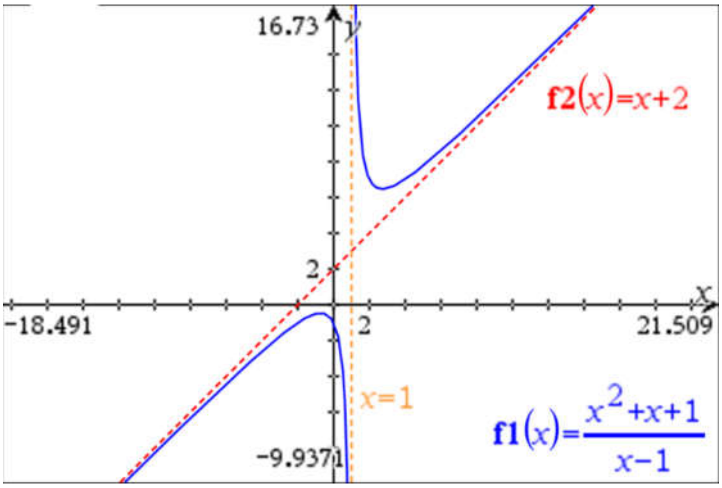
Qn	**Suggested solution**	**Markscheme**
8	***Differentiation (Implicit and Optimisation)***	**[Max mark: 17]**
(a)	$x^2y+xy^2+54=0$ Differentiating wrt. x, $x^2\frac{dy}{dx}+2xy+x\left(2y\frac{dy}{dx}\right)+y^2=0$ $\left(x^2+2xy\right)\frac{dy}{dx}+2xy+y^2=0$ When $\frac{dy}{dx}=-1$, $-\left(x^2+2xy\right)+2xy+y^2=0$ $y^2-x^2=0$ $y=\pm x$ When $y=x$, we have from equation of C: $2x^3+54=0$ $\Rightarrow x^3=-27$ $\Rightarrow y=x=-3$ When $y=-x$, $-x^3+x^3+54=0$ $\Rightarrow 54=0$ (contradiction) Therefore, there is only one point on C with gradient –1, and the point is $\left(-3,-3\right)$.	**M1 M1** – product rule & chain rule **A1** **M1** – subst. $\frac{dy}{dx}=-1$ **A1** – both answers **R1** **A1** – coordinates
(b)i)	 $\tan\frac{\pi}{6}=\frac{x}{BL}$ $BL=\sqrt{3}x$ $y=LM=a-2\sqrt{3}x$ $V=$ Base area × Height $=\frac{1}{2}\left(a-2\sqrt{3}x\right)^2\sin\frac{\pi}{3}\times x$ $=\frac{1}{2}x\left(a-2\sqrt{3}x\right)^2\left(\frac{\sqrt{3}}{2}\right)$ $=\frac{\sqrt{3}}{4}x\left(a-2\sqrt{3}x\right)^2$ (shown)	**A1** **M1 A1** **AG**

Qn	Suggested solution	Markscheme
(b)ii)	**Method 1**	
	$V=\frac{\sqrt{3}}{4}x\left(a-2x\sqrt{3}\right)^2$	
	$\frac{dV}{dx}=\frac{\sqrt{3}}{4}\left(a-2x\sqrt{3}\right)^2-3x\left(a-2x\sqrt{3}\right)$	**A1** – product rule
	$=\frac{\sqrt{3}}{4}\left(a-2x\sqrt{3}\right)\left(a-2x\sqrt{3}-4\sqrt{3}x\right)$	
	$=\frac{\sqrt{3}}{4}\left(a-2\sqrt{3}x\right)\left(a-6\sqrt{3}x\right)$	**A1** – factorised (for 1st derivative test)
	Set $\frac{dV}{dx}=0$	**M1**
	$x=\frac{a}{2\sqrt{3}}$ (rej. $\because a-2\sqrt{3}x\neq 0$) or $x=\frac{a}{6\sqrt{3}}$	**A1 A1** (with ans. rej.)
	$\begin{array}{\|c\|c\|c\|c\|}\hline x & \frac{a}{6\sqrt{3}}^{-} & \frac{a}{6\sqrt{3}} & \frac{a}{6\sqrt{3}}^{+} \\ \hline \frac{dV}{dx} & +\text{ve} & 0 & -\text{ve} \\ \hline\end{array}$	**R1**
	Therefore, V is a maximum when $x=\frac{a}{6\sqrt{3}}$.	
	Max. $V=\frac{\sqrt{3}}{4}\left(\frac{a}{6\sqrt{3}}\right)\left(a-2\sqrt{3}\left(\frac{a}{6\sqrt{3}}\right)\right)^2$	
	$=\frac{a}{24}\left(\frac{2a}{3}\right)^2=\frac{a^3}{54}$	**A1**
	Method 2	
	$V=\frac{\sqrt{3}}{4}x\left(a-2x\sqrt{3}\right)^2$	
	$=\frac{\sqrt{3}}{4}\left(12x^3-4\sqrt{3}ax^2+a^2x\right)$	
	$\frac{dV}{dx}=\frac{\sqrt{3}}{4}\left(36x^2-8\sqrt{3}ax+a^2\right)$	**A1**
	Set $\frac{dV}{dx}=0$	
	$36x^2-8\sqrt{3}ax+a^2=0$	**M1**
	$x=\frac{8\sqrt{3}a\pm\sqrt{192a^2-144a^2}}{72}$	
	$=\frac{8\sqrt{3}a\pm 4\sqrt{3}a}{72}$	
	$=\frac{\sqrt{3}a}{18}$ or $\frac{\sqrt{3}a}{6}$ (rej. $\because y=a-2\sqrt{3}x\neq 0$)	**A1 A1** (with ans. rej.)

Qn	Suggested solution	Markscheme
	$\frac{d^2V}{dx^2}=\frac{\sqrt{3}}{4}\left(72x-8\sqrt{3}a\right)=18\sqrt{3}x-6a$	**A1**
	When $x=\frac{\sqrt{3}a}{18}$,	
	$\frac{d^2V}{dx^2}=\frac{\sqrt{3}}{4}\left(-4\sqrt{3}a\right)=-3a<0$ (since $a>0$)	**R1**
	Therefore, V is a maximum when $x=\frac{\sqrt{3}a}{18}$.	
	Max. $V=\frac{\sqrt{3}}{4}\left(\frac{\sqrt{3}a}{18}\right)\left(a-2\sqrt{3}\left(\frac{\sqrt{3}a}{18}\right)\right)^2$	
	$=\frac{a}{24}\left(\frac{2a}{3}\right)^2=\frac{a^3}{54}$	**A1**

Qn	**Suggested solution**	**Markscheme**
9	***Trigonometry and Proof by Induction***	**[Max mark: 17]**
(a)	Let $\alpha = \sin^{-1} x$ and $\beta = \cos^{-1} x$ Then $\begin{cases} \sin\alpha = x & -\frac{\pi}{2} \le \alpha \le \frac{\pi}{2} \quad \Rightarrow \cos\alpha > 0 \\ \cos\beta = x & 0 \le \beta \le \pi \quad \Rightarrow \sin\beta > 0 \end{cases}$	**R1** – seen anywhere to justify $\cos\alpha = \sin\beta = \sqrt{1-x^2}$
	$\sin^{-1} x - \cos^{-1} x = \sin^{-1}(3x-2)$ [Note: $\frac{1}{3} \le x \le 1$] $\Rightarrow \sin\left(\sin^{-1} x - \cos^{-1} x\right) = 3x-2$ $\Rightarrow \sin(\alpha - \beta) = 3x-2$	**A1**
	$\sin\alpha\cos\beta - \cos\alpha\sin\beta = 3x-2$ $(x)(x) - \sqrt{1-x^2}\sqrt{1-x^2} = 3x-2$ (since $\cos\alpha > 0, \sin\beta > 0$) $x^2 - (1-x^2) = 3x-2$	**M1** – compound angle
	$2x^2 - 3x + 1 = 0$	**A1**
	$x = \frac{1}{2}$ or $x = 1$	**A1 A1**
(b)i)	$\cos(A+B) - \cos(A-B)$ $= \cos A\cos B - \sin A\sin B - (\cos A\cos B + \sin A\sin B)$ $= -2\sin A\sin B$	**M1 A1** **AG**
(b)ii)	Let P_n be the proposition $\sin 3x + \sin 5x + ... + \sin\left[(2n+1)x\right] = \frac{\cos 2x - \cos\left[2(n+1)x\right]}{2\sin x}$ for $n \in \mathbb{Z}^+, \sin x \ne 0$.	
	When $n = 1$, $LHS = \sin 3x$	**M1** – basis step
	$RHS = \frac{\cos 2x - \cos 4x}{2\sin x}$ $= \frac{2\sin 3x \sin x}{2\sin x}$	**M1** – using (b)i)
	$= \sin 3x = LHS$ Therefore P_1 is true.	**R1** – must get M1 (for hence)
	Assume P_k is true for some $k \in \mathbb{Z}^+$ i.e. $\sin 3x + \sin 5x + ... + \sin\left[(2k+1)x\right] = \frac{\cos 2x - \cos\left[2(k+1)x\right]}{2\sin x}$ When $n = k+1$,	**M1** – must assume true (for some *k*, not $\forall k$)

Qn	Suggested solution	Markscheme
	$\sin 3x+\sin 5x+...+\sin\left[(2k+1)x\right]+\sin\left[2(k+1)+1\right]x$ $=\dfrac{\cos 2x-\cos\left[2(k+1)x\right]}{2\sin x}+\sin\left[(2k+3)x\right]$ $=\dfrac{\cos 2x-\cos\left[(2k+2)x\right]+2\sin\left[(2k+3)x\right]\sin x}{2\sin x}$ $=\dfrac{\cos 2x-\cos\left[(2k+2)x\right]+\cos\left[(2k+2)x\right]-\cos\left[(2k+4)x\right]}{2\sin x}$ $=\dfrac{\cos 2x-\cos\left[2(k+2)x\right]}{2\sin x}$ $=\dfrac{\cos 2x-\cos\left[2((k+1)+1)x\right]}{2\sin x}$ Therefore, P_k is assumed true $\Rightarrow P_{k+1}$ is true.	**M1** – using P_k is true **A1** **M1** – using (b)i) **A1**
	Since P_1 is true, and P_k is assumed true $\Rightarrow P_{k+1}$ is true, by mathematical induction, P_n is true for all $n\in\mathbb{Z}^+$.	**R1** – awarded only if previous 5 marks for inductive step were obtained

Qn	**Suggested solution**	**Markscheme**
10	***Functions and Polynomials (Vieta's Formulae)***	**[Max mark: 16]**
(a)	**Method 1**	
	$y=\dfrac{x^2+x+1}{x-1},\quad x\in\mathbb{R},\ x\neq 1$	
	$x^2+(1-y)x+(1+y)=0$	**A1** – quadratic in x
	Since the quadratic equation has real solutions for x, Discriminant ≥ 0	**(R1)**
	$(1-y)^2-4(1+y)\geq 0$	
	$y^2-6y-3\geq 0$	**A1** – with $\geq$
	Solving $y^2-6y-3=0$:	
	$y=\dfrac{6\pm\sqrt{48}}{2}=3\pm 2\sqrt{3}$	**M1** – solving quad. ineq.
	$\therefore y\leq 3-2\sqrt{3}$ or $y\geq 3+2\sqrt{3}$	**A1 A1** (A1 if strict ineq.)
	Method 2	
	$y=\dfrac{x^2+x+1}{x-1}=x+2+\dfrac{3}{x-1}$	**A1**
	Set $\dfrac{dy}{dx}=1-\dfrac{3}{(x-1)^2}=0$	
	$\Rightarrow x=1\pm\sqrt{3},\ y=3\pm 2\sqrt{3}$	
	$\dfrac{d^2y}{dx^2}=\dfrac{6}{(x-1)^3}$	
	$\left.\dfrac{d^2y}{dx^2}\right\|_{x=1+\sqrt{3}}=\dfrac{6}{3\sqrt{3}}>0\ \therefore y=3+2\sqrt{3}$ min. value	
	$\left.\dfrac{d^2y}{dx^2}\right\|_{x=1-\sqrt{3}}=-\dfrac{6}{3\sqrt{3}}<0\ \therefore y=3-2\sqrt{3}$ max. value	**M1** – finding max & min points
		**G1** – sketch of graph **G1** – oblique asymptote $y=x+2$ and vertical asymptote $x=1$
	$\therefore y\leq 3-2\sqrt{3}$ or $y\geq 3+2\sqrt{3}$	**A1 A1** (A1 if strict ineq.)

Qn	Suggested solution	Markscheme
(b)i)	$\alpha+\beta+\gamma=0$ $\sum\alpha\beta=-p$ $\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2\left(\sum\alpha\beta\right)$ $\qquad\qquad=2p$ From $x^3-px-q=0$, we get $\alpha^3-p\alpha-q=0$ $\beta^3-p\beta-q=0$ $\gamma^3-p\gamma-q=0$ Adding, $\alpha^3+\beta^3+\gamma^3=p(0)+3q=3q$ Similarly, multiplying by x^2: $\alpha^5-p\alpha^3-q\alpha^2=0$ $\beta^5-p\beta^3-q\beta^2=0$ $\gamma^5-p\gamma^3-q\gamma^2=0$ Therefore, $\alpha^5+\beta^5+\gamma^5=p\left(\alpha^3+\beta^3+\gamma^3\right)+q\left(\alpha^2+\beta^2+\gamma^2\right)$ $\qquad\qquad=p(3q)+q(2p)$ $\qquad\qquad=5pq$	**M1** **A1** } No additional 2 marks if this appears in (b)ii). (b)ii) only 1 mark. **M1** **A1** **M1** **A1**
(b)ii)	$p<0$	**A1**
(b)iii)	$\alpha^5+\beta^5+\gamma^5=5pq=5$ Since $p<0$ and $pq=1$, $q<0$ Since coefficients are real, complex roots occur in complex conjugate pairs. By Vieta's Formula, $\alpha\beta\beta^*=q$ Since $\beta\beta^*=\lvert\beta\rvert^2>0$ and $q<0$, the real root $\alpha<0$	**M1** – product of roots with complex conjugates **R1** – must be $\beta\beta^*$, not just any *zw* **A1** – with working, no follow through from $p>0$

STUDENT NAME: ______________________________

TEACHER INITIALS: ________

CANDIDATE SESSION NUMBER

0	2	5	0	1	2				

EXAMINATION CODE

8	8	1	8	-	7	2	0	2

ST JOSEPH'S INSTITUTION
YEAR 6 PRELIMINARY EXAMINATION 2018

MATHEMATICS
HIGHER LEVEL
PAPER 2
Thursday

5th July 2018
2 hours
0800 – 1000 hrs

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- **Section A**: Answer all questions showing working and answers in the spaces provided in the exam paper.
- **Section B**: Answer all questions using the foolscap paper provided.
- The use of a scientific or graphical calculator is permitted in this paper.
- TI-Nspire calculators must be in Press-to-Test mode and cleared of all previous data.
- TI-84+ graphical calculators must only have permitted Apps and be ram-cleared.
- It is the responsibility of the student to ensure their calculator is examination ready.
- A clean copy of the **Mathematics HL Formulae Booklet** is required for this paper.
- Unless otherwise stated in the question all numerical answers are to be given exactly.
- The maximum mark for this examination paper is *[100 marks]*.
- This question paper consists of **11** printed pages including the Cover Sheet.

FOR MARKER USE ONLY:

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	TOTAL
										/100

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphical display calculator should be supported by suitable working; for example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are advised to show all working.

SECTION A (50 marks)

1 [Maximum mark: 7]

Consider the system of equations below where p and q are real.

$$\begin{aligned} 2x+2y+z&=p \\ x-2y+2z&=-7 \\ qx-6y+11z&=-16 \end{aligned}$$

Find the value of p and q for which the system has an infinite number of solutions.

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2 **[Maximum mark: 8]**

For events *A* and *B*, it is given that $\mathrm{P}(A\,|\,B') = \frac{4}{7}$, $\mathrm{P}(B'\,|\,A') = \frac{2}{3}$ and $\mathrm{P}(A) = \frac{11}{20}$.

Give a reason why events *A* and *B* are not independent. [1]

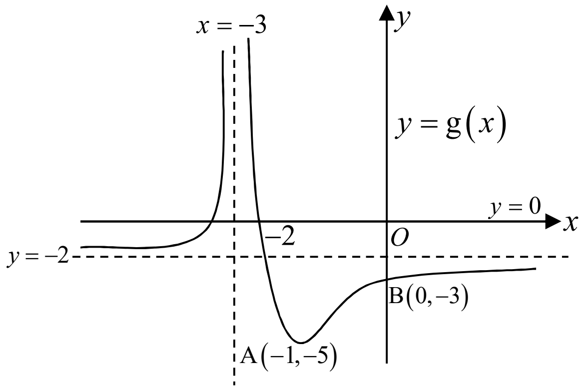
Find

(i) $\mathrm{P}\left(B \cap A'\right)$. [3]

(ii) the probability that exactly either one of A and B will occur. [4]

3 [Maximum mark: 6]

The graph of $y = g(x)$ is shown below.



(i) Sketch $y = \frac{1}{g(x)}$ on a separate diagram, indicating clearly the new coordinates of A and B, and the asymptotes. [3]

(ii) Sketch $y = -\frac{g'(x)}{\left[g(x)\right]^2}$ on a separate diagram, indicating clearly the x-intercepts and the asymptotes. [3]

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4 **[Maximum mark: 8]**

(a) It is given that, for some constant c, $\int \sec x \, \mathrm{d}x = \ln\left|\sec x + \tan x\right| + c$.

By writing $\sec^3 x = \sec x \sec^2 x$, find $\int \sec^3 x \, \mathrm{d}x$. [6]

(b) Find the value of α such that $0 < \tan^{-1}\alpha < \frac{\pi}{2}$ and $\int_0^{\tan^{-1}\alpha} \sec^3 x \, \mathrm{d}x = \sqrt{5} + \frac{1}{2}\ln(\sqrt{5}+2)$.

[2]

5 [Maximum mark: 8]

The complex number z is given by $z = r\,\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$. It is given that the complex number $w = \left(-\sqrt{3} - \mathrm{i}\right)z$.

(i) Find $|w|$ in terms of r, and $\arg w$ in terms of θ and π. [3]

(ii) Find the largest negative value of θ in terms of π such that $\dfrac{z^3}{w^*}$ is purely real. [5]

6 [Maximum mark: 5]

A circle has two fixed points A and B such that AB is the diameter of length 10 cm. A point P moves on the circle. Given that the length of the line segment AP decreases at the rate of 0.4 cm s^{-1}, find the exact rate at which the angle PAB is changing when $AP = 5$ cm.

7 [Maximum mark: 8]

Relative to the origin O, the points A, B, M and N have position vectors **a, b**, **m** and **n** respectively, where **a** and **b** are non-parallel vectors.

It is given that $\mathbf{m} = \lambda\mathbf{a} + (1 - \lambda)\mathbf{b}$ and $\mathbf{n} = 2(1 - \lambda)\mathbf{a} - \lambda\mathbf{b}$ where λ is a real parameter.
Show that $\mathbf{m} \times \mathbf{n} = \left(3\lambda^2 - 4\lambda + 2\right)\left(\mathbf{b} \times \mathbf{a}\right)$. [3]

It is given that $|\mathbf{a}| = 3$, $|\mathbf{b}| = 4$, and the angle between **a** and **b** is $\frac{\pi}{6}$. Find the smallest area of the triangle MON as λ varies. [5]

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SECTION B (50 marks)

Answer all questions on the foolscap paper provided. Please start each question on a new page.

8 [Maximum mark: 18]

(a) The fourth, seventh and sixteenth terms of a non-trivial A.P. are in geometric progression. If the first six terms of the A.P. have a sum of 12, find the common difference of the A.P. and the common ratio of the G.P. [8]

Hence determine the number of terms required for the sum of the terms of the A.P. to first exceed 1000. [2]

(b) A convergent function f(x) is defined on its maximal domain, $D_{\mathrm{f}} \subseteq \mathbb{R}$, by

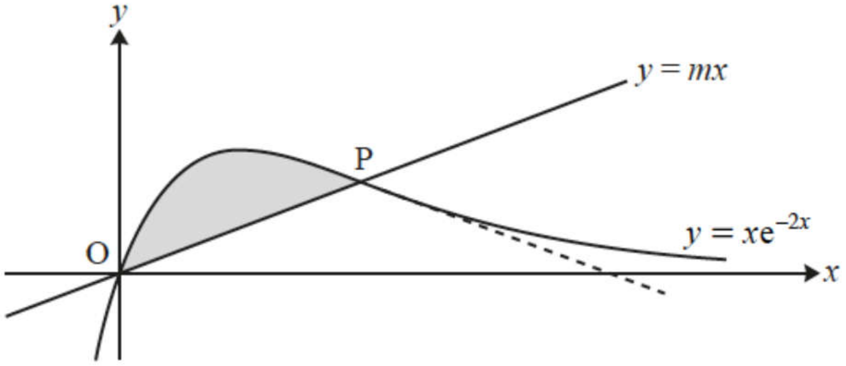
$$\mathrm{f}(x) = 1 + \frac{2}{x} + \frac{4}{x^2} + \frac{8}{x^3} + \frac{16}{x^4} + \cdots.$$

(i) Determine the maximal domain, D_{f}. [3]

(ii) Express $\mathrm{f}(x)$ in simplified rational function form. Hence sketch the graph of f(x) on D_{f}. [5]

9 [Maximum mark: 22]

The diagram below shows the curve $y = x\mathrm{e}^{-2x}$ together with the straight line $\mathrm{y = mx}$, where m is a constant with $0 < m < 1$. The curve and the line meet at O and P. The dashed line is the tangent at P.



(a) Find to 3 s.f. the coordinates of the point of inflexion of the curve $y = x\mathrm{e}^{-2x}$. [3]

(b) Show that the x-coordinate of point P is $-\frac{1}{2}\ln m$. [3]

(c) Find, in terms of m, the gradient of the tangent to the curve at P. [4]

This question continues overleaf

*Do **NOT** write solutions on this page.*

9 (continued)

The tangent and the line *OP* make the same angle with the *x*-axis.

(d) Show that $m = \mathrm{e}^{-2}$ and find the exact coordinates of P. [4]

(e) Hence, find the exact area of the shaded region enclosed between the line OP and the curve $y = x\mathrm{e}^{-2x}$. [5]

(f) Find the volume of the solid of revolution generated when the shaded area is rotated 360° about the *x*-axis, giving your answer correct to 3 s.f. [3]

10 [Maximum mark: 10]

(i) A sports club has 50 members, of whom 23 play hockey. If all the sports club members were arranged in a row at random, find

(a) the total number of arrangements in which all the hockey players would be standing together; [2]

(b) the probability that there would be a hockey player at each end of the row. [4]

(ii) As part of an experiment a computer has to assign an integer between 1 and *m* inclusive to each of *n* participants where $n \leq m$. It does this randomly never assigning the same integer twice.

If *K* is the highest value that is assigned, show that the probability of a participant being assigned the highest integer value is

$$P(K = k) = \frac{\binom{k-1}{n-1}}{\binom{m}{n}}, \quad \text{for } n \leq k \leq m.$$

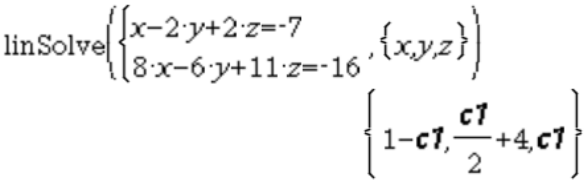
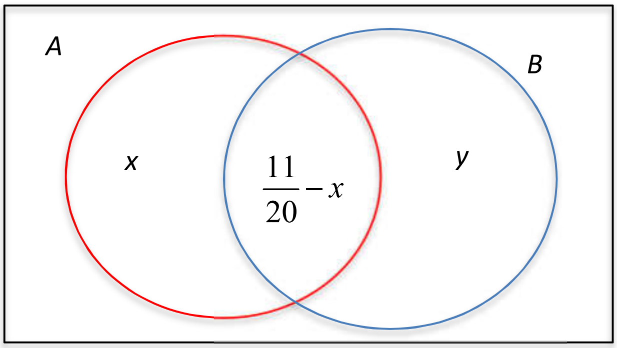
Deduce the value of

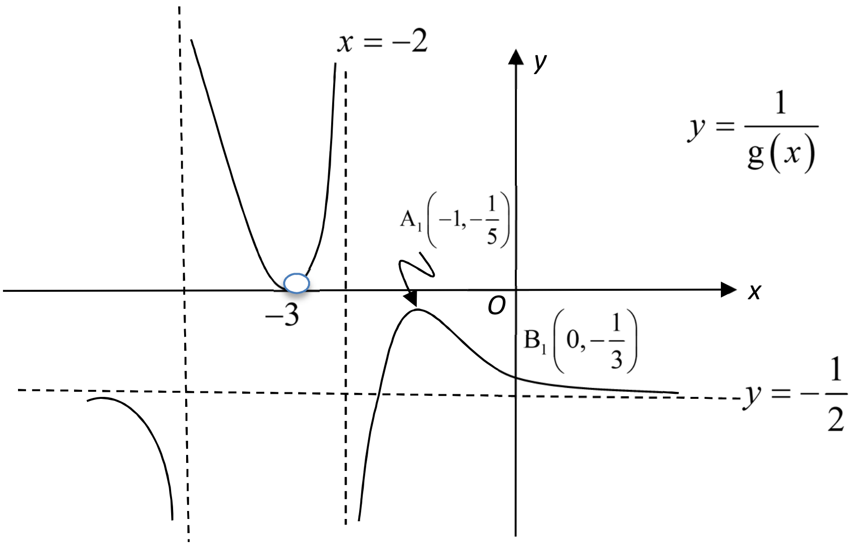
$$\sum_{k=n}^{m} \binom{k-1}{n-1}.$$

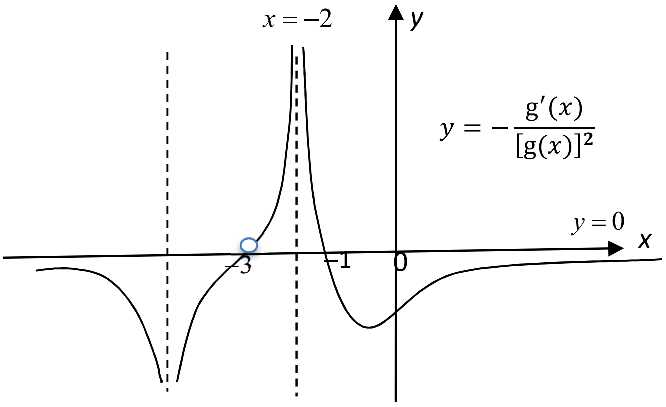
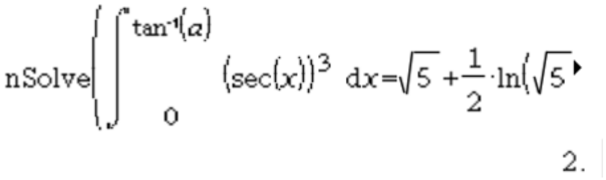
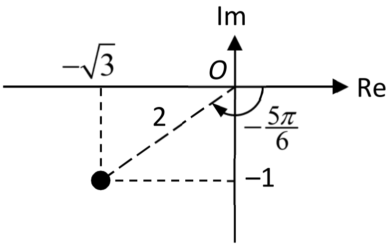
[4]

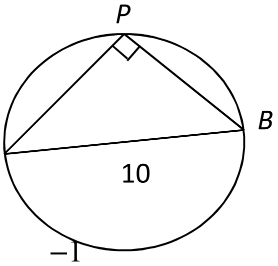
End of Paper

Year 6 HL Math Preliminary Examination 2018 Paper 2 (Markscheme)

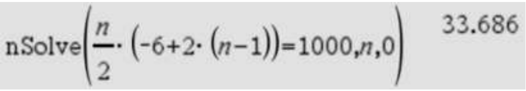
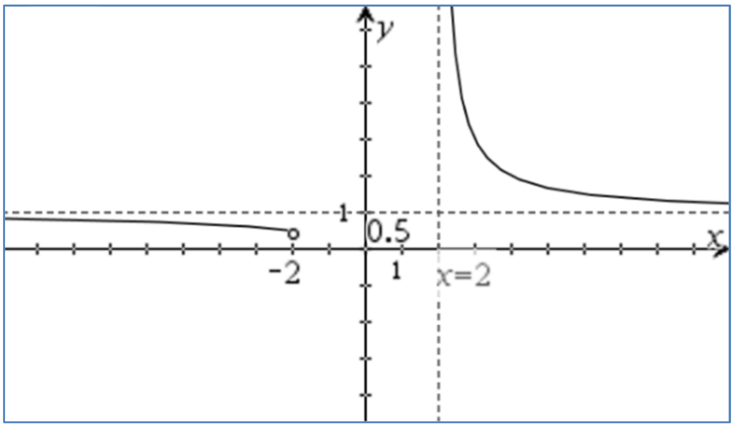
Qn	Suggested Solutions	Marks
1	**Systems of Linear Equations & Vectors**	**[Max mark: 7]**
	$\pi_1: 2x+2y+z=p$ $\pi_2: x-2y+2z=-7$ $\pi_3: qx-6y+11z=-16$	
	$\mathbf{n}_1\times\mathbf{n}_2=\begin{pmatrix}2\\2\\1\end{pmatrix}\times\begin{pmatrix}1\\-2\\2\end{pmatrix}=-3\begin{pmatrix}-2\\1\\2\end{pmatrix}$	**M1A1**
	For the system to have an infinite number of solutions, $\begin{pmatrix}-2\\1\\2\end{pmatrix}\cdot\begin{pmatrix}q\\-6\\11\end{pmatrix}=0\Rightarrow -2q-6+22=0\Rightarrow q=8$	**M1A1**
	 $\text{linSolve}\left(\begin{cases}x-2\cdot y+2\cdot z=-7\\8\cdot x-6\cdot y+11\cdot z=-16\end{cases},\{x,y,z\}\right)$ $\left\{1-c1,\frac{c1}{2}+4,c1\right\}$	**M1**
	Since (1,4,0) lies on π_1, $2(1)+2(4)+z(0)=p \quad \Rightarrow p=10$	**M1A1**
2	**Probability**	**[Max mark: 8]**
	Since $\mathrm{P}(A)\neq\mathrm{P}(A\mid B')$, events A and B' are not independent. Hence A and B are not independent.	**R1**
(i)	**Method 1** $\mathrm{P}(B'\mid A')=\frac{2}{3}\Longrightarrow\mathrm{P}(B\mid A')=\frac{1}{3}$ $\mathrm{P}(B\mid A')=\frac{1}{3}=\frac{\mathrm{P}(A'\cap B)}{\mathrm{P}(A')}\Longrightarrow\mathrm{P}(A'\cap B)=\frac{1}{3}\times\frac{9}{20}=\frac{3}{20}$	**M1** **M1A1**
	Method 2 	

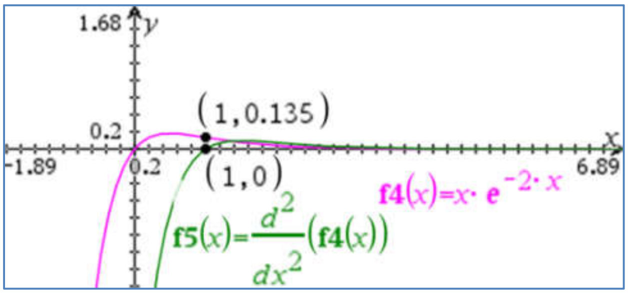
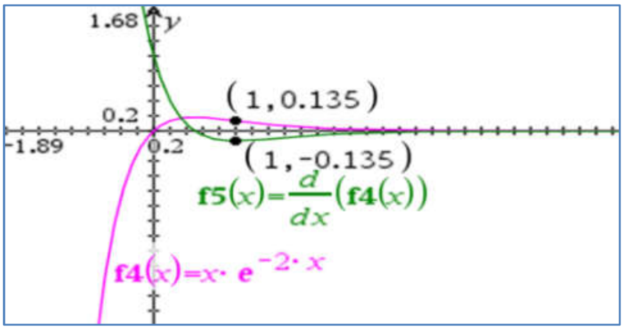
	$\mathrm{P}(B' \mid A') = \frac{2}{3}$ $\Rightarrow \frac{\mathrm{P}(B' \cap A')}{\mathrm{P}(A')} = \frac{2}{3}$ $\Rightarrow \frac{1 - \frac{11}{20} - y}{1 - \frac{11}{20}} = \frac{2}{3}$ $\Rightarrow y = \frac{3}{20}$ $\Rightarrow \mathrm{P}\left(B \cap A'\right) = \frac{3}{20}$	**M1** **A1** **A1**
(ii)	$\mathrm{P}(A \mid B') = \frac{4}{7} \Rightarrow \frac{\mathrm{P}(A \cap B')}{\mathrm{P}(B')} = \frac{4}{7}$ $\Rightarrow \frac{\mathrm{P}(A \cap B')}{1 - \mathrm{P}(B)} = \frac{4}{7}$ $\Rightarrow \frac{x}{1 - \left(\frac{11}{20} - x + y\right)} = \frac{4}{7}$ $\Rightarrow \frac{x}{1 - \left(\frac{11}{20} - x + \frac{3}{20}\right)} = \frac{4}{7}$ $\Rightarrow x = \mathrm{P}(A \cap B') = \frac{2}{5}$ P(exactly one of them will occur) = $x + y = \frac{3}{20} + \frac{2}{5} = \frac{11}{20}$	**M1** **A1** **A1** **A1**
3	**Reciprocal and Gradient Graphs**	**[Max mark: 6]**
(i)		[G1] Shape [G1] asymptotes. [G1] coordinates.

(ii)		[G1] [G1] Shape [G1] asymptotes x-intercepts
4	**Integration by parts**	**[Max mark: 8]**
(a)	$\int \sec^3 x \text{ d}x$ $=\int \sec x \sec^2 x \text{ d}x$ $=\sec x\tan x-\int \sec x\tan^2 x \text{ d}x$ $=\sec x\tan x-\int \sec x(\sec^2 x-1) \text{ d}x$ $=\sec x\tan x-\int \sec^3 x-\sec x \text{ d}x$ $=\sec x\tan x-\int \sec^3 x \text{ d}x+\ln\lvert\sec x+\tan x\rvert$ $2\int \sec^3 x \text{ d}x=\sec x\tan x+\ln\lvert\sec x+\tan x\rvert$ $\int \sec^3 x \text{ d}x=\frac{1}{2}\left(\sec x\tan x+\ln\lvert\sec x+\tan x\rvert\right)+C$	**M1A1** **M1** **A1** **M1** **A1**
(b)	$\text{nSolve}\left(\int_0^{\tan^{-1}(\alpha)} (\sec(x))^3 \text{ d}x=\sqrt{5}+\frac{1}{2}\cdot\ln(\sqrt{5}\right.$ $2.$ $\alpha=2$	**M1** **A1**
5	**Complex Numbers in Exponential Form**	**[Max mark: 8]**
(i)	**Method 1** $w=\left(-\sqrt{3}-\text{i}\right)z$ $=[2\text{e}^{\text{i}\left(-\frac{5\pi}{6}\right)}]r\text{e}^{\text{i}\theta}$ $=2r\text{e}^{\text{i}\left(-\frac{5\pi}{6}+\theta\right)}$ $\therefore \lvert w\rvert=2r,\ \arg w=-\frac{5\pi}{6}+\theta$ 	**M1** **A1A1**

	Method 2	
	$\lvert w\rvert=\left\lvert\left(-\sqrt{3}-\mathrm{i}\right)z\right\rvert$	**M1**
	$=\left\lvert\left(-\sqrt{3}-\mathrm{i}\right)\right\rvert\lvert z\rvert$	**A1**
	$=2r$	
	$\arg w=\arg\left(\left[-\sqrt{3}-\mathrm{i}\right]z\right)$	
	$=\arg\left(-\sqrt{3}-\mathrm{i}\right)+\arg z$	
	$=-\dfrac{5\pi}{6}+\theta$	**A1**
(ii)	$\arg\left(\dfrac{z^3}{w*}\right)=\arg(z^3)-\arg(w*)$	**M1**
	$=3\theta+\arg w$	
	$=3\theta+\left(-\dfrac{5\pi}{6}+\theta\right)$	
	$=4\theta-\dfrac{5\pi}{6}$	**A1**
	For $\dfrac{z^3}{w*}$ to be purely real, $\arg\left(\frac{z^3}{w^*}\right)=k\pi,\ k\in\mathbb{Z}^-$	**M1**
	$\therefore\ 4\theta-\dfrac{5\pi}{6}=-\pi$	**A1**
	$\Rightarrow\theta=-\dfrac{\pi}{24}$	
	$\therefore$ the largest negative value of θ is $-\dfrac{\pi}{24}$.	**A1**
6	**Connected Rates**	**[Max mark: 5]**
	Method 1	
	Let $AP=x$ and $\angle PAB=\theta$.	
	Given $\dfrac{\mathrm{d}x}{\mathrm{d}t}=-0.4$ cm s^{-1}, find $\dfrac{\mathrm{d}\theta}{\mathrm{d}t}$.	
	$\cos\theta=\dfrac{x}{10}\Rightarrow\theta=\cos^{-1}\left(\dfrac{x}{10}\right)$ 	**A1**
	$\Rightarrow\dfrac{\mathrm{d}\theta}{\mathrm{d}x}=\dfrac{-\frac{1}{10}}{\sqrt{1-\left(\frac{x}{10}\right)^2}}=\dfrac{-\frac{1}{10}}{\frac{1}{10}\sqrt{100-x^2}}=\dfrac{-1}{\sqrt{100-x^2}}$	**M1A1**
	At $x=5\,\text{cm}$,	
	$\dfrac{\mathrm{d}\theta}{\mathrm{dt}}=\dfrac{\mathrm{d}\theta}{\mathrm{d}x}\times\dfrac{\mathrm{d}x}{\mathrm{d}t}=\dfrac{-1}{\sqrt{100-5^2}}(-0.4)=\dfrac{2}{25\sqrt{3}}=\dfrac{2\sqrt{3}}{75}\text{rad s}^{-1}$	**M1A1**

	Method 2 Let $AP = x$ and $\angle PAB = \theta$. Given $\frac{\mathrm{d}x}{\mathrm{d}t} = -0.4$ cm s^{-1}, find $\frac{\mathrm{d}\theta}{\mathrm{d}t}$. $\cos\theta = \frac{x}{10}$ Differentiate wrt t, $(-\sin\theta)\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{1}{10}\frac{\mathrm{d}x}{\mathrm{d}t}$ When $x = 5$, $\cos\theta = \frac{1}{2} \Rightarrow \theta = 60^\circ$ $\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{-0.04}{-\sin 60^\circ} = \frac{2}{25\sqrt{3}} = \frac{2\sqrt{3}}{75}$ rad s^{-1}	**A1** **M1** **A1** **M1A1**
7	**The Vector Product**	**[Max mark: 8]**
(i)	$\mathbf{m}\times\mathbf{n} = (\lambda\mathbf{a} + (1-\lambda)\mathbf{b}) \times (2(1-\lambda)\mathbf{a} - \lambda\mathbf{b})$ $= 2\lambda(1-\lambda)(\mathbf{a}\times\mathbf{a}) - \lambda^2(\mathbf{a}\times\mathbf{b}) + 2(1-\lambda)^2(\mathbf{b}\times\mathbf{a}) - \lambda(1-\lambda)(\mathbf{b}\times\mathbf{b})$ $= 2(1-\lambda)^2(\mathbf{b}\times\mathbf{a}) - \lambda^2(\mathbf{a}\times\mathbf{b})$ since $\mathbf{a}\times\mathbf{a} = \underset{\sim}{\mathbf{0}} = \mathbf{b}\times\mathbf{b}$ $= (2(1-\lambda)^2 + \lambda^2)(\mathbf{b}\times\mathbf{a})$ since $\mathbf{a}\times\mathbf{b} = -\,\mathbf{b}\times\mathbf{a}$ $= (3\lambda^2 - 4\lambda + 2)(\mathbf{b}\times\mathbf{a})$	M1 A1 R1
(ii)	Area of triangle $MON = \frac{1}{2}\lvert\mathbf{m}\times\mathbf{n}\rvert = \frac{1}{2}\lvert(3\lambda^2 - 4\lambda + 2)(\mathbf{b}\times\mathbf{a})\rvert$ $= \frac{1}{2}\lvert 3\lambda^2 - 4\lambda + 2\rvert\lvert\mathbf{b}\times\mathbf{a}\rvert$ $= \frac{1}{2}\left\lvert 3\left(\lambda - \frac{2}{3}\right)^2 + \frac{2}{3}\right\rvert \lvert\mathbf{b}\rvert\lvert\mathbf{a}\rvert\sin\frac{\pi}{6}$ $= 3\left\lvert 3\left(\lambda - \frac{2}{3}\right)^2 + \frac{2}{3}\right\rvert$ $\therefore$ smallest area is $3 \times \frac{2}{3} = 2$ units^2	M1 A1 M1 A1 A1
8	**APs/GPs, Rational Functions**	**[Max mark: 18]**
(a)	The AP has terms $a + 3d, a + 6d, a + 15d$ $r = \frac{a+6d}{a+3d} = \frac{a+15d}{a+6d}$ $\Rightarrow a^2 + 12ad + 36d^2 = a^2 + 18ad + 45d^2$ $\Rightarrow 6ad + 9d^2 = 0$ $\Rightarrow d(2a + 3d) = 0$ or $a = -\frac{3}{2}d$ o.e. For AP: $S_6 = 12 = 3(2a + 5d)$	**M1** **A1** **M1**

	$\Rightarrow 2a = 4 - 5d$ $\therefore d(4 - 5d + 3d) = d(4 - 2d) = 0 \ \ (d \neq 0)$ $\Rightarrow d = 2$ $\boldsymbol{d} = \mathbf{2}, a = -3 \Rightarrow \boldsymbol{r} = \dfrac{\mathbf{-3 + 12}}{\mathbf{-3 + 6}} = \mathbf{3}$ $\mathrm{S}_n = \dfrac{n}{2}[-6 + 2(n - 1)] > 1000$ E.g. Using nsolve we obtain  $\mathrm{nSolve}\left(\frac{n}{2}\cdot(-6+2\cdot(n-1))=1000,n,0\right)$ 33.686 $\therefore$ required number of terms $= 34$	**A1** **M1** **A1** **A1 A1** **G1** **(or M1 if algebraic solution attempted)** **A1**
(b)(i)	$\mathrm{f}(x) = 1 + \dfrac{2}{x} + \dfrac{4}{x^2} + \dfrac{8}{x^3} + \dfrac{16}{x^4} + \cdots = \sum_{r=0}^{\infty}\left(\dfrac{2}{x}\right)^r, \lvert\mathrm{f}(x)\rvert < \infty$ f(x) represents a convergent infinite GP with common ratio $\dfrac{2}{x}$. $\therefore \ \left\lvert\dfrac{2}{x}\right\rvert < 1 \Rightarrow \lvert x\rvert > 2$ $\therefore$ maximal domain, $D_\mathrm{f} = \{x \in \mathbb{R}: x < -2 \text{ or } x > 2\}$, o.e.	**R1** **M1** **A1**
(b)(ii)	Using sum of infinite GP: $\mathrm{f}(x) = \dfrac{1}{1 - \frac{2}{x}} = \dfrac{x}{x - 2}$ or $1 + \dfrac{2}{x - 2}$ on D_f  	**M1 A1** **G1** two branches **G1** $y = 1, x = 2$ asymptotes **G1** hollow point at $(-2,0.5)$

9	**Differentiation and Integration Applications, Logs/Exp**	**[Max mark: 22]**
(a)	Using GDC for sketch of y'':  **Or** using GDC for sketch of y':  $\therefore$ Point of inflection is at $(1.00,0.135)$ to 3 s.f. **NOTE 1: M1** for correct calculation of $\frac{d^2y}{dx^2}=4\mathrm{e}^{-2x}(x-1)$. **NOTE 2: A1 only** for $(1.00,0.135)$ obtained from solving $\frac{d^2y}{dx^2}=0$.	**G1** **A1 for x =1** **Or** **G1** **A1 for x =1** **A1**
(b)	$xe^{-2x}=mx$ $\Rightarrow x(\mathrm{e}^{-2x}-m)=0$ $\Rightarrow m=\mathrm{e}^{-2x}$ (rej $x=0$) $\Rightarrow -2x=\ln m$ $\Rightarrow x=-\frac{1}{2}\ln m$	**M1** **A1** **A1** **AG**
(c)	$\frac{\mathrm{d}}{\mathrm{d}x}(xe^{-2x})=\mathrm{e}^{-2x}-2x\mathrm{e}^{-2x}$ $(\mathrm{e}^{-2x}-2x\mathrm{e}^{-2x})\vert_{x=-\frac{1}{2}\ln m}=\mathrm{e}^{\ln m}+\mathrm{e}^{\ln m}\ln m$ $=m+m\ln m=m(1+\ln m)$	**M1 A1** **M1** **A1**
(d)	$m(1+\ln m)=-m$ $\Rightarrow \ln m=-2$ ($\because 0<m<1$) $\Rightarrow m=\mathrm{e}^{-2}$ From (b) $x=-\frac{1}{2}\ln m \Rightarrow x=-\frac{1}{2}\ln \mathrm{e}^{-2}=1$ $\therefore$ P coordinates: $(1,\mathrm{e}^{-2})$.	**M1** **A1** **AG** **M1** **A1**

	Or	
	At P: $y = xe^{-2x} = mx \Rightarrow xe^{-2x} = e^{-2}x$	**M1**
	$\Rightarrow x = 1$ $\quad(\because$ at P, $x = -\frac{1}{2}\ln m \neq 0)$	
	$\therefore$ P coordinates: $(1, e^{-2})$.	**A1**
(e)	**<u>Method 1</u>**	
	Shaded area $= \int_0^1 xe^{-2x}dx - \frac{1}{2}e^{-2}$	**M1 A1**
	$= \left[-\frac{1}{2}xe^{-2x}\right]_0^1 - \int_0^1 -\frac{1}{2}e^{-2x}dx - \frac{1}{2}e^{-2}$	**M1 (correctly by parts)**
	$= \left[-\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}\right]_0^1 - \frac{1}{2}e^{-2}$ o.e.	**A1**
	$= \left(\frac{1}{4} - \frac{3}{4}e^{-2}\right) - \frac{1}{2}e^{-2} = \frac{1}{4} - \frac{5}{4}e^{-2}$	**A1**
	<u>Method 2</u>	
	Shaded area $= \int_0^1 xe^{-2x} - e^{-2}x\,dx$	**M1**
	$= \left[-\frac{1}{2}xe^{-2x}\right]_0^1 - \int_0^1 -\frac{1}{2}e^{-2x}dx - e^{-2}\left[\frac{1}{2}x^2\right]_0^1$	**M1(by parts) A1**
	$= \left[-\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}\right]_0^1 - \frac{1}{2}e^{-2}$ o.e.	**A1**
	$= \left(\frac{1}{4} - \frac{3}{4}e^{-2}\right) - \frac{1}{2}e^{-2} = \frac{1}{4} - \frac{5}{4}e^{-2}$	**A1**
(f)	$V_x = \pi\int_0^1 (xe^{-2x})^2 - (e^{-2}x)^2dx$	**M1 A1**
	$= 0.0556$ (3 s.f.)	**A1**
	Or	
	$V_x = \pi\int_0^1 (xe^{-2x})^2dx - \frac{1}{3}\pi(e^{-2})^2 \times 1$	**M1 A1**
	$= 0.074799 - 0.019180 = 0.0556$ (3 s.f.)	**A1**
10	**P&C, Probability**	**[Max mark: 10]**
(i)(a)	Treating the hockey players as one person, the number of permutations is $(50 - 23 + 1)!\,23!$	**M1**
	$= 28!\,23!$ o.e. [Accept 7.88×10^{51} (3 s.f.)]	**A1**
(i)(b)	**<u>Method 1</u>**	
	There are 23 ways to choose the hockey player on the left end, and 22 ways to choose the hockey player on the right end (or vice versa).	**M1**
	Number of ways of arranging remaining members in between is $(50 - 2)! = 48!$	**A1**
	$\therefore$ Required probability $= \frac{23\times22\times48!}{50!}$ or $\frac{^{23}C_2\times2!\times48!}{50!}$	**M1**
	$= \frac{253}{1225} = 0.207$ (3 s.f.)	**A1**

	Method 2 Probability of first and last person being a hockey player $= \text{P(1st a hockey player)} \times \text{P(last a hockey player)} = \frac{23}{50} \times \frac{22}{49}$ $= \frac{506}{2450} = \frac{253}{1225} = 0.207$ (3 s.f.) **Method 3** Required probability $= \frac{\text{Total no.of ways of selecting 2 hockey plays from23}}{\text{Total no of ways of selectiing two people from 50}}$ $= \frac{{}^{23}\mathrm{C}_2}{{}^{50}\mathrm{C}_2}$ $= \frac{253}{1225} = 0.207$ (3 s.f.)	**M1 A1 A1** **A1** **M1** **A1 A1** **A1**
(ii)	The total number of ways of assigning *n* integer values from *m* possible values with no value being assigned twice is $\binom{m}{n}$. With the highest integer value *k* assigned, the other *n* – 1 integer values can be any distinct selection from the remaining *k* – 1 . So there are $\binom{k-1}{n-1}$ ways of assigning the values distinctly with a highest value of *k*. Hence $P(K = k) = \frac{\binom{k-1}{n-1}}{\binom{m}{n}}$ Total Probability $= 1 \Rightarrow \sum_{k=n}^{m} P(K = k) = 1$ $\Rightarrow \sum_{k=n}^{m} \frac{\binom{k-1}{n-1}}{\binom{m}{n}} = 1$ $\Rightarrow \frac{1}{\binom{m}{n}} \sum_{k=n}^{m} \frac{\binom{k-1}{n-1}}{\binom{m}{n}} = 1$ $\Rightarrow \sum_{k=n}^{m} \binom{k-1}{n-1} = \binom{m}{n}$ **Or** For highest value of *K*, $\sum_{k=n}^{m} \binom{k-1}{n-1}$ gives all possible ways of assigning n values. where $\sum_{k=n}^{m} \binom{k-1}{n-1} = \binom{m}{n} \sum_{k=n}^{m} P(K = k)$ $= \binom{m}{n} \times 1 = \binom{m}{n}$	**A1** **R1** **AG** **M1** **A1** **M1** **A1**

STUDENT NAME: ____________________________

TEACHER INITIALS: _________

CANDIDATE SESSION NUMBER

0	2	5	0	1	2				

EXAMINATION CODE

8	8	1	9	-	7	2	0	1

ST. JOSEPH'S INSTITUTION
YEAR 6 PRELIMINARY EXAMINATION 2019

MATHEMATICS
HIGHER LEVEL
PAPER 1
Thursday

4 July 2019
2 hours
0800 to 1000 hrs

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- **Section A**: Answer all questions showing working and answers in the spaces provided in the exam paper.
- **Section B**: Answer all questions using the foolscap paper provided.
- The use of calculators is **not** permitted in this paper.
- A clean copy of the **Mathematics HL Formulae Booklet** is required for this paper.
- Unless otherwise stated in the question all numerical answers are to be given exactly.
- The maximum mark for this examination paper is [**100 marks**].
- This question paper consists of 11 printed pages including the Cover Sheet.
- Section A and B are to be submitted **separately.**

FOR MARKER USE ONLY:

Question:	1	2	3	4	5	6	7	8	9	10	Total
Marks:											100

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are advised to show all working.

SECTION A (50 marks)

1. [**Maximum mark: 5**]

Find an expression for g', in terms of x, when $g(x) = (\cos x)^{\cos x}$, $0 \leq x < \dfrac{\pi}{2}$.

2. [**Maximum mark: 5**]

By considering suitable graphs, determine the sum of all the real values of x such that

$$\sin x = x^{2019}.$$

3. [**Maximum mark: 7**]

Suppose the cubic function $P(x) = 2x^3 + 3x^2 - 8$ has zeros α, β, and γ. It is also given that $\alpha\beta + \alpha\gamma + \beta\gamma = 0$.

(a) Find the value of $\alpha + \beta + \gamma$ and of $\alpha\beta\gamma$. [**2**]

(b) Hence, or otherwise, find the sum and product of the zeros of $(P(2x-1))^2$. [**5**]

4. [**Maximum mark: 7**]

(a) Evaluate $\int_0^{\sqrt{3}} \tan^{-1} y \ dy$. [**4**]

(b) By the aid of a graph, deduce the value of $\int_0^{\frac{\pi}{3}} \tan x \ dx$. [**3**]

5. [**Maximum mark: 9**]

In a talent competition, there are 12 participants who have to go through multiple rounds of selection.

(a) In the first round, the participants are to form groups to compete against one another. Find the number of ways in which all 12 people can be divided into three groups of 4 people. *[You do not need to simplify your answer.]* [**3**]

(b) Only 4 people get through to the second round. In order to eliminate their competitors, each person can either team up or compete individually. Find the possible number of groupings formed in this round. [**6**]

6. **[Maximum mark: 8]**

Given that $z = \dfrac{i-1}{(\sqrt{3}+i)^2}$, where $i^2 = -1$.

(a) Find the modulus and argument of z, where $-\pi < \arg(z) \leq \pi$. **[4]**

(b) Find the third roots of the complex number z, simplifying your answers in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$. **[4]**

7. **[Maximum mark: 9]**

Let $S_n = \frac{0}{1!} + \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \frac{4}{5!} + ... + \frac{n-1}{n!}$.

Find the values of S_1, S_2, S_3 and S_4.

Make a conjecture for an expression of S_n, leaving your answer as a single fraction in terms of n.

Hence, prove your conjecture using induction for positive integer n.

Question 7 Continues on the Next Page

Do ***NOT*** *write solutions on this page.*

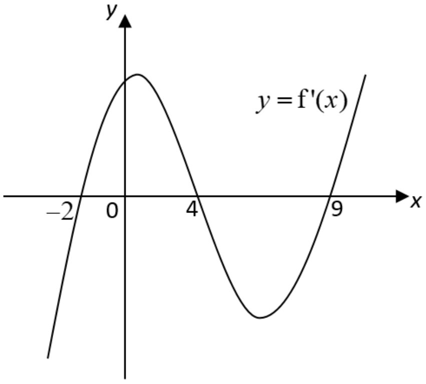
SECTION B (50 marks)

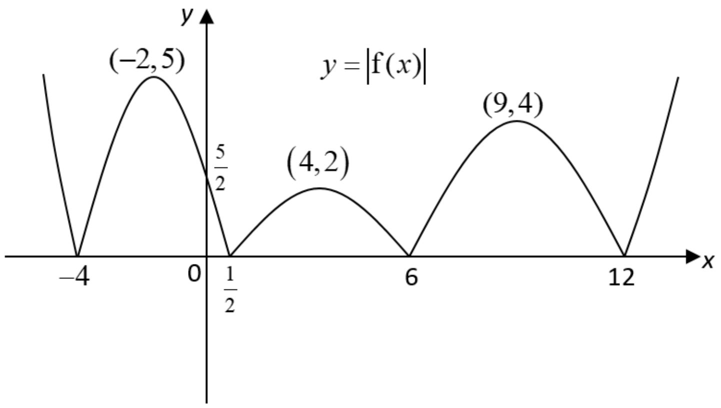
Answer all questions on the foolscap paper provided. Please start each question on a new page.

8. [**Maximum mark: 17**]

The diagrams in this question are not drawn to scale.

(a) The diagram below shows the graphs of $y = \mathrm{f}'(x)$ and $y = |\mathrm{f}(x)|$.





On a separate diagram, sketch the graphs of,

(i) $y = \mathrm{f}(x)$, and **[6]**

(ii) $y = \mathrm{f}(|x|)$, **[4]**

labeling clearly the coordinates of all turning points and axial intercepts.

(b) State an ordered set of transformations that transform
$g(x) = \sqrt{9 - 6x^2 + x^4}$, $-1 \leq x \leq 1$ onto $h(x) = (2x+1)^2$, $-1 \leq x \leq 0$. **[7]**

9. [**Maximum mark: 12**]

Let $x_1, x_2, \ldots, x_n$ and x_{n+1} be real numbers. The numbers A, B and C are defined by

$$A = \frac{1}{n}\sum_{k=1}^{n} x_k \ , \quad B = \frac{1}{n}\sum_{k=1}^{n}(x_k - A)^2 \ , \quad C = \frac{1}{n+1}\sum_{k=1}^{n+1} x_k .$$

(a) In the case of $x_k = k$, calculate the value of A when $n = 21$. **[3]**

(b) Express C in terms of A, x_{n+1} and n. **[3]**

(c) Show that **[6]**

$$B = \frac{1}{n}\sum_{k=1}^{n}\left(x_k^2\right) - A^2 \ .$$

10. [**Maximum mark: 21**]

The planes π and γ have equations $x + y + z = 1$ and $x + z = 2$ respectively, and meet in the line l.

(a) Find the value of cosine of the angle between π and γ. [**2**]

(b) Find a vector equation of line l. [**4**]

Let P be the set of planes $p_1, p_2, p_3, ...,$ such that the cartesian equation of p_n is given by

$$x + u_n y + z = S_n,$$

where u_n is the n^{th} term of a geometric progression with first term 1 and common ratio $\frac{1}{2}$, and S_n is the sum of the first n terms of the geometric progression.

(c) Write down a cartesian equation of p_k in terms of k, where $k \in \mathbb{Z}^+$. [**3**]

(d) Evaluate the value of cosine of the angle between p_1 and p_k as $k \to \infty$. [**4**]

(e) Show that l lies in all planes in P. [**5**]

(f) Let β be a plane with equation $ax + by + cz = d$. Determine, with reason, the relationship between β and two randomly chosen planes in P, if $a \neq c$. [**3**]

End of Paper

MARKSCHEME

STUDENT NAME: ______________________________

TEACHER INITIALS: __________

CANDIDATE SESSION NUMBER

0	2	5	0	1	2				

EXAMINATION CODE

8	8	1	9	-	7	2	0	1

ST. JOSEPH'S INSTITUTION
YEAR 6 PRELIMINARY EXAMINATION 2019

MATHEMATICS
HIGHER LEVEL
PAPER 1
Thursday

4 July 2019

2 hours

0800 to 1000 hrs

INSTRUCTIONS TO CANDIDATES

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- The use of calculators is **not** permitted in this paper.
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- Unless otherwise stated in the question all numerical answers are to be given exactly.
- The maximum mark for this examination paper is [**100 marks**].
- This question paper consists of 17 printed pages including the Cover Sheet.
- Section A and B are to be submitted **separately.**

FOR MARKER USE ONLY:

Question:	1	2	3	4	5	6	7	8	9	10	Total
Marks:											100

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SECTION A (50 marks)

1. [**Maximum mark: 5**]

Find an expression for g', in terms of x, when $g(x) = (\cos x)^{\cos x}, \quad 0 \leq x < \frac{\pi}{2}$.

Solution: $g(x) = (\cos x)^{\cos x}$ is the same as $\ln g(x) = (\cos x)(\ln \cos x)$. **M1**

Differentiating both sides with respect to x,

$$\frac{1}{g(x)} \frac{\mathrm{d}}{\mathrm{d}x}(g(x)) = (-\sin x)(\ln(\cos x)) + \cos x \left(\frac{1}{\cos x}\right)(-\sin x)$$

use of product rule, use of chain rule **M2**

$$g'(x) = ((\cos x)^{\cos x})(-\sin x)(\ln(\cos x) + 1)$$ **A2**

Alternative solution $g(x) = (\cos x)^{\cos x}$ is the same as $g(x) = \mathrm{e}^{(\cos x)\ \ln(\cos x)}$. **M1**

Differentiating both sides with respect to x,

$$g'(x) = \left[(\cos x)\frac{1}{\cos x}(-\sin x) \ - \ (\sin x)(\ln(\cos x))\right](\cos x)^{\cos x}$$ **M2**

$$= ((\cos x)^{\cos x})(-\sin x)(\ln(\cos x) + 1)$$ **A2**

2. [**Maximum mark: 5**]

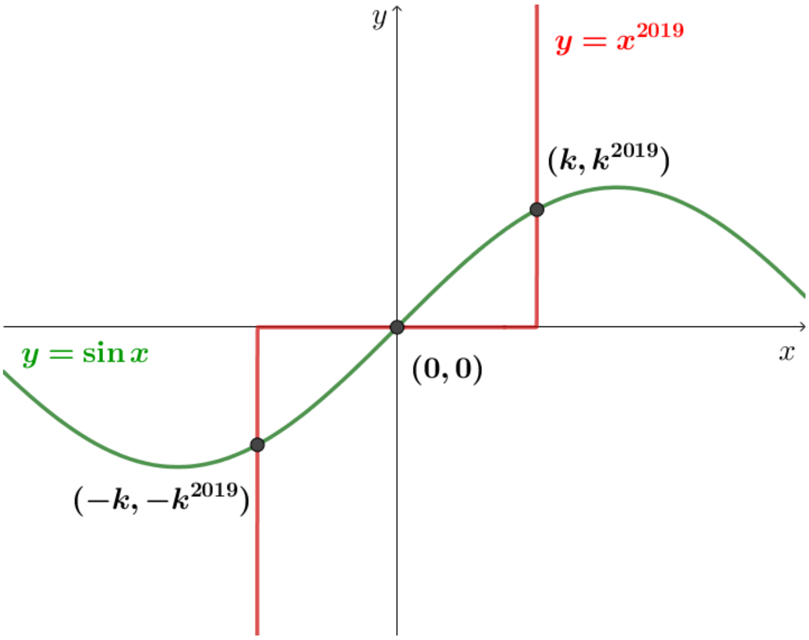
By considering suitable graphs, determine the sum of all the real values of x such that

$$\sin x = x^{2019}.$$

Solution:

The graph of $y = \sin x$ and $y = x^{2019}$ appear below.

G2 - one mark each graph



The two graphs intersect at $x = -k$, $x = 0$ and $x = k$ for some positive real number k. **A1**

This is due to both functions being odd. **R1**

Therefore, the sum of the solutions is $-k + 0 + k = 0$. **A1**

Turn Over

3. [**Maximum mark: 7**]

Suppose the cubic function $P(x) = 2x^3 + 3x^2 - 8$ has zeros α, β, and γ. It is also given that $\alpha\beta + \alpha\gamma + \beta\gamma = 0$.

(a) Find the value of $\alpha + \beta + \gamma$ and of $\alpha\beta\gamma$. [**2**]

(b) Hence, or otherwise, find the sum and product of the zeros of $(P(2x-1))^2$. [**5**]

Solution:

(a) $\alpha + \beta + \gamma = -\dfrac{3}{2}$ and $\alpha\beta\gamma = 4$ **A2**

(b) **Method 1:**

$P(2x-1)$ has zeros $\dfrac{\alpha+1}{2}, \dfrac{\beta+1}{2}, \dfrac{\gamma+1}{2}$ **R1**

and thus $(P(2x-1))^2$ has repeated zeros of $P(2x-1)$.

Hence the sum of zeros of $(P(2x-1))^2 = 2\left[\dfrac{\alpha+1}{2} + \dfrac{\beta+1}{2} + \dfrac{\gamma+1}{2}\right] = -\dfrac{3}{2} + 3 = \dfrac{3}{2}$. **M1A1**

The product of zeros of $(P(2x-1))^2$ is

$$\begin{aligned}\left[\left(\frac{\alpha+1}{2}\right)\left(\frac{\beta+1}{2}\right)\left(\frac{\gamma+1}{2}\right)\right]^2 &= \frac{1}{64}\left[(\alpha+1)(\beta+1)(\gamma+1)\right]^2 && \textbf{M1}\\ &= \frac{1}{64}\left(\alpha\beta\gamma + \alpha\beta + \alpha\gamma + \beta\gamma + \alpha + \beta + \gamma + 1\right)^2\\ &= \frac{1}{64}\left(4 + 0 - \frac{3}{2} + 1\right)^2\\ &= \frac{49}{256} && \textbf{A1}\end{aligned}$$

Method 2:

$$\begin{aligned}P(2x-1) &= 2(2x-1)^3 + 3(2x-1)^2 - 8\\ &= 16x^3 - 12x^2 - 7 && \textbf{A1}\end{aligned}$$

$(P(2x-1))^2$ has repeated zeros of $P(2x-1)$. **R1**

Hence the sum of zeros of $(P(2x-1))^2 = 2\left(\dfrac{12}{16}\right) = \dfrac{3}{2}$. **M1A1**

The product of zeros of $(P(2x-1))^2$ is $\left(\dfrac{7}{16}\right)^2 = \dfrac{49}{256}$. **A1**

Method 3:

$$\begin{aligned}\left[P(2x-1)\right]^2 &= \left[2(2x-1)^3 + 3(2x-1)^2 - 8\right]^2 && \textbf{M1}\\ &= \left(16x^3 - 12x^2 - 7\right)^2 && \textbf{A1}\\ &= 256x^6 - 384x^5 + 144x^4 - 224x^3 + 168x^2 + 49 && \textbf{A1}\end{aligned}$$

Question 3 Continues on the Next Page

Hence the sum of zeros of $(P(2x-1))^2 = \frac{384}{256} = \frac{3}{2}$. **A1**

The product of zeros of $(P(2x-1))^2 = \frac{49}{256}$. **A1**

4. [**Maximum mark: 7**]

(a) Evaluate $\int_0^{\sqrt{3}} \tan^{-1}\ y\ dy$. [**4**]

(b) By the aid of a graph, deduce the value of $\int_0^{\frac{\pi}{3}} \tan x\ dx$. [**3**]

Solution:

(a) Method 1: Using integration by parts, **M1**

$$\int_0^{\sqrt{3}} \tan^{-1} y\ \mathrm{d}y = \left[y\ \tan^{-1} y\right]_0^{\sqrt{3}} - \int_0^{\sqrt{3}} \left(\frac{y}{1+y^2}\right)\ \mathrm{d}y \quad \textbf{A1}$$

$$= \sqrt{3}\ \tan^{-1}\sqrt{3} - \left[\frac{1}{2}\ln\left(1+x^2\right)\right]_0^{\sqrt{3}} \quad \textbf{M1}$$

$$= \frac{\sqrt{3}\pi}{3} - \frac{1}{2}\ln 4 \quad \textbf{A1}$$

$$= \frac{\sqrt{3}\pi}{3} - \ln 2$$

Method 2: Sketch the graph of $y = \tan x$ and observe that **M1**

$$\int_0^{\sqrt{3}} \tan^{-1} y\ \mathrm{d}y = \sqrt{3}\left(\frac{\pi}{3}\right) - \int_0^{\frac{\pi}{3}} \tan x\ \mathrm{d}x \quad \textbf{A1}$$

$$= \frac{\pi}{\sqrt{3}} - \left[\ln|\cos x|\right]_0^{\frac{\pi}{3}} \quad \textbf{M1}$$

$$= \frac{\sqrt{3}\pi}{3} - \ln 2 \quad \textbf{A1}$$

(b) From diagram of either $y = \tan x$ or $y = \tan^{-1}$, **M1**

$\int_0^{\frac{\pi}{3}} \tan x\ dx = \left(\frac{\pi}{3}\right)(\sqrt{3}) - \int_0^{\sqrt{3}} \tan^{-1} y\ dy = \ln 2.$ **M1A1**

5. [**Maximum mark: 9**]

In a talent competition, there are 12 participants who have to go through multiple rounds of selection.

(a) In the first round, the participants are to form groups to compete against one another. Find the number of ways in which all 12 people can be divided into three groups of 4 people. *[You do not need to simplify your answer.]* [**3**]

(b) Only 4 people get through to the second round. In order to eliminate their competitors, each person can either team up or compete individually. Find the possible number of groupings formed in this round. [**6**]

Solution: (a) Number of ways is $\frac{\binom{12}{4}\binom{8}{4}\binom{4}{4}}{3!}$. **A3**

1 mark - 12C4, 1 mark - the rest of numerator, 1 mark - 3!

(b) Consider these cases: **M1**

Case 1: all are individual. Number of ways is 1. **A1**

Case 2: one group of 2 and 2 individual. Number of ways is $\binom{4}{2}$.

Case 3: one group of 3 and 1 individual. Number of ways is $\binom{4}{3}$. **A1** **case 2 and 3**

Case 4: two groups of 2. Number of ways is $\frac{\binom{4}{2}\binom{2}{2}}{2!}$ **A1**

Hence the total number of ways is

$$1+\binom{4}{2}+\binom{4}{3}+\frac{\binom{4}{2}\binom{2}{2}}{2!}=14.$$

M1A1

6. [**Maximum mark: 8**]

Given that $z = \dfrac{i-1}{(\sqrt{3}+i)^2}$, where $i^2 = -1$.

(a) Find the modulus and argument of z, where $-\pi < \arg(z) \leq \pi$. [**4**]

(b) Find the third roots of the complex number z, simplifying your answers in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$. [**4**]

Solution: (a) $|z| = \dfrac{|-1+i|}{|\sqrt{3}+i|^2} = \dfrac{\sqrt{2}}{2^2} = 2^{-3/2}$. **M1A1**

$$\begin{aligned}\arg(z) &= \arg(-1+i) - 2\arg(\sqrt{3}+i) \\ &= \frac{3\pi}{4} - 2\left(\frac{\pi}{6}\right) \\ &= \frac{5\pi}{12}\end{aligned}$$

M1

A1

(b) $z^3 = \dfrac{i-1}{(\sqrt{3}+i)^2} = 2^{\frac{-3}{2}} e^{i\left(\frac{5\pi}{12}+2k\pi\right)}, \quad k = -1, 0, 1$ **M1A1 (FT)**

Hence the roots are $z = 2^{\frac{-1}{2}} e^{i\left(\frac{5\pi}{36}+\frac{2k\pi}{3}\right)}, \quad k = -1, 0, 1$ **M1A1**

7. [**Maximum mark: 9**]

Let $S_n = \frac{0}{1!} + \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \frac{4}{5!} + ... + \frac{n-1}{n!}$.

Find the values of S_1, S_2, S_3 and S_4.

Make a conjecture for an expression of S_n, leaving your answer as a single fraction in terms of n.

Hence, prove your conjecture using induction for positive integer n.

Solution: $S_1 = \frac{0}{1!} = \frac{0}{1}$

$S_2 = \frac{1}{2!} = \frac{1}{2}$ **A1** for S1 and S2

$S_3 = \frac{1}{2!} + \frac{2}{3!} = \frac{5}{6}$

$S_4 = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} = \frac{23}{24}$ **A1** for S3 and S4

The conjecture is $S_n = \frac{n! - 1}{n!}$ **A1**

Let the proposition P_n be $S_n = \frac{n!-1}{n!}$ for $n \in \mathbb{Z}^+$, where $S_n = \frac{0}{1!} + \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \frac{4}{5!} + ... + \frac{n-1}{n!}$.

When $n = 1$, LHS $= \frac{0}{1!}$, and RHS $= \frac{0!-1}{1!} = \frac{0}{1!}$.

Since LHS = RHS, P_1 is true. **A1**

Assume that P_k is true for some $k \in \mathbb{Z}^+$, we have **R1**

$$S_k = \frac{0}{1!} + \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \frac{4}{5!} + ... + \frac{k}{-}1k! = \frac{k!-1}{k!}.$$

To prove P_{k+1}, i.e.

$$S_{k+1} = \frac{0}{1!} + \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \frac{4}{5!} + ... + \frac{k+1-1}{(k+1)!} = \frac{(k+1)!-1}{(k+1)!}.$$

Question 7 Continues on the Next Page

When $n = k+1$,

$$\begin{aligned}\textbf{LHS} &= \frac{0}{1!}+\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\frac{4}{5!}+...+\frac{k}{(k+1)!}\\ &= S_k+\frac{k}{(k+1)!}\\ &= \frac{k!-1}{k!}+\frac{k}{(k+1)!} && \textbf{A1}\\ &= \frac{(k+1)(k!)-(k+1)}{(k+1)!}+\frac{k}{(k+1)!} && \textbf{M1}\\ &= \frac{(k+1)!-k-1+k}{(k+1)!} && \textbf{A1}\\ &= \frac{(k+1)!-1}{(k+1)!} = \textbf{RHS}.\end{aligned}$$

Since P_1 is true and P_k is true $\Rightarrow P_{k+1}$ is true, by mathematical induction, P_n is true for all positive integer values of n. **R1**

*Do **NOT** write solutions on this page.*

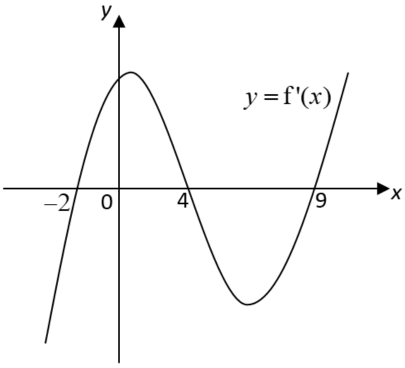
SECTION B (50 marks)

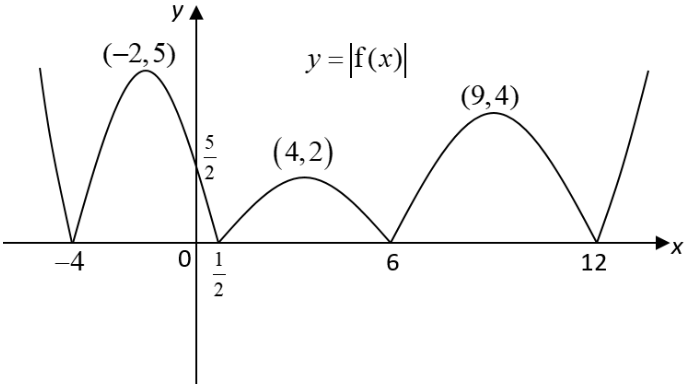
Answer all questions on the foolscap paper provided. Please start each question on a new page.

8. [**Maximum mark: 17**]

The diagrams in this question are not drawn to scale.

(a) The diagram below shows the graphs of $y = \mathrm{f}'(x)$ and $y = |\mathrm{f}(x)|$.





On a separate diagram, sketch the graphs of,

(i) $y = \mathrm{f}(x)$, and **[6]**

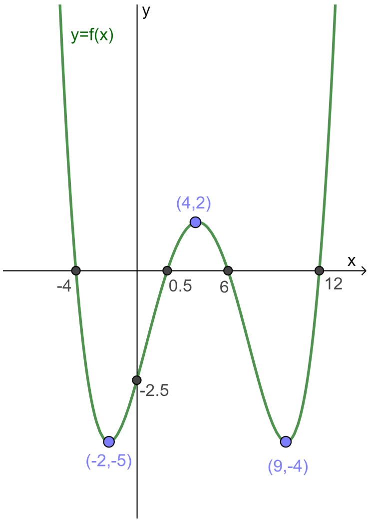
(ii) $y = \mathrm{f}(|x|)$, **[4]**

labeling clearly the coordinates of all turning points and axial intercepts.

(b) State an ordered set of transformations that transform $g(x) = \sqrt{9 - 6x^2 + x^4}$, $-1 \leq x \leq 1$ onto $h(x) = (2x+1)^2$, $-1 \leq x \leq 0$. **[7]**

Solution:

(a) (i)

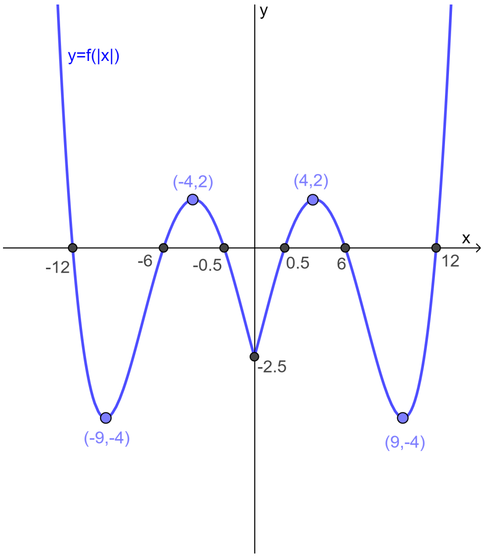


G1 - shape,

Question 8 Continues on the Next Page

G1 - x-axis intercepts, G1 - y-axis intercept
G1 - max pt (4,2), G1 - min pt (-2,-5), G1 - min pt (9,-4)

(a) (ii)



G1 - shape, G1 - axial intercepts,
G1 - max pt(4,2) and min pt (9,-4),
G1 - max pt (-4,2) and min pt (-9,-4),
If previous part answer is wrong, follow through with markscheme: M1 - $|x|$ process, A1 - corresponding points are labelled.

(b) Since $9 - 6x^2 + x^4 = (x^2 - 3)^2$ or $(3 - x^2)^2$, then
$\sqrt{9 - 6x^2 + x^4} = |x^2 - 3|$ or $\sqrt{9 - 6x^2 + x^4} = |3 - x^2|$. **M1A1**

Since $-1 \leq x \leq 1$, then $g(x) = -\sqrt{(x^2 - 3)^2} = -(x^2 - 3) = 3 - x^2$. **A1**

Hence the possible ordered transformations are
Method 1:
- Reflection about the $x-$ axis **A1**
- Translation by $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ **A2**
- Horizontal scaling by a factor of 0.5. **A1**

Method 2:
- Reflection about the $x-$ axis **A1**
- Vertical translation by 3 units **A1**
- Horizontal scaling by a factor of 0.5 **A1**
- Horizontal translation by -0.5 unit. **A1**

Method 3:

- Vertical translation by -3 **A1**
- Reflection about the $x-$ axis **A1**
- Horizontal translation by -1 unit **A1**
- Horizontal scaling by a factor of 0.5. **A1**

Method 4:

- Vertical translation by -3 **A1**
- Reflection about the $x-$ axis **A1**
- Horizontal scaling by a factor of 0.5 **A1**
- Horizontal translation by -0.5 unit. **A1**

Method 5:

- Translation by $\begin{pmatrix} -1 \\ -3 \end{pmatrix}$ **A2**
- Reflection about the $x-$ axis **A1**
- Horizontal translation by -0.5 unit. **A1**

Method 6:

- Translation by $\begin{pmatrix} -1 \\ -3 \end{pmatrix}$ **A2**
- Horizontal translation by -0.5 unit **A1**
- Reflection about the $x-$ axis. **A1**

9. [**Maximum mark: 12**]

Let $x_1, x_2, \ldots, x_n$ and x_{n+1} be real numbers. The numbers A, B and C are defined by

$$A = \frac{1}{n}\sum_{k=1}^{n} x_k \ , \quad B = \frac{1}{n}\sum_{k=1}^{n}(x_k - A)^2 \ , \quad C = \frac{1}{n+1}\sum_{k=1}^{n+1} x_k.$$

(a) In the case of $x_k = k$, calculate the value of A when $n = 21$. [**3**]

(b) Express C in terms of A, x_{n+1} and n. [**3**]

(c) Show that [**6**]

$$B = \frac{1}{n}\sum_{k=1}^{n}\left(x_k^2\right) - A^2 \ .$$

Solution:

Question 9 Continues on the Next Page

(a) When $x_k = k$ and $n = 21$,

$$A = \frac{1}{21}\sum_{k=1}^{21} k$$ **M1**

$$= \frac{1}{21}\left(\frac{21}{2}(1+21)\right)$$ **A1** use of AP formula

$$= 11$$ **A1**

(b)

$$C = \frac{1}{n+1}\sum_{k=1}^{n+1} x_k$$

$$= \frac{1}{n+1}\left[\sum_{k=1}^{n} x_k + x_{n+1}\right]$$ split summation **M1**

$$= \frac{n}{n+1}\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right) + \frac{1}{n+1}x_{n+1}$$ adjust coefficient to obtain A **M1**

$$= \frac{n}{n+1}A + \frac{1}{n+1}x_{n+1}$$ **A1**

(c)

$$B = \frac{1}{n}\sum_{k=1}^{n}(x_k - A)^2$$

$$= \frac{1}{n}\sum_{k=1}^{n}(x_k^2 - 2Ax_k + A^2)$$ **M1**

$$= \frac{1}{n}\left[\sum_{k=1}^{n}\left(x_k^2\right) - 2A\sum_{k=1}^{n} x_k + A^2\sum_{k=1}^{n} 1\right]$$

use of summation properties for second and third terms **M2**

$$= \frac{1}{n}\sum_{k=1}^{n}\left(x_k^2\right) - 2A\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right) + \frac{1}{n}A^2 n$$ manipulate to try to get A **M1**

$$= \frac{1}{n}\sum_{k=1}^{n}\left(x_k^2\right) - 2A^2 + A^2$$ **A1**

$$= \frac{1}{n}\sum_{k=1}^{n}\left(x_k^2\right) - A^2$$ shown **A1**

10. [**Maximum mark: 21**]

The planes π and γ have equations $x+y+z=1$ and $x+z=2$ respectively, and meet in the line l.

(a) Find the value of cosine of the angle between π and γ. [**2**]

(b) Find a vector equation of line l. [**4**]

Let P be the set of planes $p_1, p_2, p_3, \ldots$, such that the cartesian equation of p_n is given by

$$x + u_n y + z = S_n,$$

where u_n is the n^{th} term of a geometric progression with first term 1 and common ratio $\frac{1}{2}$, and S_n is the sum of the first n terms of the geometric progression.

(c) Write down a cartesian equation of p_k in terms of k, where $k \in \mathbb{Z}^+$. [**3**]

(d) Evaluate the value of cosine of the angle between p_1 and p_k as $k \to \infty$. [**4**]

(e) Show that l lies in all planes in P. [**5**]

(f) Let β be a plane with equation $ax+by+cz=d$. Determine, with reason, the relationship between β and two randomly chosen planes in P, if $a \neq c$. [**3**]

Solution:

(a) $\begin{pmatrix}1\\1\\1\end{pmatrix} \cdot \begin{pmatrix}1\\0\\1\end{pmatrix} = 2$ **M1**

Hence $\cos\theta = \dfrac{2}{\sqrt{3}\sqrt{2}} = \dfrac{2}{\sqrt{6}}$. **A1**

(b) $\mathbf{n} = \begin{pmatrix}1\\1\\1\end{pmatrix} \times \begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}1\\0\\-1\end{pmatrix}$ **M1A1**

Solving for a common point in π and γ, $y=-1, x=2, z=0$. **A1** **any valid point**

Hence a vector equation of l is

$$\mathbf{r} = \begin{pmatrix}2\\-1\\0\end{pmatrix} + \lambda \begin{pmatrix}1\\0\\-1\end{pmatrix}, \lambda \in \mathbb{R}.$$ **A1**

(c) $u_n = \left(\dfrac{1}{2}\right)^{n-1}$, **A1**

$S_n = \dfrac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}} = 2\left(1-\left(\dfrac{1}{2}\right)^n\right)$. **A1**

Question 10 Continues on the Next Page

Hence the Cartesian equation of p_k is

$$x + \left(\frac{1}{2}\right)^{k-1} y + z = 2\left(1 - \left(\frac{1}{2}\right)^{k}\right).$$ **A1**

(d) $p_1 : x + y + z = 1$ and $p_k : x + \left(\frac{1}{2}\right)^{k-1} y + z = 2 - \left(\frac{1}{2}\right)^{k-1}$

$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ \left(\frac{1}{2}\right)^{k-1} \\ 1 \end{pmatrix} = 2 + \left(\frac{1}{2}\right)^{k-1}$ **M1**

$\cos\theta = \dfrac{2 + \left(\frac{1}{2}\right)^{k-1}}{\sqrt{3}\sqrt{2 + \left(\frac{1}{2}\right)^{2(k-1)}}}$ **A1**

As $k \to \infty$, $\left(\frac{1}{2}\right)^{k-1} \to 0$, hence $\cos\theta \to \dfrac{2}{\sqrt{6}}$. **R1**

A1

(e) Since $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ \left(\frac{1}{2}\right)^{k-1} \\ 1 \end{pmatrix} = 0$ for all values of $k \in \mathbb{Z}^+$, **M1**

the line l is parallel to any plane in P. **A1**

Moreover,

$$\begin{aligned} \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ \left(\frac{1}{2}\right)^{k-1} \\ 1 \end{pmatrix} &= 2 - \left(\frac{1}{2}\right)^{k-1} \\ &= 2 - 2^{1-k} \\ &= 2\left(1 - 2^{-k}\right) \\ &= 2\left(1 - \left(\frac{1}{2}\right)^{k}\right). \end{aligned}$$

M1

A1

Hence, l lies in any plane in P. **R1**

(f) Any two planes in P intersect at line l, and

$$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = a - c.$$ **M1**

Since $a \neq c$, $a - c \neq 0$. Therefore β is not parallel to l and β cuts the line l at a point. **R1**

Hence β and any two planes in P meet at one point of intersection. (accept: the system of equations has a unique solution) **A1**

Question 10 Continues on the Next Page

End of Paper

STUDENT NAME: ______________________________

TEACHER NAME: ______________________________

CANDIDATE SESSION NUMBER

0	2	5	0	1	2				

EXAMINATION CODE

8	8	1	9	-	7	2	0	2

ST. JOSEPH'S INSTITUTION
YEAR 6 PRELIMINARY EXAMINATION 2019

MATHEMATICS
HIGHER LEVEL
PAPER 2
Thursday

11th July 2019
2 hours
0800 – 1000 hrs

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- **Section A**: Answer all questions showing working and answers in the spaces provided in the exam paper.
- **Section B**: Answer all questions using the writing paper provided.
- The use of a scientific or graphical calculator is permitted in this paper.
- TI-Nspire calculators must be in Press-to-Test mode and cleared of all previous data.
- TI-84+ graphical calculators must only have permitted Apps and be RAM-cleared.
- It is the responsibility of the student to ensure their calculator is examination ready.
- A clean copy of the **Mathematics HL Formulae Booklet** is required for this paper.
- Unless otherwise stated in the question all numerical answers are to be given exactly or to three significant figures.
- The maximum mark for this examination paper is *[**100** marks]*.
- This question paper consists of **12** printed pages including the Cover Sheet.
- Sections A and B are to be submitted **separately**.

FOR MARKER USE ONLY:

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	Q12	TOTAL
												/100

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphical display calculator should be supported by suitable working; for example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are advised to show all working.

SECTION A (50 marks)

1 [Maximum mark: 4]

The following table shows the values of two functions v and w and their first derivatives when $x = 0$ and $x = 1$.

x	$v(x)$	$v'(x)$	$w(x)$	$w'(x)$
0	4	3	1	5
1	–2	0.5	–1	2

Find the derivative of the composite function $(v \circ w)(\sin^{-1} 2x)$ when $x = 0$.

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2 [Maximum mark: 4]

In the expansion of $(x+3)(2x+1)^n, n \in \mathbb{Z}^+$, the coefficient of the term in x^3 is 280. Find the value of *n*.

3 **[Maximum mark: 6]**

A curve C has equation $y = f(x)$ where $f(x) = p^3 - e^{px^2} - \int_0^x (x-u)\,\mathrm{d}u$ and p is real.

The gradient of the normal to the curve C where $x = 1$ is M.

What is the greatest possible value of M as p varies?

4 **[Maximum mark: 6]**

In SJI, 60% of its alumni members liked their course of study. 70% of alumni members found jobs which they enjoyed given that they liked their course of study, while 30% of alumni members did not like their course of study and also found jobs which they did not enjoy.

Find the probability that

(i) an alumni member found a job which he did not enjoy given that he did not like his course of study, [2]

(ii) an alumni member found a job which he enjoyed, [2]

(iii) an alumni member did not like his course of study given that he found a job which he enjoyed. [2]

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5 **[Maximum mark: 8]**

Given that $\arg(w)=\frac{\pi}{6}$, $|w|=2$ and $S=1+\left(\frac{w}{4}\right)^3+\left(\frac{w}{4}\right)^6+\left(\frac{w}{4}\right)^9+\ldots.+\left(\frac{w}{4}\right)^{3r}+\ldots$

(i) show that the infinite sum *S* exists, [3]

(ii) evaluate *S*, leaving your answer in the form $\frac{a}{65}+\mathrm{i}\frac{b}{65}$. [5]

6 **[Maximum mark: 5]**

Two ships, A and B, are observed from an origin O. Relative to O, their position vectors at time *t* hours after midday are given by

$$\mathbf{r}_A = \begin{pmatrix} 4 \\ 3 \end{pmatrix} + t\begin{pmatrix} 5 \\ 8 \end{pmatrix}$$

$$\mathbf{r}_B = \begin{pmatrix} 7 \\ -3 \end{pmatrix} + t\begin{pmatrix} 0 \\ 12 \end{pmatrix}$$

where distances are measured in kilometres.

Find the minimum distance between the two ships.

7 **[Maximum mark: 6]**

Let $A(x) = 2x + 1$.
Let n be a positive integer.

Determine $A^n(x)$ where $A^n(x) = \underbrace{A(A(A \cdots A(x) \cdots))}_{n\ times}$.

Leave your answer in the simplest form possible.

Solve $A^{1000}(x) = A^{-1}(x)$, where A^{-1} is the inverse of A.

8 **[Maximum mark: 5]**

A triangle ABC is to be drawn with AB = 10cm, BC = 7cm and the angle at A equal to θ, where θ is a certain specified angle. Of the two possible triangles that could be drawn, the larger triangle has three times the area of the smaller one. Find the value of cos θ.

9 [Maximum mark: 6]

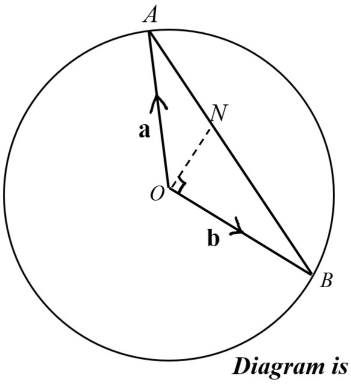


Diagram is not to scale

The above diagram shows a circle with radius r units and centre O. The points A and B on the circle are such that $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\angle AOB = 120^\circ$. The point N divides AB in the ratio $\lambda : 1 - \lambda$ and ON is perpendicular to OB.

Show that $\overrightarrow{ON} = \frac{1}{3}(2\mathbf{a}+\mathbf{b})$.

*Do **NOT** write solutions on this page.*

SECTION B (50 marks)

Answer all questions on the writing paper provided. ***Please start each question on a new page.***

10 **[Maximum mark: 11]**

(a) Consider the polynomial $P(z)=z^3-(2+i)z^2+2(2+i)z-4i$, where $z\in\mathbb{C}$ and $i^2=-1$.

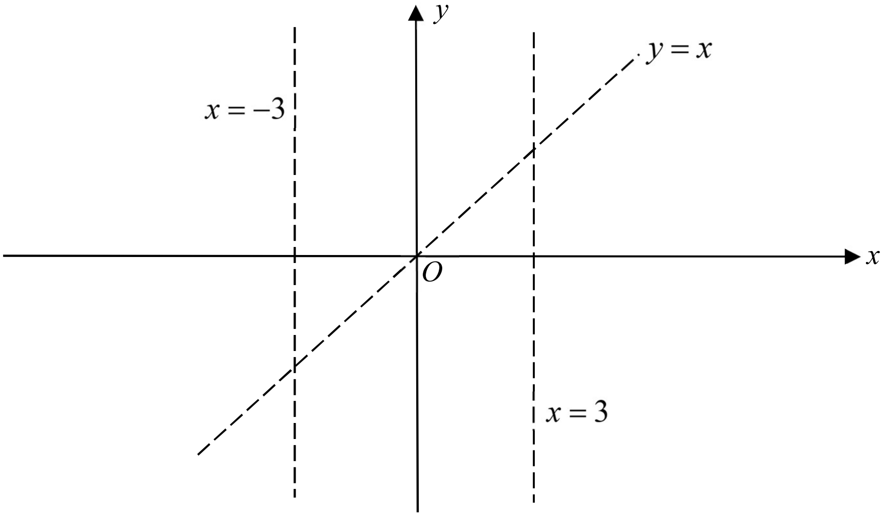
(i) Show that $z=i$ is a root of $P(z)$. [2]

(ii) Find the quadratic factor of $P(z)$. [3]

(b) Prove algebraically that the equation $-x^3+5x^2-16x+9=0$ has non-real roots. Determine the sign of the real root(s) of the equation, if any. [6]

11 **[Maximum mark: 19]**

The function *g* is given by $g(x)=x+\dfrac{1}{x^2-9}$. The graph of *g* has two vertical asymptotes $x=3$ and $x=-3$, and an oblique asymptote $y=x$ as shown in the diagram below.



(a) **(i)** Explain why $x=3$ is a vertical asymptote of the graph. [2]

(ii) Copy the diagram above and sketch the graph of $y=g(x)$ on its maximal domain, indicating the *y*-axis intercept clearly. [3]

(b) **Hence**, find the value of the positive integer q such that the polynomial $p(x)=x^3-qx^2-9x+(9q+1)$ has exactly one real root. [6]

TURN OVER

Do ***NOT*** *write solutions on this page.*

[Q11 continued]

(c) The function *h* is given by $h(x) = g(x)$ for $x < m$ such that *h* has an inverse function.

(i) Write down the largest possible value of *m*. [1]

(ii) Sketch the graphs of $y = h(x)$ and $y = h^{-1}(x)$ on the same diagram, indicating clearly the asymptotes. [3]

(iii) Find the value of $h^{-1}(-5)$. [2]

Consider the function $f(x) = \ln(-x)$ defined on its maximal domain.

(iv) Find the range of the composite function $f \circ h^{-1}$. [2]

12 [Maximum mark: 20]

(a) Using the substitution $x = 2\sin\theta$, show that

$$\int \sqrt{4-x^2}\,\mathrm{d}x = Ax\sqrt{4-x^2} + B\arcsin\left(\frac{x}{2}\right) + \text{constant},$$

where *A* and *B* are constants to be determined. [8]

The function *f* is defined by $f(x) = x\sqrt{4-x^2} + 4\arcsin\left(\frac{x}{2}\right),\ -2 \le x \le 2$.

(b) Write down an expression for the area bounded by the curve $y = f(x)$, the *x*-axis and the line $x = -2$, and find the value of this area. [3]

(c) A dish is created when the region bounded by the curve $y = f(x)$ where $0.5 \le x \le 1.5$, and the *x*-axis, is rotated by 2π radians about the *x*-axis. Find the volume of the dish. [3]

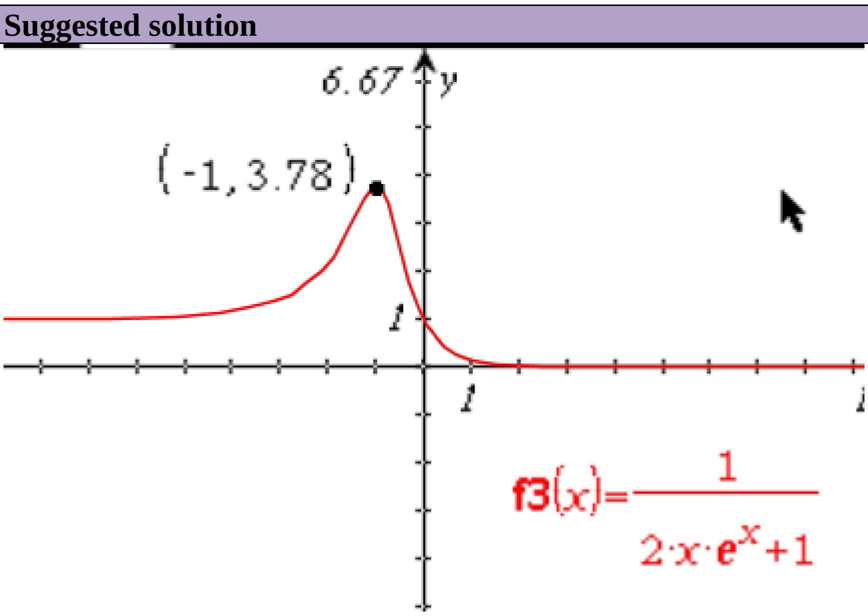
(d) Using part (a) or otherwise, find $f'(x)$ in simplified form. [3]

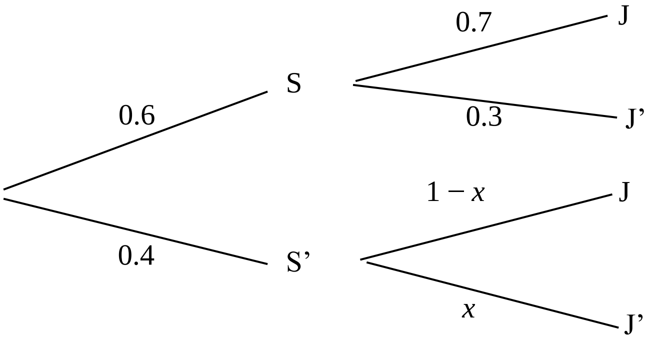
(e) Show that $f(x)$ has a point of inflexion at $x = 0$, justifying your answer. [3]

End of Paper

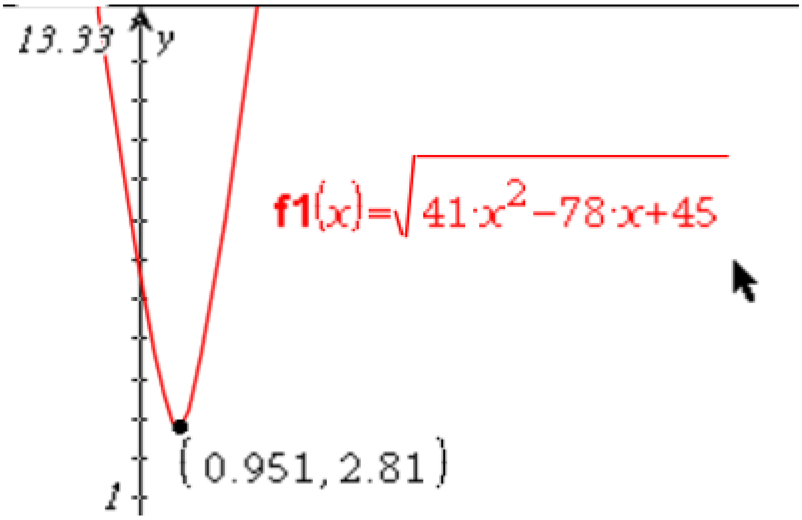
Year 6 HL Maths Prelim Exam 2019 – Paper 2 Markscheme

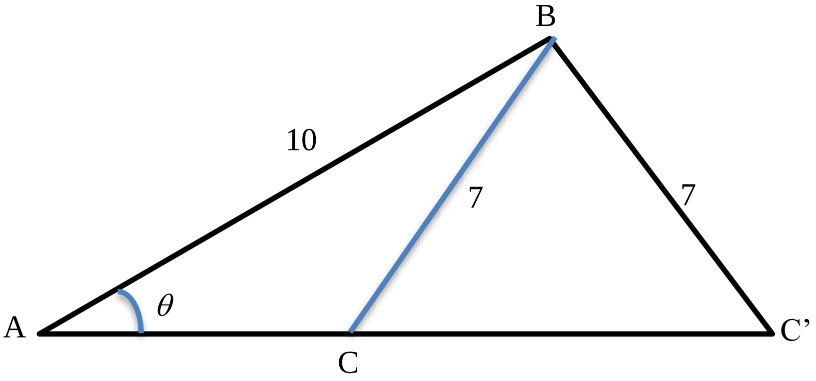
Qn	Suggested solution	Markscheme
	Section A	
1	***Differentiation – Chain Rule and Composite Function***	**Max mark: 4**
	$\frac{\mathrm{d}}{\mathrm{d}x}\left[v\circ w\left(\sin^{-1}2x\right)\right]$ $=v'\left(w\left(\sin^{-1}2x\right)\right)\cdot w'\left(\sin^{-1}2x\right)\cdot\frac{2}{\sqrt{1-4x^2}}$ $\left.\frac{\mathrm{d}}{\mathrm{d}x}\left[v\circ w\left(\sin^{-1}2x\right)\right]\right\|_{x=0}=v'\left(w(0)\right)\cdot w'(0)\cdot 2=v'(1)\cdot5\cdot2=\frac{1}{2}\cdot5\cdot2=5$	M1A1 M1A1
2	***Binomial Expansion***	**Max mark: 4**
	The general term of $(2x+1)^n$ is $\binom{n}{r}(2x)^r$. The term in x^3 is given by $\binom{n}{2}(2x)^2+3\cdot\binom{n}{3}(2x)^3$. $\binom{n}{2}(2)^2+3\cdot\binom{n}{3}(2)^3=280$  nSolve(nCr(x,2)·4+3·nCr(x,3)·2^3=280,x) 5. By GDC, $n=5$	M1A1 M1 A1
3	***Gradient of Normal***	**Max mark: 6**
	$f(x)=p^3-e^{px^2}-\int_0^x(x-u)\,\mathrm{d}u=p^3-e^{px^2}-\frac{1}{2}x^2$ $f'(x)=-2pxe^{px^2}-x$ $f'(1)=-2pe^p-1$ Gradient of normal at $x=1$: $M=\frac{1}{2pe^p+1}$	A1 M1 A1 A1

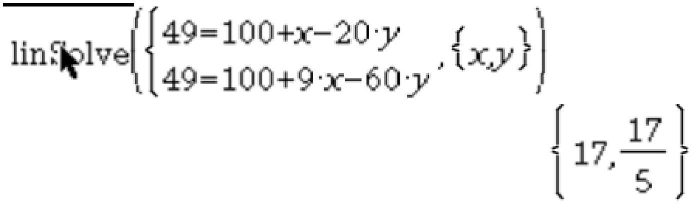
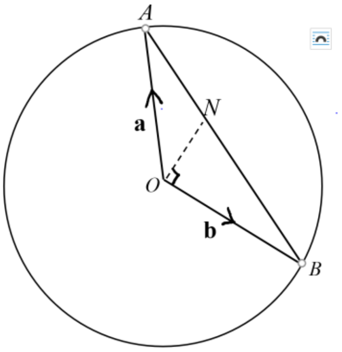
<table>
<tr><th>Qn</th><th>Suggested solution</th><th>Markscheme</th></tr>
<tr><td></td><td>


By GDC, max value of M is 3.78.</td><td>G1
A1</td></tr>
<tr><td>4</td><td><i>Probability</i></td><td>Max mark: 6</td></tr>
<tr><td></td><td>


Let S be the event that he liked his study;
J be the event that he enjoyed his job.
$x=P\left(J'\,|\,S'\right)=$ P(an alumni member found a job he did not enjoy| he did not like his course of study)</td><td></td></tr>
<tr><td>(i)</td><td>From tree diagram, 0.4x = 0.3 $\Rightarrow$ x = 0.75</td><td>M1A1</td></tr>
<tr><td>(ii)</td><td>$P(J)=0.6(0.7)+0.4(0.25)$
$=0.52$</td><td>M1
A1</td></tr>
<tr><td>(iii)</td><td>$P\left(S'\,|\,J\right)=\frac{P\left(S'\cap J\right)}{P(J)}$
$=\frac{0.1}{0.52}$
$=\frac{5}{26}=0.192\ \ (3s.f.)$</td><td>M1
A1</td></tr>
</table>

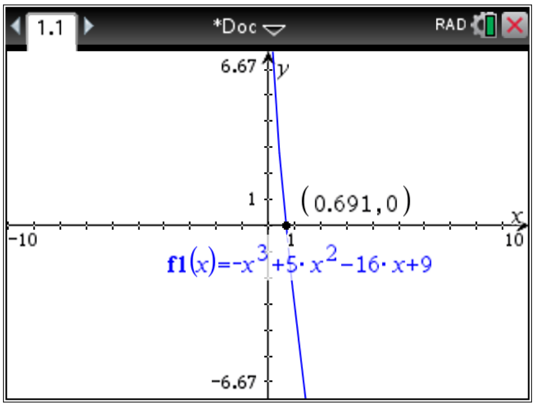
Qn	**Suggested solution**	**Markscheme**
5	***Geometric Series with Complex Numbers***	**Max mark: 8**
(i)	Common ratio = $\left(\frac{w}{4}\right)^3$	A1
	Since $\left\lvert\left(\frac{w}{4}\right)^3\right\rvert = \frac{\lvert w\rvert^3}{64} = \frac{2^3}{64} = \frac{1}{8} < 1$, the infinity sum *S* exists.	M1 A1
(ii)	$1+\left(\frac{w}{4}\right)^3+\left(\frac{w}{4}\right)^6+\left(\frac{w}{4}\right)^9+....+\left(\frac{w}{4}\right)^{3r}+...$	
	$=\frac{1}{1-\left(\frac{w}{4}\right)^3}$	M1A1
	$=\frac{1}{1-\frac{1}{8}\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}}$	M1
	$=\frac{1}{1-\frac{1}{8}i}$	A1
	$=\frac{64}{65}+\frac{8}{65}i$	A1
6	***Vectors***	**Max mark: 5**
	$\mathbf{r}_B-\mathbf{r}_A=\begin{pmatrix}3\\-6\end{pmatrix}+t\begin{pmatrix}-5\\4\end{pmatrix}$	M1
	$\lvert\mathbf{r}_B-\mathbf{r}_A\rvert=\sqrt{(3-5t)^2+(-6+4t)^2}$	M1
	$=\sqrt{41t^2-78t+45}$	A1
		G1
	Minimum distance between the 2 ships is 2.81 (3sf)	A1

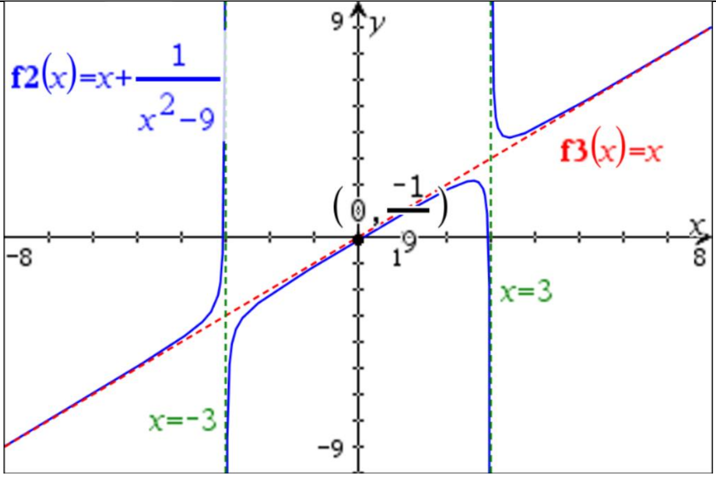
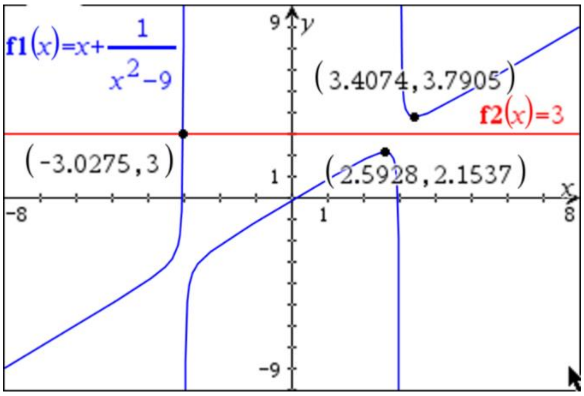
Qn	Suggested solution	Markscheme
7	***Composite Function***	**Max mark: 6**
	$A^2(x)=A(2x+1)=2(2x+1)+1=4x+3=2^2x+2^2-1$	M1A1
	$A^3(x)=A(4x+3)=2(4x+3)+1=8x+7=2^3x+2^3-1$	M1
	$A^4(x)=A(8x+7)=2(8x+7)+1=16x+15=2^4x+2^4-1$ $\vdots$	
	$A^n(x)=2^nx+2^n-1$	A1
	$A^{1000}(x)=A^{-1}(x)$ $\Rightarrow A^{1001}(x)=x$	M1
	$\Rightarrow 2^{1001}x+2^{1001}-1=x$ $\Rightarrow \left(2^{1001}-1\right)x=-\left(2^{1001}-1\right)$	
	$\Rightarrow x=-1$	A1
8	**Area of Triangle and Cosine Rule**	**Max mark: 5**
		
	Area of triangle ABC' = 3 Area of triangle ABC $\Rightarrow AC'=3AC$	A1
	By cosine rule, $7^2=10^2+(AC)^2-20(AC)\cos\theta \quad -(1)$	M1
	$7^2=10^2+(3AC)^2-20(3AC)\cos\theta \quad -(2)$	A1
	<u>Method 1</u> $(2)-(1),\quad 8(AC)^2-40(AC)\cos\theta=0$ $\Rightarrow \cos\theta=\frac{1}{5}AC \quad -(3)$	M1

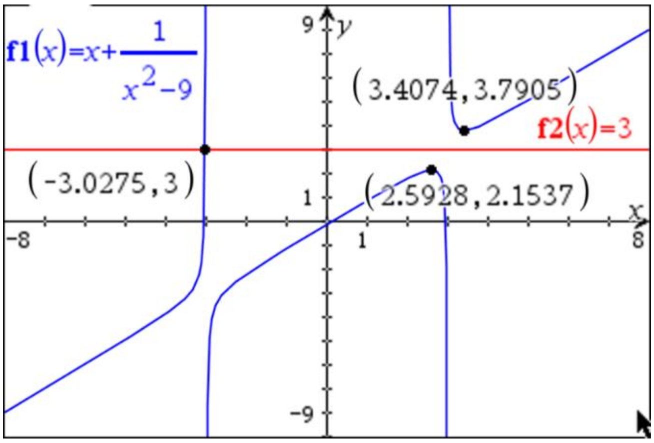
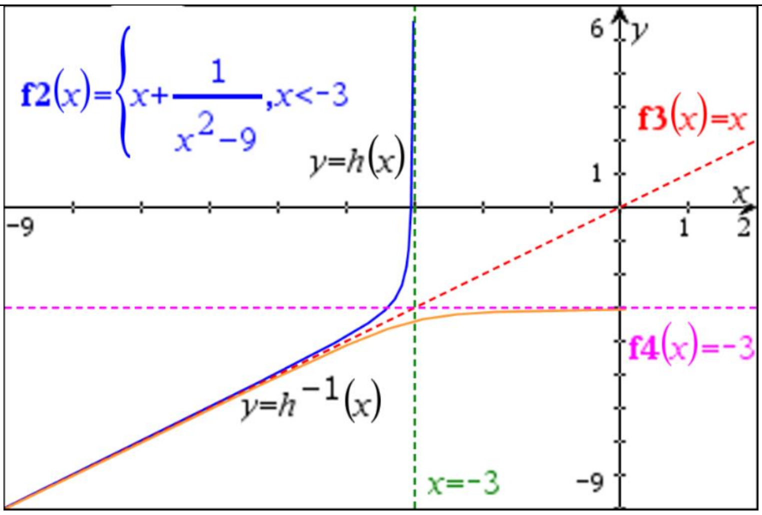
Qn	Suggested solution	Markscheme
	Subst $\cos\theta = \frac{1}{5}AC$ into (1), $(AC)^2 = 17 \Rightarrow AC = \sqrt{17}$ $\therefore \cos\theta = \frac{\sqrt{17}}{5}$	A1
	Method 2  $\text{linSolve}\left(\begin{cases}49=100+x-20\cdot y\\49=100+9\cdot x-60\cdot y\end{cases},\{x,y\}\right)$ $\left\{17, \frac{17}{5}\right\}$	M1
	By GDC, $AC\cos\theta = \frac{17}{5}$ $(AC)^2 = 17$ $\therefore \cos\theta = \frac{\sqrt{17}}{5}$	A1
	Note: Accept $\cos\theta = 0.825\ (3s.f.)$	
9	***Vectors – Ratio Theorem (and Scalar Product)***	**Max mark: 6**
		
	Method 1 $\frac{\text{Area of } \triangle\text{OBN}}{\text{Area of } \triangle\text{OAN}} = \frac{1-\lambda}{\lambda}$	M1
	$\Rightarrow \frac{\frac{1}{2}\lvert\mathbf{b}\rvert\left\lvert\overrightarrow{ON}\right\rvert}{\frac{1}{2}\lvert\mathbf{a}\rvert\left\lvert\overrightarrow{ON}\right\rvert\sin 30^\circ} = \frac{1-\lambda}{\lambda}$	A1A1
	$\Rightarrow 2 = \frac{1-\lambda}{\lambda}$ $\left(\because \lvert\mathbf{a}\rvert = \lvert\mathbf{b}\rvert = r = \text{radius of circle}\right)$	A1
	$\Rightarrow \lambda = \frac{1}{3}$	A1

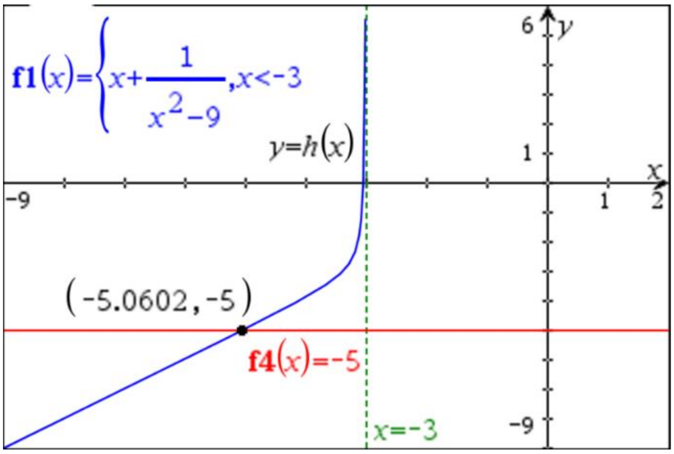
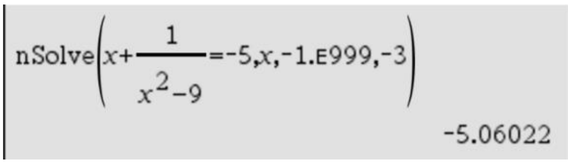
Qn	Suggested solution	Markscheme
	N divides *AB* in the ratio $\lambda : 1-\lambda$, By Ratio Thm, $\overrightarrow{ON} = \left(1-\lambda\right)\mathbf{a}+\lambda\mathbf{b} = \frac{1}{3}(2\mathbf{a}+\mathbf{b})$ (shown)	R1
	Method 2 $\angle OAN = \angle OBN = 30°$ (since ΔOAB is isosceles) $\angle AON = 30° \Rightarrow \Delta OAN$ is isosceles $BN = \frac{\lvert\mathbf{b}\rvert}{\cos 30°} = \frac{2}{\sqrt{3}}\lvert\mathbf{b}\rvert$ $AN = ON = \lvert\mathbf{b}\rvert \tan 30° = \frac{1}{\sqrt{3}}\lvert\mathbf{b}\rvert$ $\frac{BN}{AN} = 2 = \frac{1-\lambda}{\lambda}$ $\Rightarrow \lambda = \frac{1}{3}$ *N* divides *AB* in the ratio $\lambda : 1-\lambda$, By Ratio Thm, $\overrightarrow{ON} = \left(1-\lambda\right)\mathbf{a}+\lambda\mathbf{b} = \frac{1}{3}(2\mathbf{a}+\mathbf{b})$ (shown)	M1 A1 A1 A1 A1 R1
	Method 3 $\lvert\mathbf{a}\rvert = \lvert\mathbf{b}\rvert = r =$ radius of circle, $\mathbf{a}\cdot\mathbf{b} = \lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\left(120^{\circ}\right) = -\frac{1}{2}r^2$ Since *ON* is perpendicular to *OB*, $\overrightarrow{ON}\cdot\overrightarrow{OB} = 0$ $\Rightarrow \left[\left(1-\lambda\right)\mathbf{a}+\lambda\mathbf{b}\right]\cdot\mathbf{b} = 0$ $\Rightarrow \left(1-\lambda\right)\mathbf{a}\cdot\mathbf{b} + \lambda\lvert\mathbf{b}\rvert^2 = 0$ $\Rightarrow \left(1-\lambda\right)\left(-\frac{1}{2}r^2\right) + \lambda\lvert\mathbf{b}\rvert^2 = 0$ $\Rightarrow -\frac{1}{2}+\frac{3}{2}\lambda = 0$ $\Rightarrow \lambda = \frac{1}{3}$ *N* divides *AB* in the ratio $\lambda : 1-\lambda$, By Ratio Thm, $\overrightarrow{ON} = \left(1-\lambda\right)\mathbf{a}+\lambda\mathbf{b}$ Hence, $\overrightarrow{ON} = \frac{2}{3}\mathbf{a}+\frac{1}{3}\mathbf{b} = \frac{1}{3}(2\mathbf{a}+\mathbf{b})$ (shown)	A1 M1 A1 A1 A1 R1

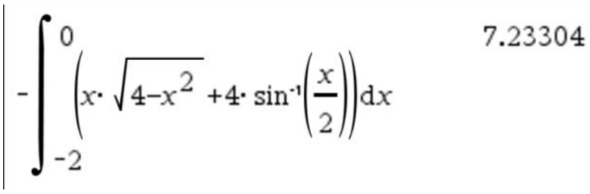
Qn	Suggested solution	Markscheme
	Section B	
10	***Factor Theorem, Polynomials, Nature of Roots***	**[Max mark: 11]**
(a)i)	$P(i)=i^3-(2+i)i^2+2(2+i)i-4i$	**M1**
	$=-i+2+i+4i-2-4i$	
	$=0$	**A1**
	By factor theorem, $z=i$ is a root of $P(z)$.	**AG**
(a)ii)	$P(z)=z^3-(2+i)z^2+2(2+i)z-4i$	
	$=(z-i)(z^2+bz+4)$	**M1** (or long division)
	Comparing coefficients of z^2: $-(2+i)=-i+b$	
	$\therefore b=-2$	**A1**
	Hence, quadratic factor is z^2-2z+4.	**A1**
(b)	**<u>Method 1 (Algebra)</u>** $\sum\alpha=5$ $\sum\alpha\beta=16$	
	$\sum\alpha^2=(\sum\alpha)^2-2(\sum\alpha\beta)$	**M1**
	$=25-32=-7<0$	**A1**
	$\sum\alpha^2<0\Rightarrow$ Not all roots are real.	**R1**
	Therefore, the equation has non-real roots.	**AG**
	Since the coefficients are real, the equation has either 3 real roots or 1 real and 2 complex conjugate roots.	
	$\alpha\beta\beta^*=9>0$	**M1** (> 0)
	Since $\beta\beta^*=\lvert\beta\rvert^2>0$, $\alpha>0$	**R1**
	The equation has one positive real root.	**A1**
	<u>Method 2 (Calculus)</u> Let $f(x)=-x^3+5x^2-16x+9$	
	$f'(x)=-3x^2+10x-16$ $=-3\left(x-\frac{5}{3}\right)^2-\frac{17}{3}<0$ (<u>Alternative</u>: use discriminant on $f'(x)$, $\Delta=-92<0$)	**M1** – 1st derivative / Δ
	Hence, there are no stationary points, i.e. f is a strictly decreasing function.	**A1** – vertex form
	Since the coefficients are real, the equation has either 3 real roots or 1 real and 2 complex conjugate roots. But since f is a strictly decreasing function, there can only be 1 real root.	**R1**
	Therefore, the equation has non-real roots.	**AG**

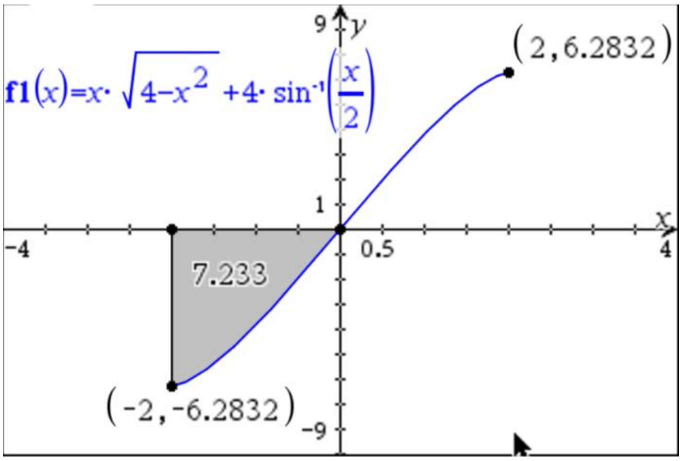
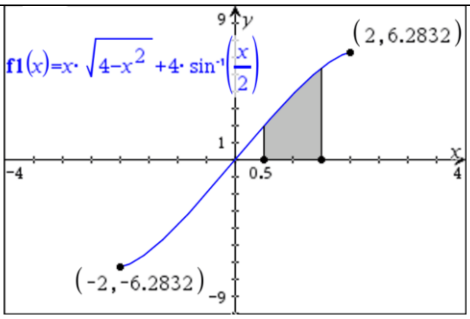
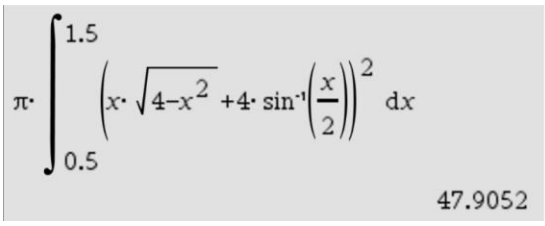
Qn	Suggested solution	Markscheme
	Further, $f(0)=9>0$ and $f(1)=-3<0$.	**M1** – IVT
	Therefore, there is only one real root between 0 and 1, and the sign of the real root is positive.	**R1** **A1**
	(*This is actually the Intermediate Value Theorem*)	
	(<u>Alternative to IVT</u>: use GDC, real root is $x = 0.691$	**M1A1**
	Therefore, the sign of the real root is positive.)	**A1**
		

Qn	Suggested solution	Markscheme
11	***Functions & Graphs, Graph of Inverse Function, Composite Function and Range***	**[Max mark: 19]**
(a)i)	As $x \to 3^-, g(x) = x + \frac{1}{x^2-9} \to -\infty$ As $x \to 3^+, g(x) = x + \frac{1}{x^2-9} \to +\infty$ Therefore, $x = 3$ is a vertical asymptote	**M1** **A1**
(a)ii)		**A2** – shape of graph with 3 branches **A1** – y-intercept $\left(0,-\frac{1}{9}\right)$ or $(0,-0.111)$
(b)	**<u>Method 1</u>** $p(x)=x^3-qx^2-9x+(9q+1)=0$ Note that $p(-3)=p(3)=1\neq 0$ i.e. x^2-9 is not a factor Dividing $p(x)=0$ by x^2-9: $\frac{x^3-qx^2-9x+(9q+1)}{x^2-9}=0$ $\frac{x(x^2-9)-q(x^2-9)+1}{x^2-9}=0$ $x+\frac{1}{x^2-9}=q$ which is $g(x)=q$  For $g(x)=q$ to have one real solution, $2.15<q<3.79$ (3 sf) Since $q\in\mathbb{Z}^+$, $q=3$	**(R1)** **M1** **A1** **(M1)** – finding local max/min points **A1** **A1**

Qn	Suggested solution	Markscheme
	Method 2  From the graph of $y=g(x)$, observe that the equation $g(x)=k$ has one real root when $2.15<k<3.79$ (3 sf). If $k\in\mathbb{Z}^+$, $k=3$. Consider $g(x)=q\Rightarrow x+\frac{1}{x^2-9}=q$ $x^3-9x+1=qx^2-9q$ $\Rightarrow x^3-qx^2-9x+9q+1=p(x)=0$ Hence, $q=3$.	**M1** – finding local max/min points **(R1)** **A1** **A1** **M1** – showing equivalence to $p(x)=0$ **A1**
(c)i)	$m=-3$	**A1**
(c)ii)		**A1** – graphs are reflections in $y=x$ **A1** – $y=h^{-1}(x)$ **A1** – asymptote $y=-3$

Qn	Suggested solution	Markscheme
(c)iii)	Let $h^{-1}(-5)=a$ Then $h(a)=-5$ By GDC, **Method 1**  $h^{-1}(-5)=-5.06$ (3 sf) **Method 2**  $\text{nSolve}\left(x+\frac{1}{x^2-9}=-5,x,-1.\text{E}999,-3\right)$ -5.06022 $h^{-1}(-5)=-5.06$ (3 sf)	**M1** **A1** **A1**
(c)iv)	$R_{h^{-1}}=(-\infty,-3)$ $D_f=(-\infty,0)$ Range of $f\circ h^{-1}=(\ln 3,\infty)$ o.e.	**M1** – finding $R_{h^{-1}}$ and D_f **A1**

Qn	Suggested solution	Markscheme
12	***Integration by Substitution, Bounded Area, Vol. of Solids of Revolution, Inflection Points***	**[Max mark: 22]**
(a)	$x=2\sin\theta$ $dx=2\cos\theta\ d\theta$ o.e. $\int\sqrt{4-x^2}\,dx=\int\sqrt{4-4\sin^2\theta}\times 2\cos\theta\ d\theta$ $=\int 4\cos^2\theta\ d\theta$ **<u>Method 1</u>**: Double-angle formula $=\int 2(\cos 2\theta+1)d\theta$ $=\sin 2\theta+2\theta+C$ $=2\sin\theta\cos\theta+2\theta+C$ **<u>Method 2</u>**: Integration by parts $\int 4\cos^2\theta\ d\theta=4\sin\theta\cos\theta-\int 4\sin\theta\,(-\sin\theta)\,d\theta$ $\int 4\cos^2\theta\ d\theta=4\sin\theta\cos\theta+4\int(1-\cos^2\theta)\,d\theta$ $\int 4\cos^2\theta\ d\theta=4\sin\theta\cos\theta+4\theta-\int 4\cos^2\theta\ d\theta$ $2\int 4\cos^2\theta\ d\theta=4\sin\theta\cos\theta+4\theta$ $\int 4\cos^2\theta\ d\theta=2\sin\theta\cos\theta+2\theta$ <u>Continuing from Method 1 or 2</u> $=2\left(\frac{x}{2}\right)\left(\sqrt{1-\frac{x^2}{4}}\right)+2\arcsin\left(\frac{x}{2}\right)+C$ $=\frac{1}{2}x\sqrt{4-x^2}+2\arcsin\left(\frac{x}{2}\right)+C$ $A=\frac{1}{2},B=2$	**A1** **M1 A1** **M1 A1** **A1** (ok w/out +C) **M1** **A1** **A1** (ok w/out +C) **A1** – 1st term (A) **A1** – 2nd term (B) (ok w/out +C)
(b)	Area $=-\int_{-2}^{0}x\sqrt{4-x^2}+4\arcsin\left(\frac{x}{2}\right)dx$ OR $\int_{0}^{2}x\sqrt{4-x^2}+4\arcsin\left(\frac{x}{2}\right)dx$ OR $\left\lvert\int_{-2}^{0}x\sqrt{4-x^2}+4\arcsin\left(\frac{x}{2}\right)dx\right\rvert$ OR $\left\lvert\int_{-2}^{0}f(x)dx\right\rvert$ o.e. **<u>Method 1 (GDC – Numerical Integral)</u>**  $-\int_{-2}^{0}\left(x\cdot\sqrt{4-x^2}+4\cdot\sin^{-1}\left(\frac{x}{2}\right)\right)dx$ 7.23304 Area = 7.23 square units (3 sf)	**A1** **M1** **A1**

Qn	Suggested solution	Markscheme
	Method 2 (GDC – Bounded Area on Graph)  Area = 7.23 square units (3 sf)	**M1** **A1**
(c)	 Volume $= \int_{0.5}^{1.5} \pi \left[x\sqrt{4-x^2} + 4\arcsin\left(\frac{x}{2}\right) \right]^2 dx$ $= 47.9$ units^3 (3 sf)  $\pi\cdot\int_{0.5}^{1.5} \left(x\cdot\sqrt{4-x^2}+4\cdot\sin^{-1}\left(\frac{x}{2}\right)\right)^2 dx$ 47.9052	**M1 A1** **A1**
(d)	**Method 1 (using part (a))** From (a), $\int \sqrt{4-x^2}dx = \frac{1}{2}x\sqrt{4-x^2} + 2\arcsin\left(\frac{x}{2}\right) + \text{c}$ $\Rightarrow 2\int \sqrt{4-x^2}dx = f(x) + \text{c}$ $\Rightarrow 2\sqrt{4-x^2} = f'(x)$ $f'(x) = 2\sqrt{4-x^2}$	**A1** **M1** **A1**

<table>
<tr><th>Qn</th><th>Suggested solution</th><th>Markscheme</th></tr>
<tr><td></td><td>

Method 2 (product and chain rule)

$$f(x)=x\sqrt{4-x^2}+4\arcsin\left(\frac{x}{2}\right)$$

$$f'(x)=x\left(\frac{-2x}{2\sqrt{4-x^2}}\right)+\sqrt{4-x^2}+\frac{2}{\sqrt{1-\left(\frac{x^2}{4}\right)}}$$

$$=-\frac{x^2}{\sqrt{4-x^2}}+\sqrt{4-x^2}+\frac{4}{\sqrt{4-x^2}}$$

$$=2\sqrt{4-x^2}$$

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M1 A1

A1

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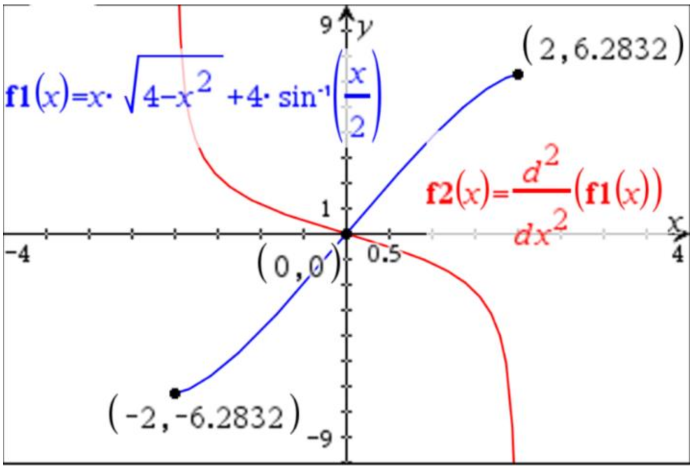
<tr><td>(e)</td><td>

Method 1: Plotting graph of $f''(x)$



From graph,

$f''(0)=0$

and $f''(x)$ changes sign through $x=0$

Hence, $f(x)$ has a point of inflection at $x=0$

Method 2: Differentiating from (d) to obtain $f''(x)$

From (a),

$f'(x)=2\sqrt{4-x^2}$

$f''(x)=-\frac{2x}{\sqrt{4-x^2}}$

$f''(x)=0\Rightarrow x=0$

When $-2\le x<0, f''(x)>0$

When $0<x\le 2, f''(x)<0$

i.e. $f''(x)$ changes sign through $x=0$

Hence, $f(x)$ has a point of inflection at $x=0$

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A1 – graph of $y=f''(x)$

A1

R1

AG

A1

A1

R1

AG

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CANDIDATE SESSION NUMBER

0	2	5	0	1	2				

STUDENT NAME: ______________________________

EXAMINATION CODE

8	8	2	0	-	7	2	0	1

TEACHER NAME: ______________________________

ST JOSEPH'S INSTITUTION
YEAR 6 PRELIMINARY EXAMINATION 2020

MATHEMATICS
HIGHER LEVEL
PAPER 1
Thursday

30th July 2020
2 hours
0800 – 1000 hrs

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- **Section A**: Answer all questions showing working and answers in the spaces provided in the exam paper.
- **Section B**: Answer all questions using the writing paper provided.
- The use of calculators is **not** permitted in this paper.
- A clean copy of the **Mathematics HL Formulae Booklet** is required for this paper.
- Unless otherwise stated in the question, all numerical answers are to be given exactly or correct to three significant figures.
- The maximum mark for this examination paper is *[**100** marks]*.
- This question paper consists of **12** printed pages including the Cover Sheet.
- Sections A and B are to be submitted **separately**.

FOR MARKER USE ONLY:

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	TOTAL
											/100

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are advised to show all working.

SECTION A (50 marks)

1 **[Maximum mark: 5]**

Let $f\left(x\right)=\dfrac{k}{x^3+1}$, $x\neq -1$ where *k* is a real constant. Given that $f^{-1}\left(8\right)=-\dfrac{1}{2}$,

(a) find the value of *k*, and, [3]

(b) find an expression for $f^{-1}\left(x\right)$. [2]

2 **[Maximum mark: 5]**

The random variable X represents the number of eggs laid in a single brood by an eagle.

X follows a Binomial distribution with mean 3 and variance $\frac{3}{4}$.

Show that the largest value that X can take is 4, and find the probability that one such eagle lays less than two eggs in a single brood.

3 **[Maximum mark: 7]**

The plane $\prod$ contains the points $A(1,2,3)$, $B(2,3,5)$ and $C(3,-2,1)$.

(a) Find the Cartesian equation of the plane $\prod$. [4]

(b) Find the coordinates of the point where the line $l : \boldsymbol{r} = \begin{pmatrix} -5 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}, \ \lambda \in \mathbb{R}$

meets the plane $\prod$. [3]

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4 **[Maximum mark: 8]**

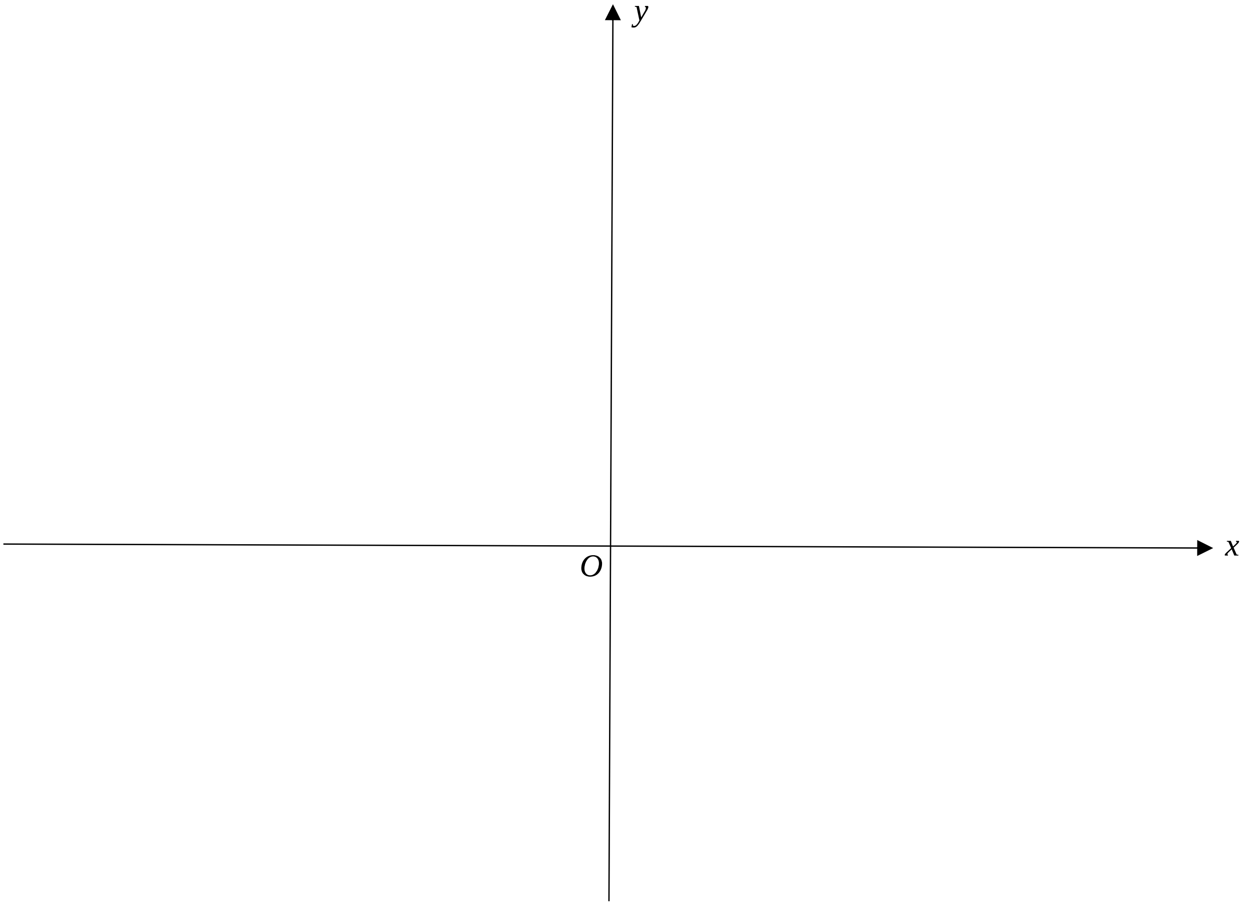
(a) Show that $\log_9\left(\cos 2x+\frac{3}{2}\right)=\log_3\sqrt{\cos 2x+\frac{3}{2}}$. [3]

(b) Let $f(x)=\log_3\sqrt{\cos 2x+\frac{3}{2}}$.

Solve the equation $3^{f\left(x+\frac{\pi}{4}\right)}=1$ for $0\le x\le\frac{\pi}{4}$. [5]

5 **[Maximum mark: 8]**

(a) On the axes below, sketch the graph of the function $y=\frac{2x-1}{x+1}$, clearly labelling all the asymptotes and the points where the curve cuts the axes. [4]



(b) **Hence**, solve the inequality

$$\ln\left(\frac{2x-1}{x+1}\right)\le 0\,.$$ [3]

(c) State the equation of the **horizontal** asymptote of the graph of $y=\ln\left(\frac{2x-1}{x+1}\right)$. [1]

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6 **[Maximum mark: 9]**

In the expansion of $\left(\dfrac{1}{2x^3}-x\right)^8$,

(a) **(i)** find the term independent of x, and [4]

(ii) find the coefficient of x^{-4}. [2]

(b) Show that there is no x^{-2} term. [1]

(c) Find the constant term in the expansion of $\left(2-3x^2\right)^2\left(\dfrac{1}{2x^3}-x\right)^8$. [2]

MORE SPACE IS AVAILABLE ON THE NEXT PAGE

7 **[Maximum mark: 8]**

The equation $x^3 + px^2 + qx + r = 0$, where $p, q, r \in \mathbb{R}$ and $p \neq 0$, has roots that are consecutive terms of an arithmetic sequence.

(a) Show that one root of the equation is $-\frac{p}{3}$ and the other roots satisfy the equation $x^2 + \frac{2p}{3}x + \frac{3r}{p} = 0$. [5]

(b) Hence or otherwise, find the value of q for which the equation $x^3 + x^2 + qx + 1 = 0$ has non-real roots, justifying your answer. [3]

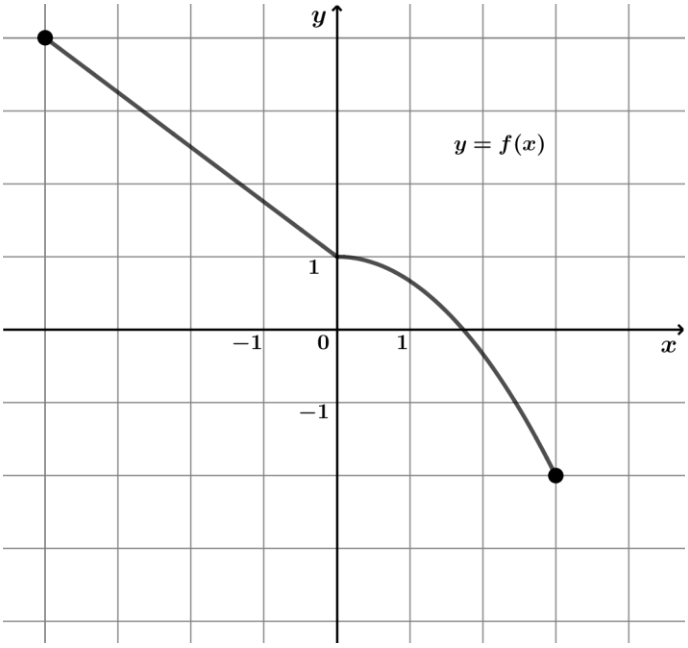
*Do **NOT** write solutions on this page.*

SECTION B (50 marks)

Answer all questions on the writing paper provided. Please start each question on a new page.

8 [Maximum mark: 13]

The graph of $y = f(x)$ is shown below.



(a) Write the domain of the following functions:

(i) $y = f^{-1}(x)$ [2]

(ii) $y = \frac{1}{2}f(4x)$ [2]

(b) Sketch the respective graphs of the following functions using the same set of axes:
(Do not sketch on the set of axes above.)

(i) $y = f^{-1}(x)$ [4]

(ii) $y = \frac{1}{2}f(4x)$ [3]

(c) Hence, find the number of solutions for which $f\left(\frac{1}{2}f(4x)\right) = x$. [2]

TURN OVER

*Do **NOT** write solutions on this page.*

9 [Maximum mark: 13]

(a) Using mathematical induction, show that for all $n \in \mathbb{Z}^+$,

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\frac{x}{1-x}\right) = \frac{n!}{(1-x)^{n+1}}, \ \ x \neq 1.$$

[7]

(b) Give a mathematical justification as to why the rational expression $\frac{1}{1-x}$ has the same n^{th} derivative as $\frac{x}{1-x}$ for all $n \in \mathbb{Z}^+$, that is,

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\frac{1}{1-x}\right) = \frac{n!}{(1-x)^{n+1}}, \ \ x \neq 1.$$

[1]

(c) Find the equation of the tangent to the curve

$$y = \frac{n!}{(1-x)^{n+1}}$$

at $x = 0$, leaving your answer in terms of n and in the form $y = mx + b$. [5]

10 [Maximum mark: 13]

(a) In a barn, 100 chicks sit peacefully in a circle. Suddenly, each chick randomly pecks the chick immediately next it, either to its left or right. What is the expected number of un-pecked chicks? [4]

(b) Chicken Little, one of the chicks, moves down the stairs on his way out of the barn. Each time he moves down, he decides randomly whether to go down 1, 2 or 3 steps.

(i) What is the probability that Chicken Little sets foot on the fourth step below his starting position on his way down the stairs? [6]

(ii) Given that Chicken Little first moves down 2 steps, what is the probability that he sets foot on the fourth step below his starting position on his way down the stairs? [3]

*Do **NOT** write solutions on this page.*

11 [Maximum mark: 11]

(a) Show that $\frac{\mathrm{d}}{\mathrm{d}x}(\tan x) = 1 + \tan^2 x$. [1]

Let I_0, I_1, I_2, ... be a sequence such that

$$I_n = \int_0^{\pi/4} \tan^n x \ \mathrm{d}x$$

for all integers $n \geq 0$.

(b) By letting $\tan^n x = (\tan^{n-2} x)(1 + \tan^2 x) - \tan^{n-2} x$, show that

$$I_n = \frac{1}{n-1} - I_{n-2}$$

for $n \geq 2$. [5]

(c) Hence, find I_4. [5]

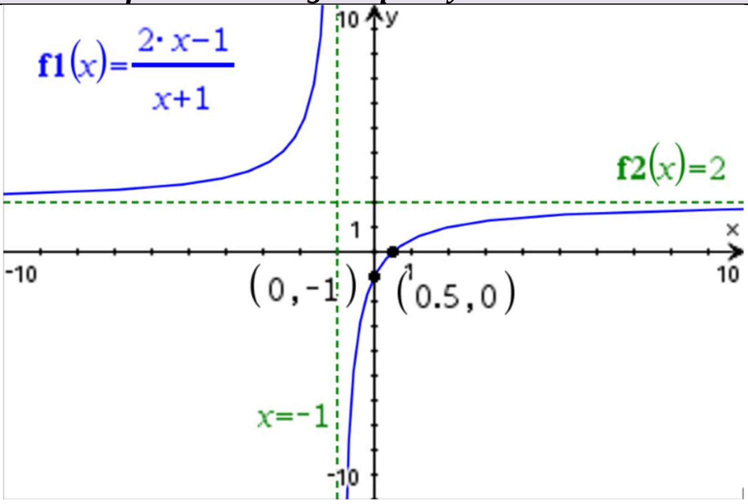
End of Paper

Year 6 HL Math Preliminary Examination 2020 Paper 1 (Markscheme)

Section A

Qn	Suggested solution	Markscheme
1	***Finding inverse function***	**[Marks: 5]**
(a)	$f^{-1}(8)=-\frac{1}{2}\Rightarrow f\left(-\frac{1}{2}\right)=8$ $\frac{k}{-\frac{1}{8}+1}=8$ o.e. $\therefore k=7$	**M1** **A1** **A1**
(b)	Let $y=\frac{7}{x^3+1}$ Interchange x and y $x=\frac{7}{y^3+1}$ $\therefore f^{-1}(x)=y=\sqrt[3]{\frac{7}{x}-1}$ or $\sqrt[3]{\frac{7-x}{x}}$ o.e	**M1** **A1** f.t. value of k Ans. must be $f^{-1}(x)$
2	***Binomial Distribution***	**[Marks: 5]**
	$X\sim B(n,p)$ $\mathrm{E}(X)=np=3$ $\mathrm{Var}(X)=3(1-p)=\frac{3}{4}$ $\Rightarrow 1-p=\frac{1}{4}$ $\Rightarrow p=\frac{3}{4}$ $\Rightarrow n=4$ Since $n=4,\ X=0,1,2,3,4$ $\therefore X=4$ is the largest value X can take. $X\sim B\left(4,\frac{3}{4}\right)$ $P(X<2)=P(X=0)+P(X=1)$ $=\left(\frac{1}{4}\right)^4+4\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)$ $=\frac{1}{256}+\frac{12}{256}$ $=\frac{13}{256}$	**M1** – E(X) and Var(X) for Bin dist **A1** **M1** – finding n **AG** **M1** – pdf for Bin **A1**

Qn	Suggested solution	Markscheme
3	***Cartesian equation of Plane + Intersection with Line***	**[Marks: 7]**
(a)	$\overrightarrow{AB}=\begin{pmatrix}1\\1\\2\end{pmatrix},\quad \overrightarrow{BC}=\begin{pmatrix}1\\-5\\-4\end{pmatrix},\quad \overrightarrow{AC}=\begin{pmatrix}2\\-4\\-2\end{pmatrix}=2\begin{pmatrix}1\\-2\\-1\end{pmatrix}$	**A1** – any two
	$\boldsymbol{n}\parallel\begin{pmatrix}1\\1\\2\end{pmatrix}\times\begin{pmatrix}1\\-2\\-1\end{pmatrix}$	**M1**
	$=\begin{pmatrix}3\\3\\-3\end{pmatrix}\parallel\begin{pmatrix}1\\1\\-1\end{pmatrix}$	**A1**
	$\prod: x+y-z=\begin{pmatrix}1\\2\\3\end{pmatrix}\bullet\begin{pmatrix}1\\1\\-1\end{pmatrix}=0$	**A1** – Cartesian form
(b)	Substitute $\boldsymbol{r}=\begin{pmatrix}-5+2\lambda\\1-\lambda\\2+3\lambda\end{pmatrix}$ into $\prod$:	
	$(-5+2\lambda)+(1-\lambda)-(2+3\lambda)=0$	**M1**
	$\therefore\lambda=-3$	**A1** f.t. eqn of $\prod$
	Point of intersection is $(-11,4,-7)$	**A1** – coordinates
4	***Laws of Log + Trigo equation with exponential***	**[Marks: 8]**
(a)	$\log_9\left(\cos 2x+\frac{3}{2}\right)=\frac{\log_3\left(\cos 2x+\frac{3}{2}\right)}{\log_3 9}$	**M1 A1** – change of base
	$=\frac{1}{2}\log_3\left(\cos 2x+\frac{3}{2}\right)$	**A1**
	$=\log_3\sqrt{\cos 2x+\frac{3}{2}}$	**AG**
(b)	$3^{f\left(x+\frac{\pi}{4}\right)}=1$ $3^{\log_3\sqrt{\cos\left[2\left(x+\frac{\pi}{4}\right)\right]+\frac{3}{2}}}=1$ $\sqrt{\cos\left[2\left(x+\frac{\pi}{4}\right)\right]+\frac{3}{2}}=1$ $\boxed{\begin{array}{l}0\le x\le\frac{\pi}{4}\\ \frac{\pi}{2}\le 2\left(x+\frac{\pi}{4}\right)\le\pi\end{array}}$	**M1 A1**
	$\cos\left[2\left(x+\frac{\pi}{4}\right)\right]=-\frac{1}{2}$ OR $\sin(2x)=\frac{1}{2}$ $2\left(x+\frac{\pi}{4}\right)=\frac{2\pi}{3}$ OR $2x=\frac{\pi}{6}$	**M1 A1**
	$x=\frac{\pi}{12}$ OR $x=\frac{\pi}{12}$	**A1**

Qn	Suggested solution	Markscheme
	Alternative Method $3^{f\left(x+\frac{\pi}{4}\right)}=1$ $3^{\log_3\sqrt{\cos\left[2\left(x+\frac{\pi}{4}\right)\right]+\frac{3}{2}}}=1$ By (a), $\log_9\left(\cos\left[2\left(x+\frac{\pi}{4}\right)\right]+\frac{3}{2}\right)=\log_3 1=0$	
	$\cos\left[2\left(x+\frac{\pi}{4}\right)\right]+\frac{3}{2}=9^0=1$	**M1 A1**
	$\cos\left[2\left(x+\frac{\pi}{4}\right)\right]=-\frac{1}{2}$ OR $\sin(2x)=\frac{1}{2}$	
	$2\left(x+\frac{\pi}{4}\right)=\frac{2\pi}{3}$ OR $2x=\frac{\pi}{6}$	**M1 A1**
	$x=\frac{\pi}{12}$ OR $x=\frac{\pi}{12}$	**A1**
5	***Rational function + Log inequality***	**[Marks: 8]**
(a)		**A1** – shape **A1** – V.A. at $x=-1$ **A1** – H.A. at $y=2$ (must be labelled with equations) **A1** – $\left(\frac{1}{2},0\right)$ & $(0,-1)$
(b)	For $\ln\left(\frac{2x-1}{x+1}\right)\le 0$,	
	$0<\frac{2x-1}{x+1}\le 1$	**M1** – both bounds
	Note that $\frac{2x-1}{x+1}=1$ when $x=2$ $\therefore \frac{1}{2}<x\le 2$ or $x\in\left(\frac{1}{2},2\right]$ o.e.	**A1 A1** – with correct inequality sign, f.t. *x*-intercept in (a)
(c)	$y=\ln 2$	**A1** f.t. H.A. $y=b$ in (a) only if $b>0$

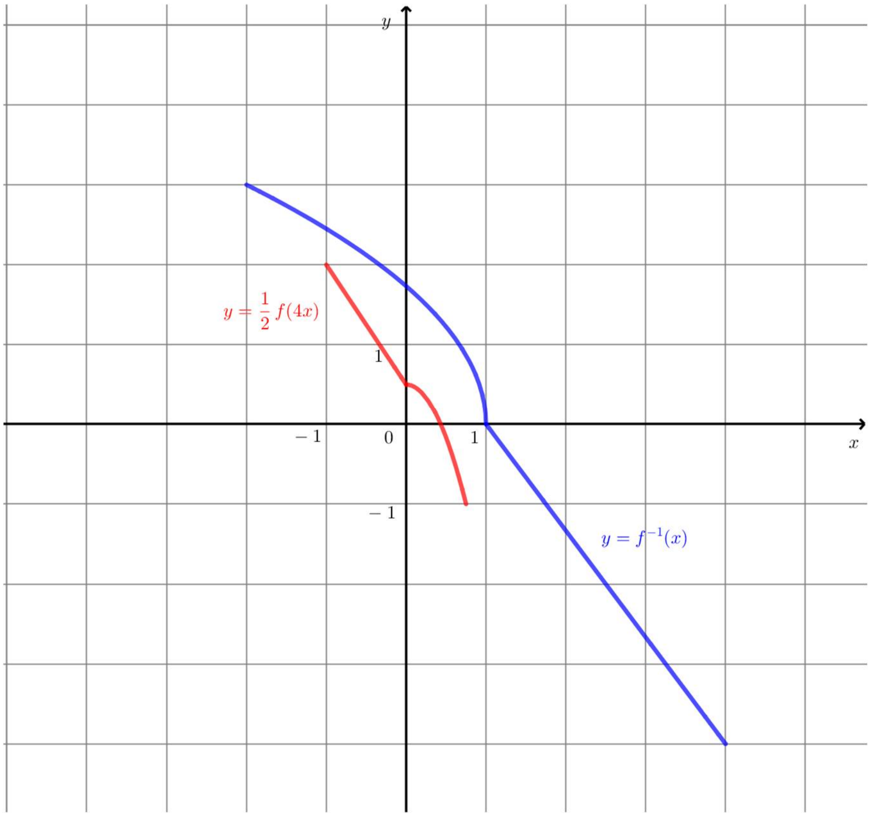
Qn	Suggested solution	Markscheme
6	***Binomial Theorem***	**[Marks: 9]**
(a)i)	General term $=\binom{8}{r}\left(\frac{1}{2x^3}\right)^{8-r}(-x)^r$ or $\binom{8}{k}\left(\frac{1}{2x^3}\right)^{k}(-x)^{8-k}$ Power of $x=4r-24$ or $8-4k$ Term independent of $x=\binom{8}{6}\left(\frac{1}{2}\right)^2$ $=\frac{8(7)}{2}\left(\frac{1}{4}\right)=7$	**M1** **A1** **M1** – correct term from $r=6$ or $k=2$ **A1**
(a)ii)	Coefficient of $x^{-4}=-\binom{8}{5}\left(\frac{1}{2}\right)^3$ $=-\frac{8(7)(6)}{3!}\left(\frac{1}{8}\right)=-7$	**M1** – correct term from $r=5$ or $k=3$ **A1**
(b)	If $4r-24=-2,\ r=\frac{22}{4}\notin\mathbb{N}$ or $k=\frac{10}{4}\notin\mathbb{N}$ There is no x^{-2} term.	**R1** **AG**
(c)	$(2-3x^2)^2\left(\frac{1}{2x^3}-x\right)^8=(4-12x^2+9x^4)\left(\ldots+7-\frac{7}{x^4}+\ldots\right)$ Constant term $=4(7)+9x^4\left(-\frac{7}{x^4}\right)$ $=28-63=-35$	**M1** – both terms **A1** f.t. 7 and –7 in (i)
7	***Polynomials***	**[Marks: 8]**
(a)	**Method 1** Let the 3 roots be α,β,γ. $\begin{cases}\alpha+\beta+\gamma=-p & \text{---}(1)\\ \alpha\beta\gamma=-r & \text{---}(2)\\ \beta-\alpha=\gamma-\beta\Rightarrow\alpha+\gamma=2\beta & \text{---}(3)\end{cases}$ From (1) and (3), $3\beta=-p$ $\therefore\beta=-\frac{p}{3}$ is a root. (shown) From (1), $\alpha+\gamma=-p-\beta=-\frac{2p}{3}$ From (2), $\alpha\gamma=-\frac{r}{\beta}=\frac{3r}{p}$ The other two roots, α and γ, are solutions of $x^2-(\alpha+\gamma)x+\alpha\gamma=0$ $\Rightarrow x^2+\frac{2p}{3}x+\frac{3r}{p}=0$ (shown)	**M1** – Sum and Product of roots **A1** – terms of AP (only if leads to $-\frac{p}{3}$) **AG** **A1** **A1** **M1** **AG**

Qn	Suggested solution	Markscheme
	Method 2 Let the 3 roots be $a-d, a,$ and $a+d$. $(a-d)+a+(a+d)=-p$ $\Rightarrow 3a=-p$ $\therefore a=-\frac{p}{3}$ is one of the roots. (shown) $(a-d)+(a+d)=2a=-\frac{2p}{3}$ $(a-d)(a+d)=-\frac{r}{a}=\frac{3r}{p}$ $\therefore a-d$ and $a+d$ are zeros of $x^2-\left(-\frac{2p}{3}\right)x+\frac{3r}{p}=0$ (shown)	**A1** – terms of AP (only if leads to $-\frac{p}{3}$) **M1** – Sum of roots **AG** **A1** **A1** **M1** **AG**
	Method 3 Let the 3 roots be $a-d, a,$ and $a+d$. $(a-d)+a+(a+d)=-p$ $\Rightarrow 3a=-p$ $\therefore a=-\frac{p}{3}$ is one of the roots. (shown) $x^3+px^2+qx+r=\left(x+\frac{p}{3}\right)\left(x^2+Ax+B\right)$ By inspection (or long division), Coefficient of x^2: $\frac{p}{3}+A=p$ $A=\frac{2p}{3}$ Constant: $\frac{p}{3}(B)=r$ $B=\frac{3r}{p}$ The other two roots are solutions of $x^2+\frac{2p}{3}x+\frac{3r}{p}=0$ (shown)	**A1** – terms of AP (only if leads to $-\frac{p}{3}$) **M1** – Sum of roots **AG** **M1** **A1** **A1** **AG**
(b)	**Method 1** $p=1$ and $r=1$: $x^3+x^2+qx+1=\left(x+\frac{1}{3}\right)\left(x^2+\frac{2}{3}x+3\right)$ Comparing coefficient of x : $q=\frac{1}{3}\left(\frac{2}{3}\right)+3=\frac{29}{9}$	**M1** **A1**

Qn	Suggested solution	Markscheme
	Method 1(a) Consider discriminant of $x^2+\frac{2}{3}x+3$: $\Delta=\frac{4}{9}-4(3)=-\frac{104}{9}<0$ $\therefore x^3+x^2+\frac{29}{9}x+1=0$ has non-real roots. Method 1(b) $\sum\alpha^2=\left(\sum\alpha\right)^2-2\sum_{\alpha\neq\beta}\alpha\beta$ $=(-1)^2-2\left(\frac{29}{9}\right)=-\frac{49}{9}<0$ $\therefore x^3+x^2+\frac{29}{9}x+1=0$ has non-real roots.	**R1** **R1**
	Method 2 $p=1$ and $r=1$: Consider discriminant of $x^2+\frac{2}{3}x+3$: $\Delta=\frac{4}{9}-4(3)=-\frac{104}{9}<0$ $\therefore x^3+x^2+\frac{29}{9}x+1=0$ has non-real roots. Substitute $x=-\frac{1}{3}$ (which is a root from (a)) into equation: $\left(-\frac{1}{3}\right)^3+\left(-\frac{1}{3}\right)^2+\left(-\frac{1}{3}\right)q+1=0$ $\Rightarrow q=\frac{29}{9}$	**R1** **M1** **A1**
	Method 3 $p=1$ and $r=1$: Product of the 3 roots: $-\frac{1}{3}\left(-\frac{1}{3}-d\right)\left(-\frac{1}{3}+d\right)=-1$ $\Rightarrow\frac{1}{9}-d^2=3\Rightarrow d^2=-\frac{26}{9}<0$ $\therefore x^3+x^2+qx+1=0$ has non-real roots $q=\left(-\frac{1}{3}\right)\left(-\frac{1}{3}-d\right)+\left(-\frac{1}{3}\right)\left(-\frac{1}{3}+d\right)+\left(-\frac{1}{3}-d\right)\left(-\frac{1}{3}+d\right)$ $=\frac{2}{9}+\frac{1}{9}-d^2=\frac{29}{9}$	**R1** **M1** **A1**

Qn	Suggested solution	Markscheme
	Method 4 $x^2+\frac{2}{3}x+\left(q-\frac{2}{9}\right)$ $x+\frac{1}{3}\,\big)\,x^3+x^2+qx+1$ $x^3+\frac{1}{3}x^2$ $\frac{2}{3}x^2$ $\frac{2}{3}x^2+\frac{2}{9}x$ $\left(q-\frac{2}{9}\right)x$ $\left(q-\frac{2}{9}\right)x+\frac{1}{3}q-\frac{2}{27}$ $-\frac{1}{3}q+\frac{29}{27}$	**M1**
	Since $x+\frac{1}{3}$ is a factor, $-\frac{1}{3}q+\frac{29}{27}=0 \Rightarrow q=\frac{29}{9}$	**A1**
	Consider discriminant of $x^2+\frac{2}{3}x+\left(q-\frac{2}{9}\right)$: $\Delta=\frac{4}{9}-4(3)=-\frac{104}{9}<0$ $\therefore x^3+x^2+\frac{29}{9}x+1=0$ has non-real roots.	**R1**

Section B

Qn	Suggested solution	Markscheme
8	***Functions with Transformation and Inverse***	**[marks: 13]**
(a)	$D_{f^{-1}} = R_f$ $= \{y \in \mathbb{R} \mid -2 \le y \le 4\}$ or $[-2,\ 4]$ $D_{\frac{1}{2}f(4x)} = \{y \in \mathbb{R} \mid -4 \le 4x \le 3\}$ $= \{y \in \mathbb{R} \mid -1 \le x \le \frac{3}{4}\}$ or $\left[-1, \frac{3}{4}\right]$	**(M1)** **A1** **(M1)** - transformation **A1**
(b)	 Note that the two graphs do not intersect because the zero of the blue (inverse) function is 1 while the zero of the red function is strictly less than 1.	For f^{-1}: **A1A1** – A1 correct shape; A1 correct endpoints for the linear part **A1A1** – A1 correct shape; A1 correct endpoints for the non-linear part For $\frac{1}{2}f(2x)$: **A1A1** – A1 correct shape; A1 correct endpoints for the linear part **A1**– correct shape and endpoints for the non-linear part
(c)	The equation $f\left(\frac{1}{2}f(4x)\right) = x$ is equivalent to $\frac{1}{2}f(4x) = f^{-1}(x)$. Since the two graphs do not intersect, the number of solutions is zero. Note: The composition $f\left(\frac{1}{2}f(4x)\right)$ is well defined over $D_{f(4x)}$ as the range of the inner function is inside the domain of the outer function, i.e., $R_{\frac{1}{2}f(4x)} = [-1, 2] \subset D_f = [-4, 3]$.	**M1** **A1** – A0 for just guessing.

Qn	Suggested solution	Markscheme
9	***Mathematical Induction + Derivatives and Tangent Lines***	**[marks: 13]**
(a)	Let P_n be the statement $$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\frac{x}{1-x}\right)=\frac{n!}{(1-x)^{n+1}} \text{ for all } n\in\mathbb{Z}^+.$$ For $n=1$: $$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{1-x}\right)=\frac{1(1-x)-x(-1)}{(1-x)^2}=\frac{1}{(1-x)^2}=\frac{1!}{(1-x)^{1+1}}$$ Thus, P_1 is true. Assume P_k is true for some $k\in\mathbb{Z}^+$, i.e., $\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left(\frac{x}{1-x}\right)=\frac{k!}{(1-x)^{k+1}}$. Now, we prove that P_{k+1} is also true (using P_k). $$\frac{\mathrm{d}^{k+1}}{\mathrm{d}x^{k+1}}\left(\frac{x}{1-x}\right)=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left(\frac{x}{1-x}\right)\right)=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{k!}{(1-x)^{k+1}}\right)$$ $$=k!\,(-(k+1))(1-x)^{(-(k+1)-1)}(-1)$$ $$=\frac{(k+1)!}{(1-x)^{k+2}}$$ Therefore, since P_1 is true and P_{k+1} is true whenever P_k is true, by mathematical induction, P_n is true for all $n\in\mathbb{Z}^+$.	**A1** – correct derivative **M1** - $n=k$ **M1** – split kth derivative **A1** – use of inductive assumption **M1A1** – attempt + correct derivative **R1** – only award if everything else is correct
(b)	Since $$\frac{1}{1-x}=\frac{x}{1-x}+1,$$ we get $$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\frac{1}{1-x}\right)=\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\frac{x}{1-x}+1\right)=\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\frac{x}{1-x}\right).$$ Graphically, $y=\frac{1}{1-x}$ is a vertical translation of $y=\frac{x}{1-x}$ and so will have the same derivatives.	**R1** – realization that one is a vertical translate of the other.
(b)	Note that the point of tangency is $(0,n!)$ Gradient at $(0,n!)$ is simply $\frac{(n+1)!}{(1-0)^{n+2}}=(n+1)!$. Therefore, the equation of the tangent line is $$y-n!=(n+1)!\,(x-0)\Leftrightarrow y=(n+1)!\,x+n!$$	**A1** **(M1)A1** **M1A1**

Qn	Suggested solution	Markscheme
10	***Basic Probability + Conditional Probability + Expectation***	**[marks: 13]**
(a)	Any chick is between two other chicks. The only way for a chick to not get pecked is if the chick on its left pecks left and the chick on its right pecks right. Thus, $P(\text{left chick pecks left and right chick pecks right}) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$	**M1A1**
	Alternatively, there are three ways of getting pecked: $P(\text{left chick pecks and right chick pecks}) = \left(\frac{1}{2}\right)^2$; $P(\text{left chick pecks and right chick does not peck}) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$; $P(\text{left chick does not peck and right chick pecks}) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$.	OR
	Thus, the probability of not getting pecked is $1 - \frac{3}{4} = \frac{1}{4}$.	M1A1
	As there are 100 chicks, the expected number of un-pecked chicks is $100 \times \frac{1}{4} = 25$.	**M1A1**
(b)	(i) There are four cases in which Chicken Little steps on the 4th step:	**M1 –** at least 2 cases correctly identified
	$1 + 1 + 1 + 1 \Longrightarrow \left(\frac{1}{3}\right)^4 = 1/81$	**A1**
	$1 + 1 + 2 \text{ or } 1 + 2 + 1 \text{ or } 2 + 1 + 1 \Longrightarrow 3\left(\frac{1}{3}\right)^2\left(\frac{1}{3}\right) = \frac{1}{9}$	**A1**
	$1 + 3 \text{ or } 3 + 1 \Longrightarrow 2\left(\frac{1}{3}\right)\left(\frac{1}{3}\right) = \frac{2}{9}$	**A1**
	$2 + 2 \Longrightarrow \left(\frac{1}{3}\right)\left(\frac{1}{3}\right) = \frac{1}{9}$	**A1**
	Thus, the probability is $\frac{1}{81} + \frac{1}{9} + \frac{2}{9} + \frac{1}{9} = \frac{37}{81}$	**A1**
	(ii) Given he first takes 2 steps, then the conditional probability is $\dfrac{\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)\left(\frac{1}{3}\right)}{\frac{1}{3}} = \dfrac{\frac{1}{27} + \frac{1}{9}}{\frac{1}{3}} = \left(\frac{1}{3}\right)^2 + \frac{1}{3} = \frac{4}{9}$	**M1** – correct formula seen **A1** – correct numerator **A1** – final answer

Qn	Suggested solution	Markscheme
11	***Integration + Trigonometric function***	**[marks: 11]**
(a)	$\frac{\mathrm{d}}{\mathrm{d}x}(\tan x) = \sec^2 x = 1 + \tan^2 x$	**A1** – $\sec^2 x$ **AG**
(b)	$I_n = \int_0^{\frac{\pi}{4}} \tan^n x\, dx$ $= \int_0^{\frac{\pi}{4}} \tan^{n-2} x\,(1+\tan^2 x)dx - \int_0^{\frac{\pi}{4}} \tan^{n-2} x\, dx$ $= \int_0^{\frac{\pi}{4}} \tan^{n-2} x\, d(\tan x) - \int_0^{\frac{\pi}{4}} \tan^{n-2} x\, dx$ $= \left.\frac{\tan^{n-1} x}{n-1}\right\|_0^{\frac{\pi}{4}} - I_{n-2}$ $= \frac{1}{n-1} - I_{n-2}$	**M1** – split sum **M1A1** – first term **A1** – I_{n-2} **A1** – first term correct substitution **AG**
(c)	$I_0 = \int_0^{\frac{\pi}{4}} \tan^0 x\, dx = \int_0^{\frac{\pi}{4}} dx = \frac{\pi}{4}$ $I_2 = \frac{1}{2-1} - I_0 = 1 - \frac{\pi}{4}$ $I_4 = \frac{1}{4-1} - \left(1 - \frac{\pi}{4}\right) = \frac{\pi}{4} - \frac{2}{3}$ Alternatively, $I_4 = \frac{1}{4-1} - I_2$ $= \frac{1}{3} - \left(\frac{1}{2-1} - I_0\right)$ $= \frac{1}{3} - 1 + \int_0^{\pi/4} \mathrm{d}x$ $= -\frac{2}{3} + \frac{\pi}{4}$	**A1** **M1A1** **M1A1** OR A1 – correct use of (b) on I_4 M1 – correct expression for I_2 M1A1 – correct way to get $\pi/4$ A1 – final answer

STUDENT NAME: ____________________________

TEACHER INITIALS: ________

CANDIDATE SESSION NUMBER

0	2	5	0	1	2				

EXAMINATION CODE

8	8	2	0	-	7	2	0	1

ST JOSEPH'S INSTITUTION
YEAR 6 PRELIMINARY EXAMINATION 2020

MATHEMATICS
HIGHER LEVEL
PAPER 2
Thursday

6th August 2020
2 hours
0800 – 1000 hrs

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- **Section A**: Answer all questions showing working and answers in the spaces provided in the exam paper.
- **Section B**: Answer all questions using the foolscap paper provided.
- The use of a scientific or graphical calculator is permitted in this paper.
- TI-Nspire calculators must be in Press-to-Test mode and cleared of all previous data.
- TI-84+ graphical calculators must only have permitted Apps and be ram-cleared.
- It is the responsibility of the student to ensure their calculator is examination ready.
- A clean copy of the **Mathematics HL Formulae Booklet** is required for this paper.
- Unless otherwise stated in the question, all numerical answers are to be given exactly or correct to three significant figures.
- The maximum mark for this examination paper is *[100 marks]*.
- This question paper consists of **12** printed pages including the Cover Sheet.
- Submit Sections A and B separately.

FOR MARKER USE ONLY:

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	TOTAL
											/100

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphical display calculator should be supported by suitable working; for example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are advised to show all working.

SECTION A (50 marks)

1 [Maximum mark: 5]

The probability distribution function of a discrete random variable X is defined by

$$\mathrm{P}\left(X = x\right) = Cx\left(8 - x\right) \text{ for } x = 1, 2, 3, 4$$

(a) Find the value of C. [3]

(b) Find $\mathrm{E}(X)$. [2]

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2 [Maximum mark: 7]

(a) If $z = \mathrm{i}(1+\mathrm{i})\left(-\sqrt{3}\mathrm{i}-1\right)$, find the modulus and argument of z, given that $-\pi < \arg z \le \pi$. [4]

(b) Given that $(x+\mathrm{i}y)^2 = b+\mathrm{i}$, where $x,\ y,\ b \in \mathbb{R}$, find the value of $x^2 - xy - y^2$ in terms of b. [3]

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3 [Maximum mark: 7]

Year 5 students were asked to sign up for a CAS project on a particular afternoon. Assume that the arrival of each student is independent and the number of students who signed up per 10-minute interval can be modelled by a Poisson distribution with mean 2.7 .

(a) Find the most likely number of students who signed up in the first half hour. [3]

(b) Given that there are exactly 12 students who signed up for the CAS project during the 2-hour sign-up period, find the probability that exactly 4 of them signed up during the first 20 minutes. [4]

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4 [Maximum mark: 8]

An arithmetic sequence has first term a and common difference d, where a and d are non-zero real numbers. The ninth, tenth and thirteenth terms of the arithmetic sequence form the first three terms of a geometric sequence.

(a) Show that $a = -\frac{15}{2}d$. [3]

(b) The sum of the first n terms of the arithmetic sequence is denoted by S_n.

Find the value of S_{16}. [2]

(c) Given that the k^{th} term of the arithmetic sequence is the fourth term of the geometric sequence, find the value of k. [3]

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5 [Maximum mark: 4]

A student working on a coding project studies 11-digit quaternary sequences. A quaternary sequence is a sequence formed using the digits 0, 1, 2 or 3.

Examples of quaternaries are *12030112210, 00231110312, 32103210321* and *00000000000.*

Find the number of ways that the 11-digit quaternary sequences can be formed with

(a) no restriction, [1]

(b) at least two consecutive digits the same. [3]

6 [Maximum mark: 6]

(a) Find the complex roots z_1 and z_2 of the equation $z^2+(7+\mathrm{i})z+24+7\mathrm{i}=0$. [3]

(b) It is given that $A_1A_2A_3A_4$ forms a square centred at the origin, where A_n represents, on an Argand diagram, the complex number z_n for $n=1,\ 2,\ 3,\ 4$.

Find the complex numbers z_3 and z_4. [3]

7 [Maximum mark: 8]

Lines L_1 and L_2 have vector equations

$$\mathbf{r}=\begin{pmatrix}9\\-2\\15\end{pmatrix}+\lambda\begin{pmatrix}-1\\1\\-2\end{pmatrix},\ \lambda\in\mathbb{R}\ \text{ and }\ \mathbf{r}=\begin{pmatrix}3\\2\\4\end{pmatrix}+\mu\begin{pmatrix}1\\1\\0\end{pmatrix},\ \mu\in\mathbb{R}\ \text{ respectively.}$$

(a) Justify that L_1 and L_2 are skew lines. [3]

(b) Find the perpendicular distance between the lines L_1 and L_2, giving your answer in exact form. [5]

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8 [Maximum mark: 5]

(a) Prove that $\frac{\sin(A-B)}{\cos A\cos B}=\tan A-\tan B$. [1]

(b) Hence find an expression for

$$\sum_{k=1}^{N}\frac{\sin x}{\cos\left[(k+1)x\right]\cos(kx)}$$

, in terms of N and x. [4]

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*Do **NOT** write solutions on this page.*

SECTION B (50 marks)

Answer all questions on the foolscap paper provided. Please start each question on a new page.

9 [Maximum mark: 15]

It is given that $g(t)=\dfrac{6}{3\mathrm{e}^{t}+\mathrm{e}^{-t}}$, where $t\in\mathbb{R}$.

(a) By using the substitution $u=\mathrm{e}^{t}$, show that $\int g(t)\ \mathrm{d}t=2m\ \arctan\left(m\mathrm{e}^{t}\right)+c$, where m is a real constant to be determined and c is the constant of integration. [5]

An object moves with velocity $v(t)=1-g(t),\ t\geq 0$, where t is the time in seconds after it passes through the origin.

(b) Find the time when the acceleration is the greatest. [2]

(c) Find an expression for the displacement in terms of t, giving your answer in exact form. [4]

(d) The object returns to the origin some time later. Find the total distance that the object has travelled when it returns to the origin. [4]

*Do **NOT** write solutions on this page.*

10 [Maximum mark: 14]

The continuous random variable X has probability density function given by

$$f(x)=\begin{cases} \dfrac{3k}{2}, & 1\le x\le 2 \\ \dfrac{k^2}{2}(x-5)^2, & 2<x\le 5 \\ 0, & \text{otherwise} \end{cases}$$, where k is a positive constant.

(a) Show that $k=\dfrac{1}{3}$. [3]

(b) Find $\mathrm{E}(X)$. [2]

(c) Find the interquartile range of X . [4]

(d) Find $\mathrm{P}(X<3)$. [2]

Ten independent observations of X are taken and the random variable Y is the number of observations such that $X<3$.

(e) Find $\mathrm{P}(Y<3)$. [3]

*Do **NOT** write solutions on this page.*

11 [Maximum mark: 21]

(a) Let $z = \cos\theta + \mathrm{i}\sin\theta$.

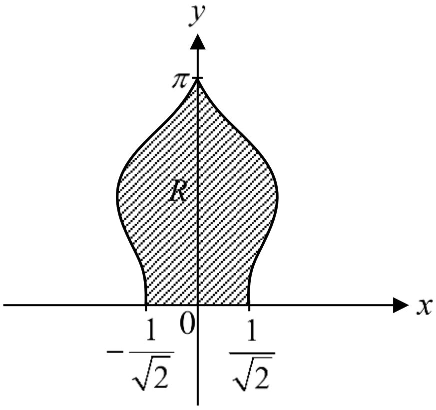
(i) Show that $\left(z^n - \dfrac{1}{z^n}\right) = 2\mathrm{i}\sin n\theta$, where $n \in \mathbb{Z}$.

(ii) Use the binomial theorem to expand $\left(z - \dfrac{1}{z}\right)^3$, giving your answer in terms of z.

(iii) Hence show that $4\sin^3\theta = 3\sin\theta - \sin 3\theta$. [6]

The following diagram shows part of the graph of $4x^2 = 4\sin^3 y + \cos y + 1$ for $0 \le y \le \pi$.

The graph cuts the x-axis at $\left(-\dfrac{1}{\sqrt{2}}, 0\right)$ and $\left(\dfrac{1}{\sqrt{2}}, 0\right)$, and the y-axis at the point $(0, \pi)$.



(b) **(i)** Find an expression for $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y.

(ii) Show that the gradient of the curve at the point $\left(-\dfrac{\sqrt{5}}{2}, \dfrac{\pi}{2}\right)$ is $4\sqrt{5}$. [5]

The shaded region R is the area bounded by the curve and the x-axis.

(c) Find the area of R. [4]

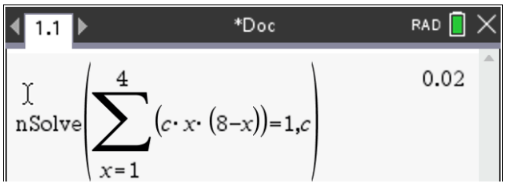
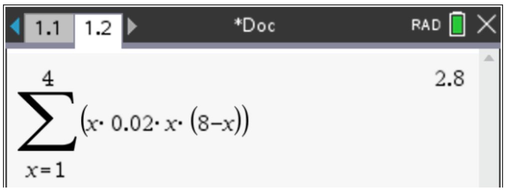
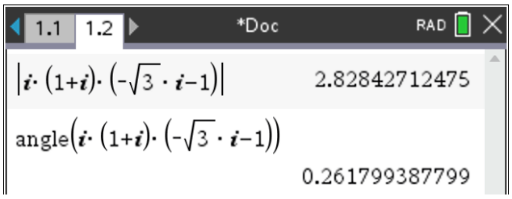
The region R is now rotated about the y-axis, through π radians, to form a solid.

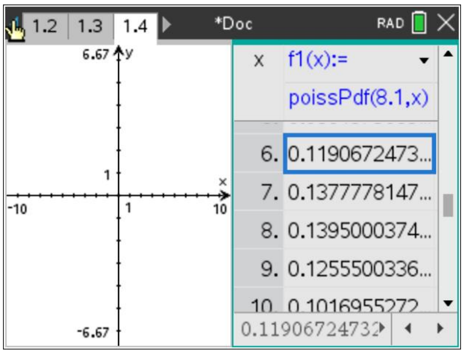
(d) Find the volume of the solid formed in exact form. [6]

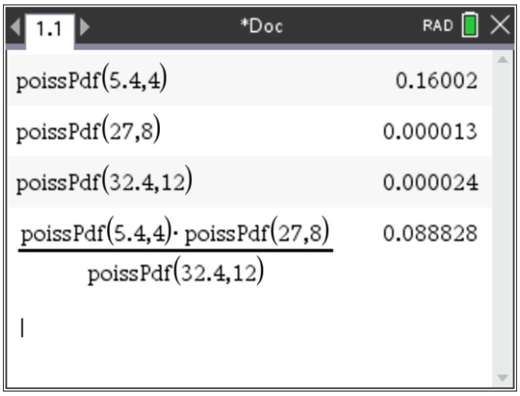
End of Paper

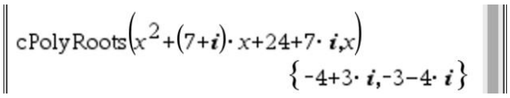
Year 6 HL Math Preliminary Examination 2020 Paper 2 (Mark Scheme)

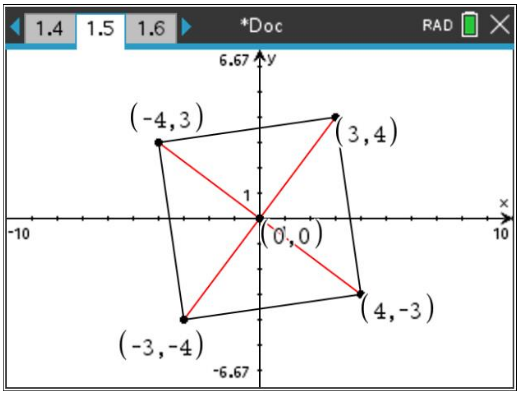
Section A

Qn	Suggested Solutions	Marks
1	**Discrete Random Variables**	**[Max mark: 5]**
(a)	1.1 *Doc RAD $\text{nSolve}\left(\sum_{x=1}^{4}(c\cdot x\cdot(8-x))=1,c\right)$ 0.02 $\sum_{\text{all } x} \mathrm{P}(X=x)=1$ $\sum_{x=1}^{4} Cx(8-x)=1$ Using GDC, $C=0.02$	**M1** **(M1)** **A1**
(b)	**Method 1 (by GDC)** 1.1 1.2 *Doc RAD $\sum_{x=1}^{4}(x\cdot 0.02\cdot x\cdot(8-x))$ 2.8 $\mathrm{E}(X)=\sum_{\text{all } x} x\cdot \mathrm{P}(X=x)$ $=\sum_{x=1}^{4} x\cdot 0.02x(8-x)$ $=2.8$ (by GDC)	**GDC Method** **M1** **A1**
	Method 2 (Analytic) $\mathrm{E}(X)=\sum_{\text{all } x} x\cdot \mathrm{P}(X=x)$ $=\sum_{x=1}^{4} x\cdot 0.02x(8-x)$ $=0.02\sum_{x=1}^{4} x^2(8-x)$ $=0.02\left[1(7)+4(6)+9(5)+16(4)\right]$ $=2.8$	**Analytic** **M1** **A1**
2	**Complex Numbers – Cartesian Algebra**	**[Max mark: 7]**
(a)	**Method 1 (by GDC)** 1.1 1.2 *Doc RAD $\lvert i\cdot(1+i)\cdot(-\sqrt{3}\cdot i-1)\rvert$ 2.82842712475 $\text{angle}(i\cdot(1+i)\cdot(-\sqrt{3}\cdot i-1))$ 0.261799387799 $\lvert z\rvert=2.83$ (to 3 s.f.) $\arg z=0.262$ (to 3 s.f.)	**Method 1** **(M1) A1** **(M1) A1**

Qn	Suggested Solutions	Marks
	<u>Method 2 (Analytic)</u>	**<u>Method 2</u>**
	$\|z\| = \left\|\mathrm{i}(1+\mathrm{i})\left(-\sqrt{3}\mathrm{i}-1\right)\right\|$	
	$= \|\mathrm{i}\|\|1+\mathrm{i}\|\left\|-\sqrt{3}\mathrm{i}-1\right\|$	**M1**
	$= 1\cdot\sqrt{2}\cdot 2 = 2\sqrt{2}$	**A1**
	$\arg z = \arg \mathrm{i} + \arg(1+\mathrm{i}) + \arg\left(-\sqrt{3}\mathrm{i}-1\right)$	
	$= \frac{\pi}{2} + \frac{\pi}{4} - \frac{2\pi}{3}$	**M1**
	$= \frac{\pi}{12}$	**A1**
(b)	$(x+\mathrm{i}y)^2 = x^2 + 2xy\mathrm{i} - y^2$	**M1**
	Comparing real and imaginary components, $x^2 - y^2 = b$ and $xy = \frac{1}{2}$	**M1**
	$\therefore x^2 - xy - y^2 = b - \frac{1}{2}$	**A1**
3	**Poisson Distribution**	**[Max mark: 7]**
(a)	Let X be the r.v. 'the number of signups in an half-hour interval'. $X \sim \mathrm{Po}(8.1)$	**M1**
	 x: f1(x):= poissPdf(8.1,x) 6. 0.1190672473... 7. 0.1377778147... 8. 0.1395000374... 9. 0.1255500336... 10. 0.1016955272 0.11906724732 **<u>Method 1</u>** Using GDC, $\mathrm{P}(X=7)=0.138$ $\mathrm{P}(X=8)=0.140$ $\mathrm{P}(X=9)=0.126$	**M1 (Method 1)**
	<u>Method 2</u> Reasoning that mode of Poisson distribution is the largest integer $< \lambda$ when $\lambda \notin \mathbb{Z}$. When $\lambda \in \mathbb{Z}$, the distribution is bi-modal, and mode $= \lambda$ and $\lambda - 1$.	**M1 (Method 2)**
	That is, mode = $\lfloor 8.1 \rfloor$. Mode $= 8$	**A1**

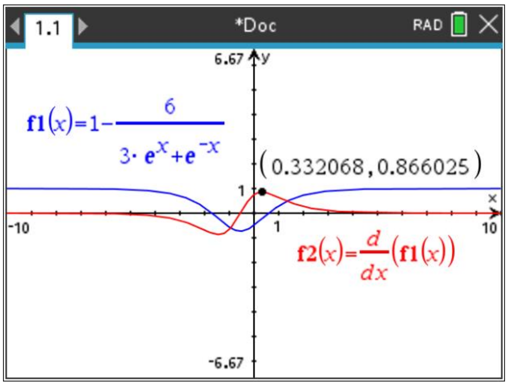
Qn	**Suggested Solutions**	**Marks**
(b)	Let V, W and Y be the number of signups in the first 20 minutes, next 100 minutes and during the 2-hour period (120 min) respectively. $V \sim \text{Po}(5.4)$, $W \sim \text{Po}(27)$, $Y \sim \text{Po}(32.4)$	**M1**
	 1.1 *Doc RAD poissPdf(5.4,4) 0.16002 poissPdf(27,8) 0.000013 poissPdf(32.4,12) 0.000024 $\frac{\text{poissPdf}(5.4,4)\cdot \text{poissPdf}(27,8)}{\text{poissPdf}(32.4,12)}$ 0.088828 $\text{P}(V=4\mid Y=12)$ $=\dfrac{\text{P}(V=4\cap Y=12)}{P(Y=12)}$ $=\dfrac{\text{P}(V=4\cap W=8)}{P(Y=12)}$ $=\dfrac{\text{P}(V=4)\times \text{P}(W=8)}{\text{P}(Y=12)}$ $=\dfrac{0.1600..\times 0.0000131..}{0.00002371..}$ $=0.0888$ (to 3 s.f.)	**M1** **(M1)** **A1**
4	**AP GP**	**[Max mark: 8]**
(a)	Using common ratio, $r=\dfrac{a+9d}{a+8d}=\dfrac{a+12d}{a+9d}$ $(a+9d)^2=(a+8d)(a+12d)$ $a^2+18ad+81d^2=a^2+20ad+96d^2$ $2ad+15d^2=0$ $2a+15d=0 \quad (\because d\neq 0)$ $a=-\dfrac{15}{2}d$	**M1** **M1** **A1**
(b)	$S_{16}=\dfrac{16}{2}(2a+15d)$ $=\dfrac{16}{2}(0) \quad (\because 2a+15d=0)$ $=0$	**M1 o.e.** **A1**
(c)	$r=\dfrac{a+9d}{a+8d}=\dfrac{a+(k-1)d}{a+12d}$	**M1 o.e.**

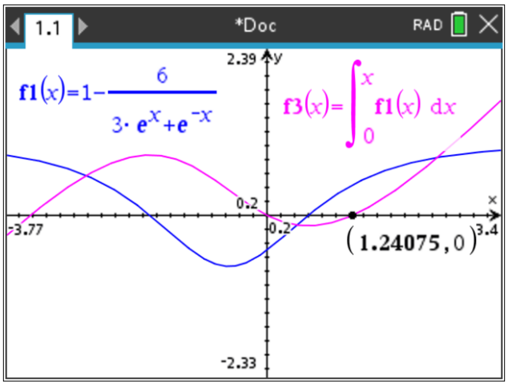
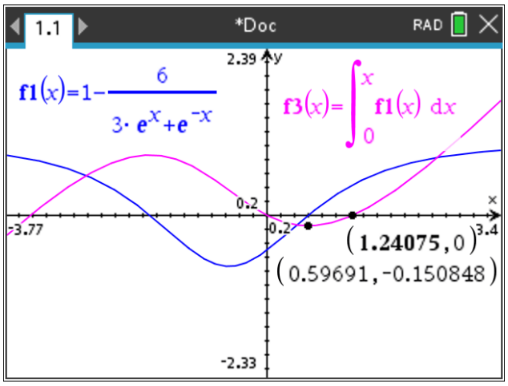
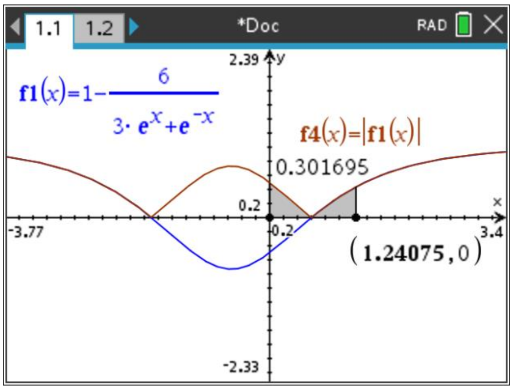
Qn	Suggested Solutions	Marks
	$\frac{2a+18d}{2a+16d}=\frac{2a+(2k-2)d}{2a+24d}$ $\frac{(2a+15d)+3d}{(2a+15d)+d}=\frac{(2a+15d)+(2k-17)d}{(2a+15d)+9d}$ $\frac{3d}{d}=\frac{(2k-17)d}{9d} \quad (\because 2a+15d=0)$ $\frac{(2k-17)}{9}=3$ Solving, $k=22$	**A1** (r = 3) **A1**
5	**Permutations & Combinations**	**[Max mark: 4]**
(a)	No. of ways = 4^{11} = 4194304	**A1**
(b)	No. of ways = Total without restriction – No consecutive digits are the same = 4 194 304 − 4 × 3^{10} = 3 958 108	**M1 M1 -** 4 × 3^{10} **A1**
6	**Complex Numbers – Argand Diagram**	**[Max mark: 6]**
(a)	 cPolyRoots$(x^2+(7+i)\cdot x+24+7\cdot i,x)$ $\{-4+3\cdot i,-3-4\cdot i\}$ By GDC (roots of polynomial), the two roots are $-4+3i$ and $-3-4i$	**(M1)** **A1 A1**
(b)	<u>**Method 1 (Using rotation in complex numbers)**</u> $z_3=\mathrm{i}(-3-4\mathrm{i})$ $=4-3\mathrm{i}$ $z_4=-\mathrm{i}(-4+3\mathrm{i})$ $=3+4\mathrm{i}$ *Note that the answers can be swapped.	<u>**Method 1**</u> **M1** **A1** **A1**

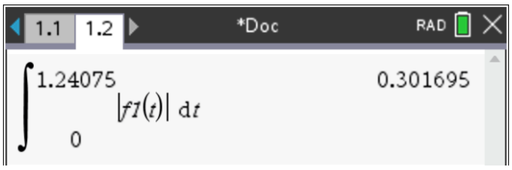
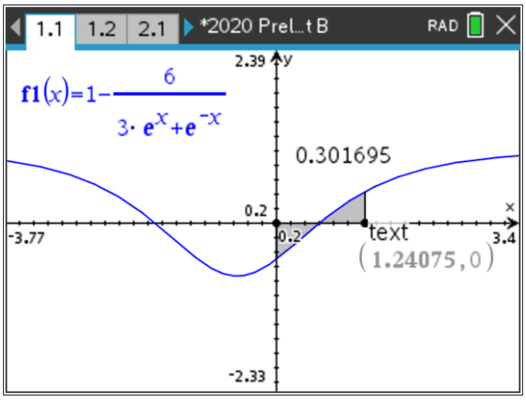
Qn	Suggested Solutions	Marks
	Method 2 (Geometry on Argand)  By any geometrical method of rotating about the origin, The other two complex numbers z_3 and z_4 are $4-3i$ and $3+4i$.	**Method 2** **(M1)** **A1 A1**
7	**Vectors**	**[Max mark: 8]**
(a)	L_1 and L_2 are not parallel since $\begin{pmatrix}-1\\1\\-2\end{pmatrix}$ is not parallel to $\begin{pmatrix}1\\1\\0\end{pmatrix}$. Use GDC to check if the two lines have any solutions (3 equations, 2 unknowns) Let $\begin{pmatrix}9\\-2\\15\end{pmatrix}+\lambda\begin{pmatrix}-1\\1\\-2\end{pmatrix}=\begin{pmatrix}3\\2\\4\end{pmatrix}+\mu\begin{pmatrix}1\\1\\0\end{pmatrix}$ That is, $9-\lambda=3+\mu$ $-2+\lambda=2+\mu$ $15-2\lambda=4$ Using GDC, no solution found. Hence, the two lines are non-parallel and non-intersecting, and are therefore skew.	**R1** **M1 o.e.** **A1** **AG**
(b)	$n=\begin{pmatrix}-1\\1\\-2\end{pmatrix}\times\begin{pmatrix}-1\\1\\0\end{pmatrix}=2\begin{pmatrix}-1\\1\\1\end{pmatrix}$	**M1 A1**

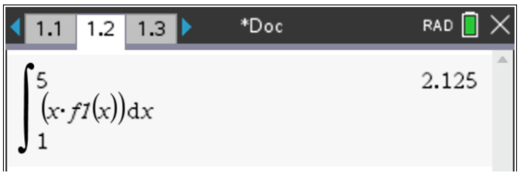
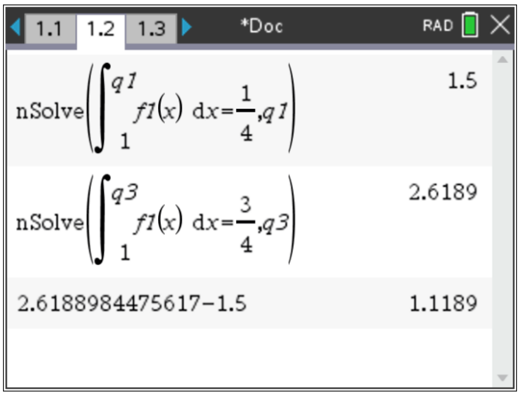
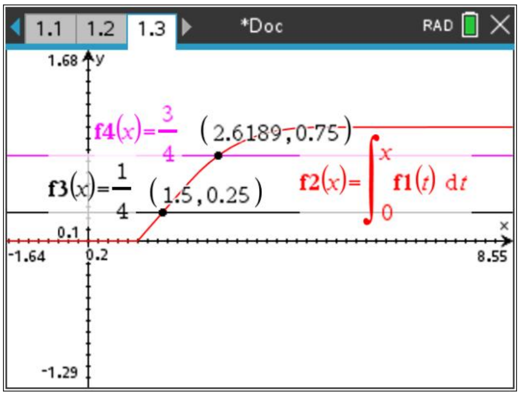
Qn	Suggested Solutions	Marks
	Let $\overrightarrow{OA}=\begin{pmatrix}9\\-2\\15\end{pmatrix}$ and $\overrightarrow{OB}=\begin{pmatrix}3\\2\\4\end{pmatrix}$, then $\overrightarrow{AB}=\begin{pmatrix}-6\\4\\-11\end{pmatrix}$	**M1 (for $\overrightarrow{AB}$)**
	Perpendicular distance $=\dfrac{\left\lvert\begin{pmatrix}-6\\4\\-11\end{pmatrix}\cdot\begin{pmatrix}-1\\1\\1\end{pmatrix}\right\rvert}{\sqrt{3}}$	**M1 ft**
	$=\dfrac{1}{\sqrt{3}}$	**A1**
8	**Trigonometry & Sigma Notation**	**[Max mark: 5]**
(a)	$\dfrac{\sin(A-B)}{\cos A\cos B}$	
	$=\dfrac{\sin A\cos B-\cos A\sin B}{\cos A\cos B}$	**M1**
	$=\tan A-\tan B$ (shown)	**AG**
(b)	$\displaystyle\sum_{k=1}^{N}\frac{\sin x}{\cos\left[(k+1)x\right]\cos(kx)}$	
	$\displaystyle=\sum_{k=1}^{N}\frac{\sin\left[(k+1)x-kx\right]}{\cos\left[(k+1)x\right]\cos(kx)}$	**(M1)**
	$\displaystyle=\sum_{k=1}^{N}\left\{\tan\left[(k+1)x\right]-\tan(kx)\right\}$	**A1**
	$=\left\{\tan(2x)-\tan(x)\right\}+\left\{\tan(3x)-\tan(2x)\right\}+\ldots$ $+\left\{\tan\left[(N+1)x\right]-\tan(Nx)\right\}$	**(M1)**
	$=\tan\left[(N+1)x\right]-\tan(x)$	**A1**

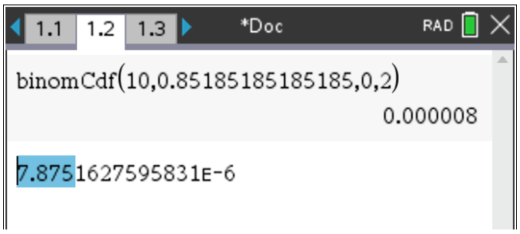
Section B

Qn	Suggested Solutions	Marks
9	**Calculus – Integration by Substitution & Kinematics**	**[Max mark: 15]**
(a)	Let $u=\mathrm{e}^t$, then $\frac{\mathrm{d}u}{\mathrm{d}t}=\mathrm{e}^t$.	**M1**
	That is, $\frac{\mathrm{d}u}{\mathrm{d}t}=u$.	
	$\therefore \int \frac{6}{3\mathrm{e}^t+\mathrm{e}^{-t}}\,\mathrm{d}t=\int \frac{6}{3u+u^{-1}}\cdot\frac{1}{u}\,\mathrm{d}u$	**M1**
	$=\int \frac{6}{3u^2+1}\,\mathrm{d}u\left(=\int \frac{2}{u^2+\left(\frac{1}{\sqrt{3}}\right)^2}\,\mathrm{d}u\right)$	**A1**
	$=2\cdot\sqrt{3}\tan^{-1}\left(\sqrt{3}u\right)+c$ $\left(\text{Use of } \int \frac{1}{a^2+x^2}\,\mathrm{d}x=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C\right)$	**M1**
	$=2\sqrt{3}\tan^{-1}\left(\sqrt{3}\mathrm{e}^t\right)+c,$ where c is an arbitrary constant.	**A1**
(b)	Realising the need to find (1) max pt on $a=\frac{\mathrm{d}}{\mathrm{d}t}(v)$ or (2) zero on $\frac{\mathrm{d}a}{\mathrm{d}t}=\frac{\mathrm{d}^2}{\mathrm{d}t^2}(v)$.	
		
	E.g. Using GDC, sketch the graph of $\frac{\mathrm{d}v}{\mathrm{d}t}$ for max.	**M1**
	Max acceleration occurs when $t=0.332068$ $=0.332$ (to 3 sf)	**A1**
(c)	$s=\int v\ \mathrm{d}t$	
	$=\int 1-\frac{6}{3\mathrm{e}^t+\mathrm{e}^{-t}}\,\mathrm{d}t$	
	Using (a), we have $s=t-2\sqrt{3}\tan^{-1}\left(\sqrt{3}\mathrm{e}^t\right)+c'$	**M1 (ft. *m*) A1**
	Given $s=0$ when $t=0$, we have	

Qn	Suggested Solutions	Marks
	$c' = 2\sqrt{3}\tan^{-1}\left(\sqrt{3}e^{0}\right)$ $= 2\sqrt{3}\left(\frac{\pi}{3}\right) = \frac{2\sqrt{3}\pi}{3}$ Hence, $s = t - 2\sqrt{3}\tan^{-1}\left(\sqrt{3}e^{t}\right) + \frac{2\sqrt{3}\pi}{3}$	**M1** **A1 (ft. *m*)**
(d)	Using GDC, either plot the graph of $s(t)$ or use nsolve, based on the expression in **(c)** or the integral graph of $v(t)$.	
	 From the graph, the object returns to origin when $t = 1.24075...$ or turns around when $t = 0.59691...$ Next, to find the total distance travelled, using GDC:	**M1 (any valid method)** **A1 (ft. from (c))**
	<u>Method 1</u> Find the further distance reached, i.e. (1) min. pt of $s(t)$, or (2) when $v(t) = 0$. Total distance travelled $= 2\times\lvert -0.150848\rvert$ $= 0.3016..$ $= 0.302$ (to 3 s.f.) 	**<u>Method 1</u>** **M1** **A1**
	<u>Method 2</u> Plot the graph of $\lvert v(t)\rvert$. Total distance travelled $= \int_0^{1.24075}\lvert v(t)\rvert \mathrm{d}t$ $= 0.3016..$ $= 0.302$ (to 3 s.f.) 	**<u>Method 2</u>** **M1** **A1**

Qn	Suggested Solutions	Marks
	Method 3	**Method 3**
	Using definite integral: Total distance travelled $=\int_0^{1.24075}\lvert v(t)\rvert \mathrm{d}t$  $\int_0^{1.24075} \lvert f1(t)\rvert \, dt$ 0.301695	**M1**
	$=0.3016..$ $=0.302$ (to 3 s.f.)	**A1**
	Method 4 	**Method 3**
	Using bounded area on the graph of $v(t)$ and the x-axis.	**M1**
	Total distance travelled $=0.3016..$ $=0.302$ (to 3 s.f.)	**A1**
10	**Continuous Random Variable**	**[Max mark: 14]**
(a)	Since X is a random variable, $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x=1$. Therefore,	
	$\int_1^2 \frac{3k}{2}\mathrm{d}x+\int_2^5 \frac{k^2}{2}(x-5)^2\,\mathrm{d}x=1$	**M1**
	$\left[\frac{3kx}{2}\right]_1^2+\left[\frac{k^2}{2}\cdot\frac{(x-5)^3}{3}\right]_2^5=1$ $\frac{3k}{2}+\frac{9k^2}{2}=1$	**M1 (steps requires as this is a 'show' question)**
	$9k^2+3k-2=0$ $(3k-1)(3k+2)=0$	**A1 (quadratic equation)**
	$k=\frac{1}{3}$ or $k=-\frac{2}{3}$ (rej, since $k>0$)	
	$\therefore k=\frac{1}{3}$	**AG**

Qn	Suggested Solutions	Marks
(b)	 $\int_{1}^{5}(x\cdot f1(x))dx$ 2.125 $E(X)=\int_{-\infty}^{\infty} x\,f(x)\,dx$ $=\int_{1}^{5} x\,f(x)\,dx$ $=2.125$ (exact)	 **M1 (any mtd)** **A1**
(c)	**Method 1**  $\text{nSolve}\left(\int_{1}^{q1} f1(x)\,dx=\frac{1}{4},q1\right)$ 1.5 $\text{nSolve}\left(\int_{1}^{q3} f1(x)\,dx=\frac{3}{4},q3\right)$ 2.6189 2.6188984475617−1.5 1.1189 Using GDC (nsolve), obtain the lower and upper quartile. $IQR=Q3-Q1$ $=2.6189-1.5$ $=1.1189...$ $=1.12$ (to 3 s.f.)	**Method 1** **M1 (any valid method, incl. analytical)** **A1 A1** **A1**
	Method 2  Using GDC, plot the cumulative distribution function, $F(t)=\int_{1}^{t} f(x)\,dx$. $F(Q1)=0.25$ $\Rightarrow Q1=1.5$ $F(Q3)=0.75$ $\Rightarrow Q3=2.6189$ $IQR=Q3-Q1$ $=1.12$ (to 3 s.f.)	**Method 2** **M1** **A1** **A1** **A1**
(d)	$\mathrm{P}(X<3)=\int_{1}^{3} f(x)\,dx$ $=0.851852...=0.852$ (to 3 s.f.)	**M1** **A1**

Qn	**Suggested Solutions**	**Marks**
(e)	 binomCdf(10,0.85185185185185,0,2) 0.000008 7.8751627595831E-6 $Y \sim \mathrm{B}(10, 0.851852...)$ $\mathrm{P}(Y < 3) = \mathrm{P}(Y \leq 2)$ $= 0.000007875...$ $= 0.00000788$ (to 3 s.f.) $\left(= 7.88 \times 10^{-6}\right)$	**M1** **M1** **A1**
11	**Complex – Trigo, Calculus Techniques & Applications**	**[Max mark: 21]**
(a)	**(i)** Given $z = \cos\theta + \mathrm{i}\sin\theta$ By De Moivre's Theorem, $z^n = \cos n\theta + \mathrm{i}\sin n\theta$ Similarly, $\frac{1}{z^n} = z^{-n} = \cos(-n\theta) + \mathrm{i}\sin(-n\theta)$ $\therefore \frac{1}{z^n} = \cos n\theta - \mathrm{i}\sin n\theta$ That is, $\left(z^n - \frac{1}{z^n}\right) = 2\mathrm{i}\sin n\theta$ (shown) **(ii)** $\left(z - \frac{1}{z}\right)^3 = z^3 - 3z^2\left(\frac{1}{z}\right) + 3z\left(\frac{1}{z}\right)^2 - \left(\frac{1}{z}\right)^3$ $= z^3 - 3z + \frac{3}{z} - \frac{1}{z^3}$ **(iii)** Using parts **(i)** and **(ii)**, $(2\mathrm{i}\sin\theta)^3 = \left(z^3 - \frac{1}{z^3}\right) - 3\left(z - \frac{1}{z}\right)$ $-8\mathrm{i}\sin^3\theta = 2\mathrm{i}\sin 3\theta - 3(2\mathrm{i}\sin\theta)$ $4\sin^3\theta = 3\sin\theta - \sin 3\theta$	 **M1** **M1** **AG** **M1** **A1** **M1** **A1** **AG**
(b)	**(i)** $4x^2 = 4\sin^3 y + \cos y + 1$ Differentiate implicitly w.r.t. x, $8x = 12\sin^2 y \cdot \cos y \cdot \frac{\mathrm{d}y}{\mathrm{d}x} - \sin y \cdot \frac{\mathrm{d}y}{\mathrm{d}x}$ $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{8x}{12\sin^2 y\cos y - \sin y}$ $\left(= \frac{8x}{\sin y(12\sin y\cos y - 1)}\right)$	 **M1 M1 M1** **A1**

Qn	Suggested Solutions	Marks
	Alt. $\frac{\mathrm{d}}{\mathrm{d}x}\left(4x^2\right)=\frac{\mathrm{d}}{\mathrm{d}x}\left(3\sin y-\sin 3y+\cos y+1\right)$ (applying (a))	
	Then,	
	$8x=3\cos y\cdot\frac{\mathrm{d}y}{\mathrm{d}x}-3\cos 3y\cdot\frac{\mathrm{d}y}{\mathrm{d}x}-\sin y\cdot\frac{\mathrm{d}y}{\mathrm{d}x}$	**M1 M1 M1**
	$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{8x}{3\cos y-3\cos 3y-\sin y}$	**A1**
	(ii) At $\left(-\frac{\sqrt{5}}{2},\frac{\pi}{2}\right)$, $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{8\left(-\frac{\sqrt{5}}{2}\right)}{12(1)(0)-(1)}=4\sqrt{5}$	**M1 AG**
(c)	Shaded area	
	$=2\int_0^{\pi}x\,\mathrm{d}y$	
	$=2\int_0^{\pi}\frac{1}{2}\sqrt{4\sin^3 y+\cos y+1}\,\mathrm{d}y$	**M1 A1**
	$=4.76201...=4.76$ (to 3 s.f.)	**(M1) A1**
(d)	**Method 1:** (Using (a)'s identity)	**Method 1**
	Volume of revolution	
	$=\pi\int_0^{\pi}x^2\,\mathrm{d}y=\pi\int_0^{\pi}\frac{1}{4}\left(4\sin^3 y+\cos y+1\right)\mathrm{d}y$	**M1 A1**
	$=\frac{\pi}{4}\int_0^{\pi}\left(\left(3\sin y-\sin 3y\right)+\cos y+1\right)\mathrm{d}y$	**M1 (Identity)**
	$=\frac{\pi}{4}\left[-3\cos y+\frac{1}{3}\cos 3y+\sin y+y\right]_0^{\pi}$	**M1**
	$=\frac{\pi}{4}\left\{\left[3-\frac{1}{3}+0+\pi\right]-\left[-3+\frac{1}{3}\right]\right\}$	**A1**
	$=\frac{\pi}{4}\left\{\frac{16}{3}+\pi\right\}\ \left(=\pi\left(\frac{4}{3}+\frac{\pi}{4}\right)=\frac{\pi\left(3\pi+16\right)}{12}\right)$	**A1**
	Method 2: (Direct)	**Method 1**
	Volume of revolution	
	$=\pi\int_0^{\pi}x^2\,\mathrm{d}y=\pi\int_0^{\pi}\frac{1}{4}\left(4\sin^3 y+\cos y+1\right)\mathrm{d}y$	**M1 A1**
	$=\frac{\pi}{4}\int_0^{\pi}\left(\sin y\left(1-\cos^2 y\right)+\cos y+1\right)\mathrm{d}y$	**M1 (Pythogorean Identity)**

Qn	Suggested Solutions	Marks
	$=\frac{\pi}{4}\left\{\int_0^{\pi}\sin y\ \mathrm{d}y-\int_0^{\pi}\sin y\left(\cos^2 y\right)\mathrm{d}y+\int_0^{\pi}\cos y\ \mathrm{d}y+\int_0^{\pi}1\mathrm{d}y\right\}$	
	$=\frac{\pi}{4}\left\{\left[-\cos y\right]_0^{\pi}-\left[-\frac{1}{3}\cos^3 y\right]_0^{\pi}+\left[\sin y\right]_0^{\pi}+\left[y\right]_0^{\pi}\right\}$	**M1**
	$=\frac{\pi}{4}\left\{-\left[\cos y\right]_0^{\pi}+\left[\frac{1}{3}\cos^3 y\right]_0^{\pi}+\left[\sin y\right]_0^{\pi}+\left[y\right]_0^{\pi}\right\}$	**A1**
	$=\frac{\pi}{4}\left\{-\left[-1-1\right]+\left[-\frac{1}{3}-\frac{1}{3}\right]+0+\pi\right\}$	
	$=\pi\left(\frac{4}{3}+\frac{\pi}{4}\right)$	**A1**

STUDENT NAME: ______________________________

TEACHER NAME: ______________________________

ST JOSEPH’S INSTITUTION
YEAR 6 PRELIMINARY EXAMINATION 2021

MATHEMATICS: ANALYSIS AND APPROACHES
HIGHER LEVEL
PAPER 1
Tuesday

29 June 2021
2 hours
0800 – 1000 hrs

INSTRUCTIONS TO CANDIDATES

- Write your name and your teacher’s name in the spaces provided.
- Do not open this examination paper until instructed to do so.
- **Section A**: Answer all questions showing working and answers in the spaces provided in the exam paper.
- **Section B**: Answer all questions using the writing paper provided.
- The use of calculators is **not** permitted in this paper.
- A clean copy of the **Mathematics: Analysis and Approaches formula booklet** is required for this paper.
- Unless otherwise stated in the question, all numerical answers are to be given exactly or correct to three significant figures.
- The maximum mark for this examination paper is *[**110** marks]*.
- This question paper consists of **12** printed pages including the Cover Sheet.
- Sections A and B are to be submitted **separately**.

FOR MARKER USE ONLY:

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	TOTAL
											/110

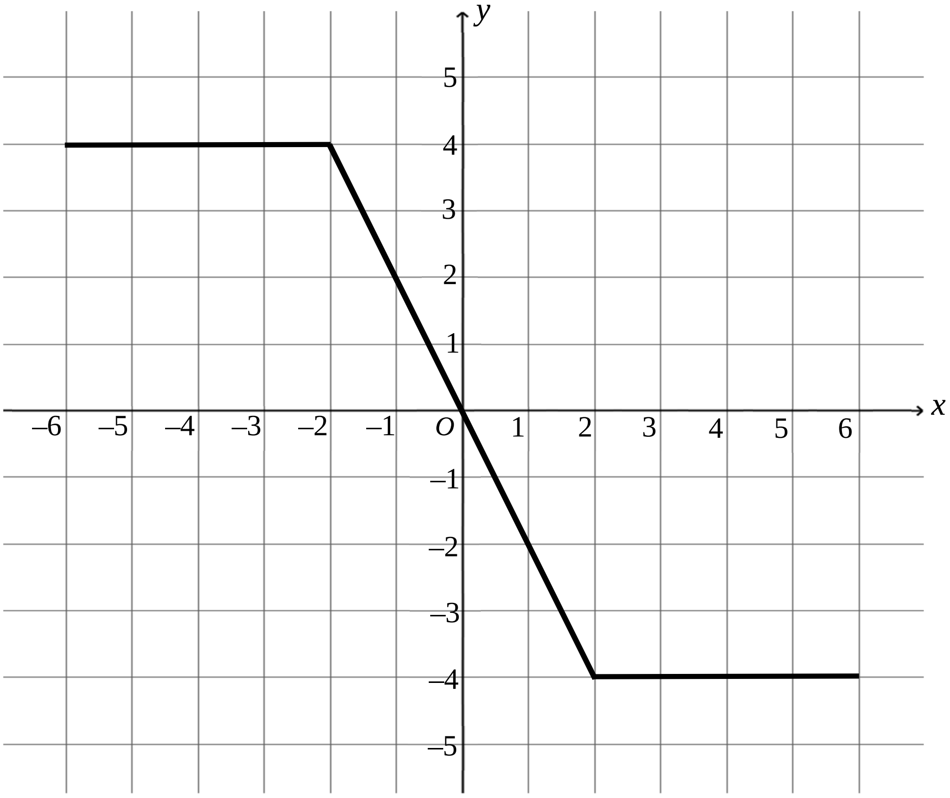
Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are advised to show all working.

SECTION A (55 marks)

Answer **all** questions in the spaces provided.

1 **[Maximum mark: 5]**

The graph of $f(x)=|x-2|-|x+2|$, $-6 \le x \le 6$, is given below.



Find the value of

(a) $f'(1.5)$, [2]

(b) $f''(-1)$, [1]

(c) $\int_{-1}^{3} f(x)\,\mathrm{d}x$. [2]

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2 **[Maximum mark: 6]**

Find the values of *x*, where $0 \leq x \leq \pi$, for which the vectors $\begin{pmatrix} \frac{1}{2} \\ \cos x \\ -1 \end{pmatrix}$ and $\begin{pmatrix} \sqrt{3} \\ 2\cos x \\ 1 \end{pmatrix}$ are perpendicular, leaving your answers in terms of π.

TURN OVER

3 **[Maximum mark: 8]**

The function f is defined by $f\left(x\right)=\dfrac{\mathrm{e}^{x}-\mathrm{e}^{-x}}{2}$ for $x\in\mathbb{R}$.

(a) Find the Maclaurin expansion of f up to and including the term in x^5. [5]

(b) Hence, find the value of $\lim\limits_{x\to0}\dfrac{x-f\left(x\right)}{x^3}$. [3]

4 **[Maximum mark: 9]**

(a) Find the roots of the equation $27w^3 + 125 = 0,\ w \in \mathbb{C}$, giving your answers in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$. [6]

(b) The roots found in **(a)** are represented by the vertices of a triangle in an Argand diagram. Find the area of the triangle. [3]

TURN OVER

5 [Maximum mark: 6]

(a) Show that the sum of the geometric series $1-x^2+x^4+\ldots+(-1)^{n-1}x^{2(n-1)}$ can be expressed as $\sum_{k=0}^{n-1}(-1)^k x^{2k}=\frac{1}{1+x^2}+\frac{(-1)^{n-1}x^{2n}}{1+x^2}$. [2]

(b) Hence, by using integrals, find the value of $\sum_{k=0}^{n-1}\left[\frac{(-1)^k}{2k+1}\right]-(-1)^{n-1}\int_0^1\frac{x^{2n}}{1+x^2}\,\mathrm{d}x$. [4]

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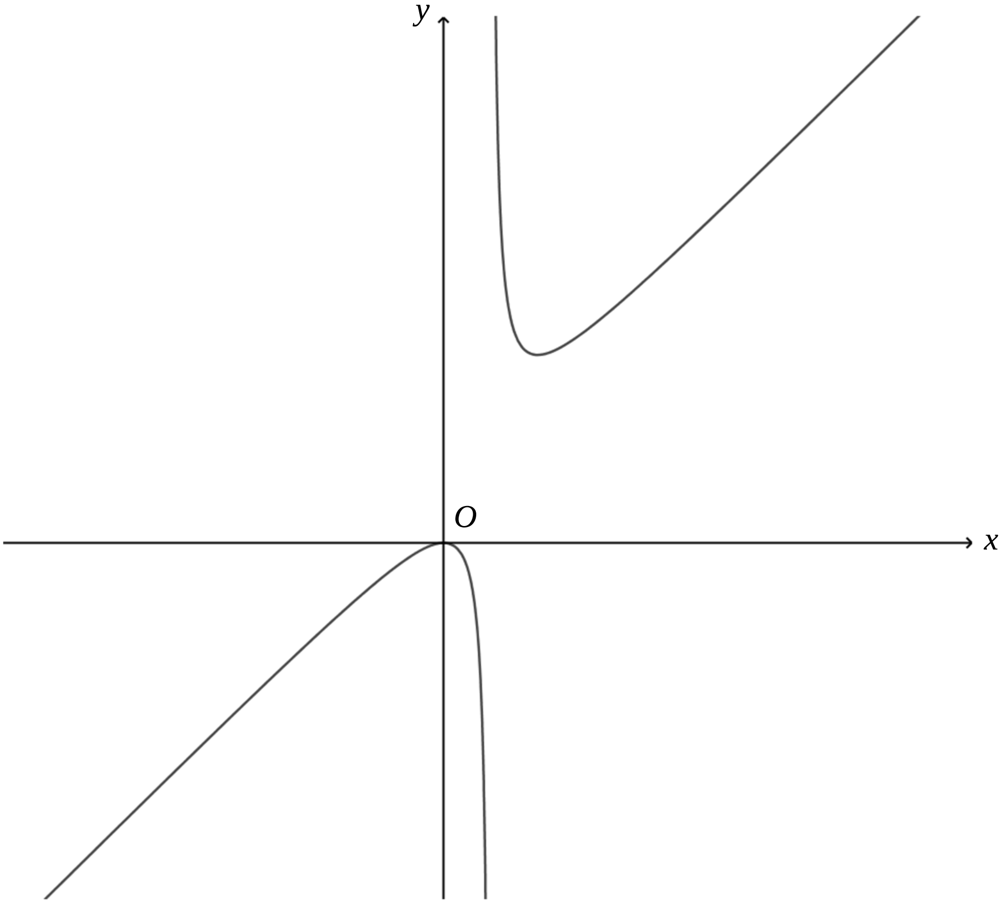
6 **[Maximum mark: 12]**

Let the function g be defined by $g\left(x\right)=\frac{x^2}{x-3}$ on its maximal domain. The graph of $y=g\left(x\right)$ is given below.



(a) Find the asymptotes of $y=g\left(x\right)$. Draw and label them on the graph given above. [4]

(b) Describe the sequence of transformations that map the graph of $y=g\left(x\right)$ to the graph of $y=3x+5+\frac{3}{x-1}$. [4]

(c) State the number of zero(s) of $y=3x+5+\frac{3}{x-1}$ and write down the equation of the new asymptotes. Hence, justify that the zero(s) lie in the interval $\left]-\frac{5}{3},1\right[$. [4]

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7 **[Maximum mark: 9]**

The function f is defined by $f(x) = x^4 - 2\cos x$, where $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.

The function h is defined by $h(x) = (f \circ g)(x)$ where $g(x)$ is a continuous and differentiable function with the following properties:

- g is increasing for all values of x,
- g has a stationary point of inflexion at $x = 0$,
- g is concave upward for $x > 0$, and concave downward for $x < 0$,
- $g(x) > 0$ for $x > 0$, and $g(x) < 0$ for $x < 0$.

(a) Show that $h''(0) = 0$. [4]

(b) By completing the table below with the positive '+' and negative '–' signs for the respective derivative values, explain whether h has a point of inflexion at $x = 0$. [5]

x	$f'(x)$	$f''(x)$	$g(x)$	$g'(x)$	$g''(x)$
0^-			–	+	–
0^+			+	+	+

TURN OVER

*Do **NOT** write solutions on this page.*

SECTION B (55 marks)

*Answer **all** questions on the writing paper provided.* ***Please start each question on a new page.***

8 [Maximum mark: 10]

The graph of $y = f(x)$ for $-6 \leq x \leq 6$ is shown below, where the graph passes through the points $(-6,-5)$, $(-3,-4)$, $(0,0)$, $(3,3)$ and $(6,4)$.

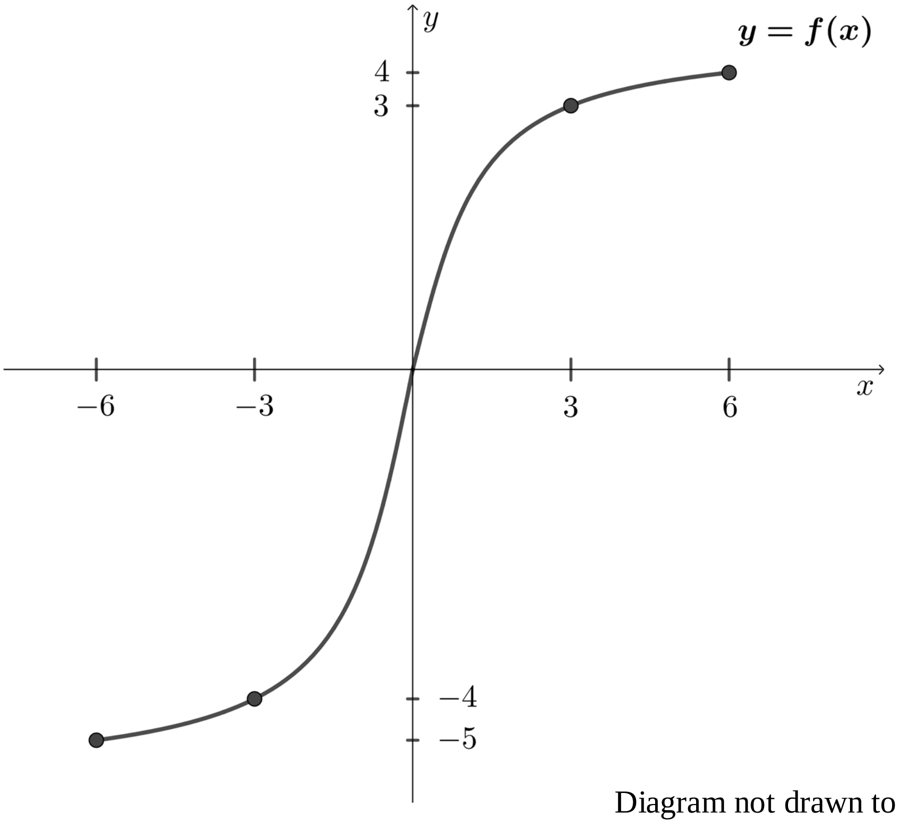


Diagram not drawn to scale

On different sets of axes, sketch the graph of the following. Indicate any axial intercepts and asymptotes. Also, label the endpoints of each resulting graph clearly.

(a) $y = f(|x|), \ -6 \leq x \leq 6.$ [3]

(b) $y = \dfrac{1}{f(0.5x)}, \ -6 \leq x \leq 6, \ x \neq 0.$ [4]

(c) $y = \left[f(x) \right]^2, \ -6 \leq x \leq 6.$ [3]

TURN OVER

*Do **NOT** write solutions on this page.*

9 **[Maximum mark: 17]**

The plane Π has normal vector $\mathbf{n} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$ and $\mathrm{Q}(-3,1,-2)$ lies on the plane.

The coordinates of P are $(2,1,2)$.

(a) Show that Π has equation $2x-3y+z=-11$. [2]

(b) Determine whether P lies on Π. [2]

(c) Find an equation of the line, ℓ, perpendicular to Π passing through point P, leaving your answer in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d},\ \ \lambda \in \mathbb{R}$. [3]

(d) Show that ℓ intersects Π at the point $\mathrm{F}(0,4,1)$. [3]

(e) Hence, find the distance between Π and P. [3]

(f) Find $\overrightarrow{PQ}$. [2]

(g) Find $\left|\overrightarrow{PQ} \bullet \hat{\mathbf{n}}\right|$, where $\hat{\mathbf{n}}$ is a unit vector of $\mathbf{n}$. [1]

(h) Comment on the geometrical interpretation of the answer in **(g)**. [1]

10 **[Maximum mark: 14]**

(a) Prove or disprove the statement:

“ $f(x) = x^3 - x^2 - x + 1$ is a one-one function for all $x \in \mathbb{R}$.” [3]

(b) Using the principle of mathematical induction, prove that [8]

$$\left(\sum_{r=1}^{n} r\right)^2 = \sum_{r=1}^{n} r^3, \quad n \geq 2.$$

(c) **Hence**, show that [3]

$$\sum_{r=1}^{n} (r+1)^3 = \left(\sum_{r=1}^{n+1} r\right)^2 - 1, \quad n \geq 2.$$

TURN OVER

*Do **NOT** write solutions on this page.*

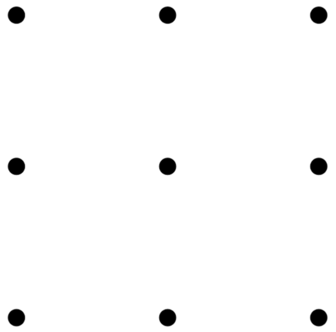
11 [Maximum mark: 14]

(a) How many 5-letter passwords can be formed from A, B, C, D, E, F, G

i. if the letters in each password must be distinct?

ii. if the letters in each password must be distinct and A, B, C, D can only occur as the first, third or fifth letters while E, F, G as the second or fourth letters? [5]

(b) Consider the following set of points that form a perfect square grid.



Find

i. the number of straight lines which pass through 3 points;

ii. the number of triangles whose vertices are points in the diagram above; and,

iii. the number of rectangles whose vertices are points in the diagram above. [9]

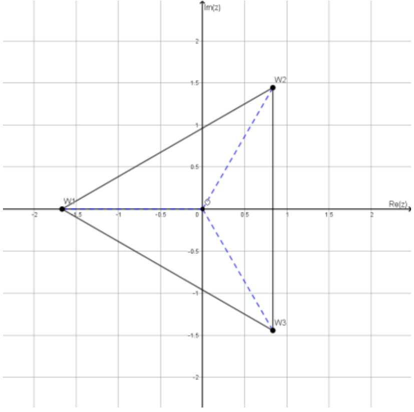
End of Paper

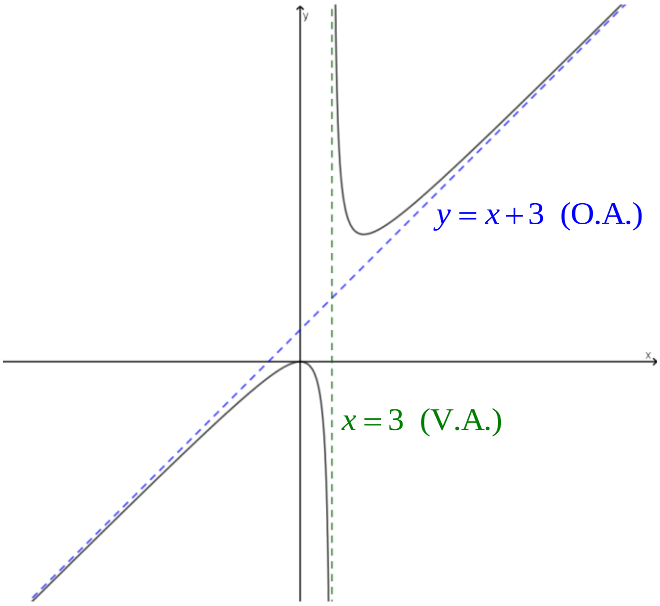
Year 6 HL MAA Preliminary Examination 2021 Paper 1 (Markscheme)

Section A

Qn	Suggested solution	Markscheme
1	***Absolute value function graph + Definite integral***	**[Marks: 5]**
(a)	$f\,'(1.5)=-2$	**M1 A1**
(b)	$f\,''(-1)=0$	**A1**
(c)	$\int_{-1}^{3} f(x)\,\mathrm{d}x=\frac{1}{2}(1)(2)-\frac{1}{2}(1+3)(4)$	**M1 – must subtract**
	$=-7$	**A1**
2	***Scalar Product + Double angle formula (Trigo)***	**[Marks: 6]**
	$\begin{pmatrix} 1/2 \\ \cos x \\ -1 \end{pmatrix}\bullet\begin{pmatrix} \sqrt{3} \\ 2\cos x \\ 1 \end{pmatrix}=0$	**M1** – dot product = 0
	$\frac{\sqrt{3}}{2}+2\cos^2 x-1=0$	**A1**
	$2\cos^2 x-1=-\frac{\sqrt{3}}{2}$	
	$\cos(2x)=-\frac{\sqrt{3}}{2}$	**M1 A1** – double angle
	$2x=\frac{5\pi}{6}$ or $\frac{7\pi}{6}$	
	$x=\frac{5\pi}{12}$ or $\frac{7\pi}{12}$	**A1, A1**
3	***Maclaurin series + application to Limits***	**[Marks: 8]**
(a)	<u>**Method 1**</u> $f(x)=\frac{1}{2}\left(\mathrm{e}^{x}-\mathrm{e}^{-x}\right)$	
	$=\frac{1}{2}\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\ldots\right)$	**M1** **A1**
	$-\frac{1}{2}\left(1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}-\frac{x^5}{5!}+\ldots\right)$	**A1**
	$\approx x+\frac{x^3}{3!}+\frac{x^5}{5!}$	**A2, 1, 0**

Qn	Suggested solution	Markscheme
	Method 2 $f(x)=\dfrac{\mathrm{e}^{x}-\mathrm{e}^{-x}}{2}$ $f'(x)=\dfrac{\mathrm{e}^{x}+\mathrm{e}^{-x}}{2}$ $f''(x)=\dfrac{\mathrm{e}^{x}-\mathrm{e}^{-x}}{2}=f(x)$ $f(0)=f''(0)=f^{(4)}(0)=0$ $f'(0)=f^{(3)}(0)=f^{(5)}(0)=1$ $\therefore\ f(x)\approx x+\dfrac{x^{3}}{3!}+\dfrac{x^{5}}{5!}$	**M1** – finding derivatives **A1** **A1** **A2, 1, 0**
(b)	$\lim\limits_{x\to 0}\dfrac{x-f(x)}{x^{3}}=\lim\limits_{x\to 0}\dfrac{x-\left(x+\dfrac{x^{3}}{3!}+\dfrac{x^{5}}{5!}\right)}{x^{3}}$ $=-\lim\limits_{x\to 0}\left(\dfrac{\dfrac{x^{3}}{3!}+\dfrac{x^{5}}{5!}}{x^{3}}\right)$ $=-\lim\limits_{x\to 0}\left(\dfrac{1}{3!}+\dfrac{x^{2}}{5!}\right)$ $=-\dfrac{1}{6}$	**M1** – use of expansion from (a), not L'Hopital's (No A1 at this line) **A1** f.t. (simplified with division) **A1** f.t.
4	***Roots of a complex number + Area of triangle***	**[Marks: 9]**
(a)	**Method 1** $w^{3}=-\dfrac{125}{27}=\dfrac{125}{27}\mathrm{e}^{\mathrm{i}\pi}$ $=\dfrac{125}{27}\mathrm{e}^{\mathrm{i}(\pi+2n\pi)},\quad n=-1,0,1$ $w=\dfrac{5}{3}\mathrm{e}^{\mathrm{i}\left(\frac{2n+1}{3}\right)\pi},\quad n=-1,0,1$ $=\dfrac{5}{3}\mathrm{e}^{\mathrm{i}\pi},\ \dfrac{5}{3}\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}$ or $\dfrac{5}{3}\mathrm{e}^{-\mathrm{i}\frac{\pi}{3}}$	**A1** – $\arg\left(w^{3}\right)=\pi$ **M1 A1** **M1** **A2, 1, 0**
	Method 2 $w^{3}=-\dfrac{125}{27}=\left(-\dfrac{5}{3}\right)^{3}$ $w=-\dfrac{5}{3}$ is a root $\Rightarrow(3w+5)$ is a factor.	

Qn	Suggested solution	Markscheme
	$27w^3+125=(3w+5)(9w^2+bw+25)$	**M1** – find quad factor
	Comparing coefficient of *w*, $5b+75=0 \Rightarrow b=-15$	
	$27w^3+125=(3w+5)(9w^2-15w+25)=0$	**A1** – $9w^2-15w+25$
	$\therefore 9w^2-15w+25=0$	
	$w=\dfrac{15\pm\sqrt{225-4(225)}}{18}=\dfrac{15\pm15\left(\sqrt{3}\mathrm{i}\right)}{18}=\dfrac{5\pm5\left(\sqrt{3}\mathrm{i}\right)}{6}$	**A1** – Cartesian form
	$\therefore w=\dfrac{5}{3}\mathrm{e}^{\mathrm{i}\pi},\ \dfrac{5}{3}\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}$ or $\dfrac{5}{3}\mathrm{e}^{-\mathrm{i}\frac{\pi}{3}}$	**M1** – convert to Euler **A2, 1, 0**
(b)	Note the triangle on the Argand diagram: 	
	<u>Method 1</u>	
	Area of triangle $=3\times\dfrac{1}{2}\left(\dfrac{5}{3}\right)^2\sin\dfrac{2\pi}{3}$	**M1** (must have x3) **A1** (must be $\sin\dfrac{2\pi}{3}$)
	$=3\times\dfrac{1}{2}\left(\dfrac{5}{3}\right)^2\left(\dfrac{\sqrt{3}}{2}\right)=\dfrac{25\sqrt{3}}{12}$	**A1**
	<u>Method 2</u>	
	Area of triangle $=\dfrac{1}{2}\left(\dfrac{5\sqrt{3}}{3}\right)^2\sin\dfrac{\pi}{3}$	**M1 A1** ($\dfrac{5\sqrt{3}}{3}$ sides)
	$=\dfrac{1}{2}\left(\dfrac{25}{3}\right)\left(\dfrac{\sqrt{3}}{2}\right)=\dfrac{25\sqrt{3}}{12}$	**A1**
	<u>Method 3</u>	
	Area of triangle $=\dfrac{1}{2}\left(2\times\dfrac{5\sqrt{3}}{6}\right)\left(\dfrac{5}{3}+\dfrac{5}{6}\right)$	**M1 A1**
	$=\dfrac{1}{2}\left(\dfrac{5\sqrt{3}}{3}\right)\left(\dfrac{15}{6}\right)=\dfrac{25\sqrt{3}}{12}$	**A1**

Qn	**Suggested solution**	**Markscheme**
5	***Geometric series + Definite integral with arctan***	**[Marks: 6]**
(a)	$\sum_{k=0}^{n-1}(-1)^k x^{2k} = \frac{1\left[1-\left(\left(-1\right)x^2\right)^n\right]}{1-\left(-x^2\right)}$ $= \frac{1+\left(-1\right)^{n-1}x^{2n}}{1+x^2} = \frac{1}{1+x^2}+\frac{\left(-1\right)^{n-1}x^{2n}}{1+x^2}$	**M1** – geometric sum with ***n*** terms **A1** – $r=-x^2$ **AG** (if errors in simplification, get only 1m out of 2)
(b)	From (a), $\sum_{k=0}^{n-1}(-1)^k x^{2k} - \frac{\left(-1\right)^{n-1}x^{2n}}{1+x^2} = \frac{1}{1+x^2}$ Integrate wrt. x from $x=0$ to $x=1$: $\left[\sum_{k=0}^{n-1}(-1)^k\frac{x^{2k+1}}{2k+1}\right]_0^1 - \int_0^1\frac{\left(-1\right)^{n-1}x^{2n}}{1+x^2}\mathrm{d}x = \int_0^1\frac{1}{1+x^2}\mathrm{d}x$ $\sum_{k=0}^{n-1}(-1)^k\frac{1}{2k+1} - \left(-1\right)^{n-1}\int_0^1\frac{x^{2n}}{1+x^2}\mathrm{d}x = \left[\arctan\left(x\right)\right]_0^1$ $\sum_{k=0}^{n-1}\left[\frac{\left(-1\right)^k}{2k+1}\right] - \left(-1\right)^{n-1}\int_0^1\frac{x^{2n}}{1+x^2}\mathrm{d}x = \frac{\pi}{4}$	**M1** – integration **A1** – summation term **A1** – arctan term **A1** (no f.t.)
6	***Rational function + Graph transformations***	**[Marks: 12]**
(a)	$y=\frac{x^2}{x-3}$ $=x+3+\frac{9}{x-3}$ 	**M1 A1** **A1 A1**

Qn	**Suggested solution**	**Markscheme**
(b)	$y=3x+5+\frac{3}{x-1}$ $=\left(\left(3x\right)+3\right)+2+\frac{9}{\left(3x\right)-3}$ $=g\left(3x\right)+2$ **OR** $g\left(x\right)=x+3+\frac{9}{x-3}$ $g\left(3x\right)=3x+3+\frac{9}{3x-3}$ $g\left(3x\right)+2=3x+5+\frac{3}{x-1}$ Horizontal scaling with factor $\frac{1}{3}$ followed by Translation by $\begin{pmatrix}0\\2\end{pmatrix}$ o.e. **OR** Translation by $\begin{pmatrix}0\\2\end{pmatrix}$ followed by Horizontal scaling with factor $\frac{1}{3}$ o.e.	**M1** **A1** **A1 A1**
(c)	Number of zeros = 2 New asymptotes are $x=1$ and $y=3x+5$ The zeros lie between the *x*-intercepts of the new asymptotes, which are at $\left(-\frac{5}{3},0\right)$ and $\left(1,0\right)$.	**A1** **A1 A1** **R1** – must have correct *x*-intercept
7	***Differentiation with points of inflexion***	**[Marks: 9]**
(a)	**<u>Method 1</u>** $h\left(x\right)=\left(f\circ g\right)\left(x\right)$ $h'\left(x\right)=f'\left[g\left(x\right)\right].g'\left(x\right)$ $h''\left(x\right)=f'\left[g\left(x\right)\right].g''\left(x\right)+g'\left(x\right).\left[f''\left(g\left(x\right)\right).g'\left(x\right)\right]$ $h''\left(0\right)=f'\left[g\left(0\right)\right].g''\left(0\right)+f''\left[g\left(0\right)\right].\left[g'\left(0\right)\right]^2$ $=0$ (shown) since $g'\left(0\right)=g''\left(0\right)=0$ (stationary point of inflexion)	**M1** – Chain rule **M1 A1** – product rule **A1** – correct use of both $g'\left(0\right)=g''\left(0\right)=0$ **AG**

<table>
<tr><th>Qn</th><th>Suggested solution</th><th>Markscheme</th></tr>
<tr><td></td><td>

Method 2

$h(x)=\left[g(x)\right]^4-2\cos\left[g(x)\right]$

$h'(x)=\left(4\left[g(x)\right]^3+2\sin\left[g(x)\right]\right).g'(x)$

$h''(x)=\left(4\left[g(x)\right]^3+2\sin\left[g(x)\right]\right).g''(x)$

$+g'(x).\left(12\left[g(x)\right]^2+2\cos\left[g(x)\right]\right).g'(x)$

$h''(0)=\left(4\left[g(0)\right]^3+2\sin\left[g(0)\right]\right).g''(0)$

$+\left(12\left[g(0)\right]^2+2\cos\left[g(0)\right]\right).\left[g'(0)\right]^2$

$=0$ (shown)

</td><td>

M1 – Chain rule

M1 – product rule

A1

A1 $g'(0)=g''(0)=0$

AG

</td></tr>
<tr><td>(b)</td><td>

$f(x)=x^4-2\cos x$

$f'(x)=4x^3+2\sin x=\begin{cases}>0 \text{ for } x=0^+\\ <0 \text{ for } x=0^-\end{cases}$

$f''(x)=12x^2+2\cos x>0$ for $x=0^+$ and $x=0^-$

<table>
<tr><th></th><th>$f'(x)$</th><th>$f''(x)$</th><th>$g(x)$</th><th>$g'(x)$</th><th>$g''(x)$</th></tr>
<tr><td>$x=0^-$</td><td>–</td><td>+</td><td>–</td><td>+</td><td>–</td></tr>
<tr><td>$x=0^+$</td><td>+</td><td>+</td><td>+</td><td>+</td><td>+</td></tr>
</table>

Method 1

$h''(x)=f'\left[g(x)\right].g''(x)+f''\left[g(x)\right].\left[g'(x)\right]^2$

$h''(0^-)>0$

$h''(0^+)>0$

Since $h''(x)$ has the same sign on either side of $x=0$ i.e. there is no change in concavity, there is no point of inflexion at $x=0$.

Method 2

$h'(x)=f'\left[g(x)\right].g'(x)$

<table>
<tr><td>x</td><td>0^-</td><td>0</td><td>0^+</td></tr>
<tr><td>$h'(x)$</td><td>–</td><td>0</td><td>+</td></tr>
<tr><td>Tan</td><td>\</td><td>__</td><td>/</td></tr>
</table>

h has no point of inflexion at $x=0$.

(In fact, *h* has a local minimum point at $x=0$.)

</td><td>

M1

A1 – considering signs for 0^+ and 0^-

A1

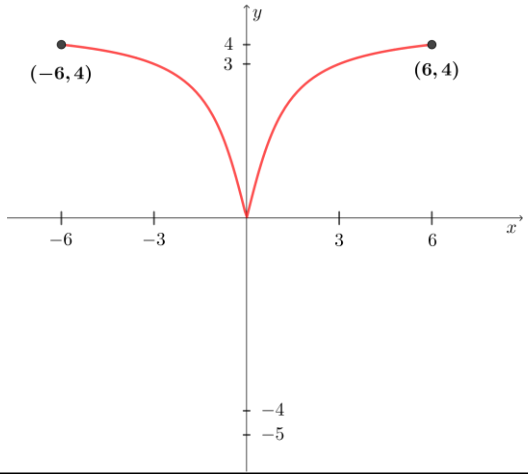
R1
A1

M1 A1

A1

</td></tr>
</table>

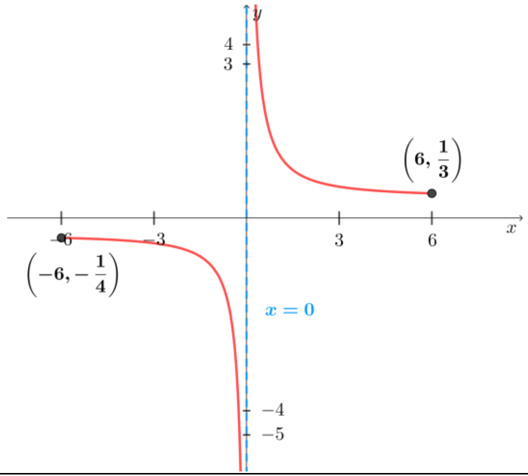
Section B

<table>
<tr><th>Qn</th><th>Suggested solution</th><th>Markscheme</th></tr>
<tr><td>8</td><td><i>Graph Sketching</i></td><td>[marks:]</td></tr>
<tr><td>(a)</td><td>


</td><td>A1 – shape;
A0 if not sharp at the origin

A1 - right endpoint

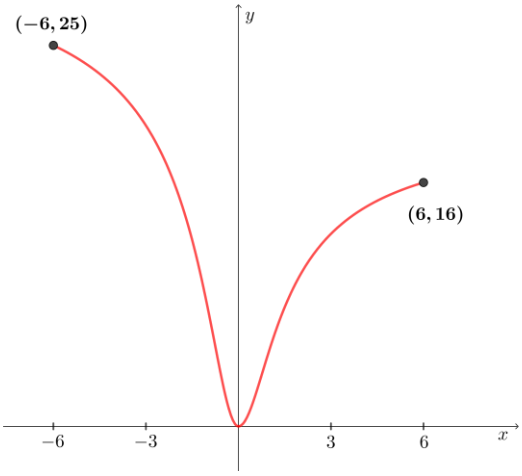
A1 - left endpoint

(A0A0A0 – for non-function)</td></tr>
<tr><td>(b)</td><td>


</td><td>A1 - shape

A1 – vertical asymptote

A1 - right endpoint

A1 – left endpoint</td></tr>
<tr><td>(c)</td><td>


</td><td>A1 – shape
(condone if sharp at the origin)

(A0 if both endpoints have the same y-value)

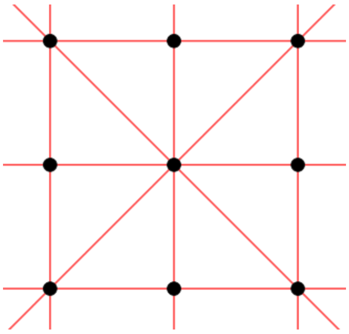
A1 - right endpoint

A1 - left endpoint</td></tr>
</table>

Qn	Suggested solution	Markscheme
9	***Vectors – Lines & Planes***	**[marks: 17]**
(a)	$\Pi:\begin{pmatrix}x\\y\\z\end{pmatrix}\bullet\begin{pmatrix}2\\-3\\1\end{pmatrix}=\begin{pmatrix}-3\\1\\-2\end{pmatrix}\bullet\begin{pmatrix}2\\-3\\1\end{pmatrix}\Rightarrow 2x-3y+z=-6-3-2=-11$	**M1** – correct use of equation **A1** – dot product **AG**
(b)	$\begin{pmatrix}2\\1\\2\end{pmatrix}\bullet\begin{pmatrix}2\\-3\\1\end{pmatrix}\Rightarrow 4-3+2=3\neq-11$, therefore, P does <u>not</u> lie on Π.	**M1** **A1** - not
(c)	$\ell:\mathbf{r}=\begin{pmatrix}2\\1\\2\end{pmatrix}+\lambda\begin{pmatrix}2\\-3\\1\end{pmatrix},\quad\lambda\in\mathbb{R}$	**R1** – use of direction vector **A1**
(d)	$2(2+2\lambda)-3(1-3\lambda)+(2+\lambda)=-11$ $4+4\lambda-3+9\lambda+2+\lambda=-11$ $3+14\lambda=-11$ $\lambda=-1$ So the point of intersection is $(2-2,1+3,2-1)=(0,4,1)$	**M1** **A1** **A1**
(e)	$\text{dist}(\Pi,P)=\left\lvert\overrightarrow{FP}\right\rvert=\left\lvert\begin{pmatrix}2-0\\1-4\\2-1\end{pmatrix}\right\rvert=\left\lvert\begin{pmatrix}2\\-3\\1\end{pmatrix}\right\rvert=\sqrt{2^2+(-3)^2+1^2}=\sqrt{14}$ Note: This is a "hence" question.	**(R1)** – $=\lvert\overrightarrow{FP}\rvert$ **M1A1**
(f)	$\overrightarrow{PQ}=\begin{pmatrix}-3-2\\1-1\\-2-2\end{pmatrix}=\begin{pmatrix}-5\\0\\-4\end{pmatrix}$	**M1A1**
(g)	$\hat{\mathbf{n}}=\frac{\mathbf{n}}{\lvert\mathbf{n}\rvert}=\frac{1}{\sqrt{2^2+(-3)^2+1^2}}\begin{pmatrix}2\\-3\\1\end{pmatrix}=\frac{1}{\sqrt{14}}\begin{pmatrix}2\\-3\\1\end{pmatrix}$ $\left\lvert\overrightarrow{PQ}\bullet\hat{\mathbf{n}}\right\rvert=\left\lvert\begin{pmatrix}-5\\0\\-4\end{pmatrix}\bullet\frac{1}{\sqrt{14}}\begin{pmatrix}2\\-3\\1\end{pmatrix}\right\rvert=\frac{1}{\sqrt{14}}\lvert-10-4\rvert=\sqrt{14}$	**A1** **A1**
(h)	$\left\lvert\overrightarrow{PQ}\bullet\hat{\mathbf{n}}\right\rvert=\text{dist}(\Pi,P)$	**A1**

Qn	Suggested solution	Markscheme
10	***Proving via Counterexample, Mathematical Induction & Series***	**[marks: 14]**
(a)	$f(1)=1-1-1+1=0$ $f(-1)=-1-1+1+1=0$ $f(-1)=f(1)$ and $-1\neq 1$ means f is not one-one.	**M1** – any valid counterexample **R1** **A1** - not
(b)	Let $P(n)$ be the statement $\displaystyle\left(\sum_{r=1}^{n} r\right)^2=\sum_{r=1}^{n} r^3$, for $n\geq 2$. $P(2)$ is true since $\displaystyle\left(\sum_{r=1}^{2} r\right)^2=(1+2)^2=9$ and $\displaystyle\sum_{r=1}^{2} r^3=1^3+2^3=9$. Assume $P(k)$ is true for some $k\geq 2$, i.e., $\displaystyle\left(\sum_{r=1}^{k} r\right)^2=\sum_{r=1}^{k} r^3, \quad \exists\, k\geq 2.$ We now prove that $P(k+1)$ is also true: $\displaystyle\left(\sum_{r=1}^{k+1} r\right)^2=\left(\sum_{r=1}^{k} r+(k+1)\right)^2$ $\displaystyle=\left(\sum_{r=1}^{k} r\right)^2+2(k+1)\left(\sum_{r=1}^{k} r\right)+(k+1)^2$ $\displaystyle=\sum_{r=1}^{k} r^3+2(k+1)\left(\sum_{r=1}^{k} r\right)+(k+1)^2$ $\displaystyle=\sum_{r=1}^{k} r^3+2(k+1)\left(\frac{1}{2}k(k+1)\right)+(k+1)^2$ $\displaystyle=\sum_{r=1}^{k} r^3+k(k+1)^2+(k+1)^2$ $\displaystyle=\sum_{r=1}^{k} r^3+(k+1)^2(k+1)$ $\displaystyle=\sum_{r=1}^{k} r^3+(k+1)^3$ $\displaystyle=\sum_{r=1}^{k+1} r^3$ OR	**A1** **A1** **M1** – split sum **M1** – correct expansion **A1** – use of inductive assumption **M1** – use of identity **M1** – factor to get the correct form

Qn	Suggested solution	Markscheme
	(working the other way around) $\sum_{r=1}^{k+1} r^3 = \sum_{r=1}^{k} r^3 + (k+1)^3$ $= \left(\sum_{r=1}^{k} r\right)^2 + (k+1)^3$ $= \left(\frac{1}{2}k(k+1)\right)^2 + (k+1)^3$ $= \frac{1}{4}(k+1)^2\left(k^2+4(k+1)\right)$ $= \frac{1}{4}(k+1)^2(k+2)^2$ $= \left(\frac{1}{2}(k+1)(k+1+1)\right)^2$ $= \left(\sum_{r=1}^{k+1} r\right)^2$ Therefore, since $P(2)$ is true and $P(k+1)$ is true whenever $P(k)$ for any $k \geq 2$ is true, by mathematical induction, $P(n)$ is true for all $n \geq 2$.	**M1** – split sum **A1** – use of inductive assumption **M1** – use of identity **M1** – factor **M1** – factor to get the correct form **R1**
(c)	$\sum_{r=1}^{n}(r+1)^3 = 2^3+3^3+\cdots+(n+1)^3 = \left(\sum_{r=1}^{n+1} r^3\right) - 1$ $= \left(\sum_{r=1}^{n+1} r\right)^2 - 1$ Note: $\sum_{r=1}^{n+1} r^3 = \left(\sum_{r=1}^{n+1} r\right)^2$ but $\sum_{r=1}^{n}(r+1)^3 \neq \left(\sum_{r=1}^{n}(r+1)\right)^2$	**M1** – recognize sum **A1 -** ±1 **A1** – n+1 /correct index

Qn	Suggested solution	Markscheme
11	***Permutation & Combination***	**[marks: 14]**
(a)	i. $^{7}P_{5}=7\times6\times5\times4\times3=2520$ ii. $4\times3\times3\times2\times2=144$	**M1** - use of multiplication **A1 -** 2520 **A1 -** 1st, 3rd, 5th **A1** - 2nd, 4th **A1** - 144
(b)	i. Lines passing through 3 points:  Total number of lines = 3 + 3 + 2 = 8	**(M1)** – “cases” seen, e.g., horizontal, vertical **and** diagonal lines considered **A1** **[N2]**
	ii. Number of triangles, including the degenerate cases (triangles with collinear vertices) $^{9}C_{3}=\frac{9!}{3!6!}=\frac{9\times8\times7\times6!}{3!6!}=3\times4\times7=84.$ Number of degenerate triangles = number of lines passing through 3 points = 8 Thus, total number of triangles = $84-8=76$.	**(M1A1)** **(M1)** – removing degenerate triangles **A1** **[N4]**
	iii. Number of rectangles with vertical or horizontal sides: $^{3}C_{2}\times{}^{3}C_{2}=\frac{3!}{2!1!}\times\frac{3!}{2!1!}=3\times3=9.$ Diamond = 1 Total number of rectangles = 9 + 1 = 10	**(M1)** - $^{3}C_{2}$ **(A1)** **A1** **[N3]**

STUDENT NAME ______________________________

TEACHER NAME: ______________________________

ST. JOSEPH'S INSTITUTION
YEAR 6 PRELIMINARY EXAMINATION 2021

MATHEMATICS: ANALYSIS AND APPROACHES
HIGHER LEVEL
PAPER 2
Tuesday

6 July 2021
2 hours
0800 – 1000 hrs

INSTRUCTIONS TO CANDIDATES

- Write your name and teacher's name in the spaces provided.
- Do not open this examination paper until instructed to do so.
- **Section A:** Answer all questions showing working and answers in the spaces provided in the exam paper.
- **Section B:** Answer all questions using the foolscap paper provided.
- The use of a scientific or examination graphical calculator is permitted in this paper.
- TI-Nspire calculators must be in Press-to-Test mode and cleared of all previous data.
- TI-84+ graphical calculators must only have permitted apps and be ram cleared.
- A clean copy of the **Mathematics: Analysis and Approaches formula booklet** is required for this paper.
- Unless otherwise stated in the question all numerical answers must be given exactly or to three significant figures.
- The maximum mark for this examination paper is *[**110** marks]*.
- This question paper consists of **11** printed pages including the Cover Sheet**.**
- Sections A and B are to be submitted **separately**.

FOR MARKER USE ONLY:

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	TOTAL
											/110

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, for example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are advised to *show all working.*

SECTION A (55 marks)

Answer **all** questions in the **spaces** provided.

1 [Maximum mark: 6]

The 4th term of an arithmetic sequence is 3142 and the 21st term is 1884.

(a) Find the first term and common difference of this sequence. [3]

(b) Calculate the largest value of n for which the sum of the first n terms of the sequence is positive. [3]

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TURN OVER

2 **[Maximum mark: 10]**

Consider the function f defined by

$$f : x \mapsto \frac{1}{1-x},\ x \in \mathbb{R} \setminus \{0,1\}.$$

It is given that $f^n(x) = \underbrace{(f \circ f \circ \ldots \circ f)}_{n \text{ times}}(x)$, for all $n \in \mathbb{Z}^+$.

(a) Find the function f^{-1}, indicating clearly its domain. [3]

(b) Find an expression for $f^2(x)$. [3]

(c) Deduce the expression for $f^3(x)$. [2]

(d) Find $f^{2021}(2)$. [2]

TURN OVER

3 **[Maximum mark: 9]**

(a) Find the expansion of $\sqrt{\frac{1-x^2}{2+x}}$ in ascending powers of x, up to and including the term in x^2, where $|x|<1$. [6]

(b) By substituting $x=\frac{1}{4}$ to your answer in (a), estimate the value of $\sqrt{\frac{5}{6}}$, expressing your answer in the form $\frac{a}{b}$, where a and b are integers to be determined. [3]

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TURN OVER

4 **[Maximum mark: 6]**

(a) In Singapore, car financing loans are based on **simple** interest on the principal amount loaned.

In order to buy a car, Mr Heng takes out a loan of $90,000 on a **simple** annual interest of 1.99% for 8 years. How much will he have to pay altogether towards the loan? [2]

(b) The value of a new car depreciates 15% per year for the first 5 years, and 10% per year thereafter.

Mr Heng buys his new car for $180,000 and intends to sell it exactly 8 years later.

(i) How much will he receive when he sells his car? [2]

(ii) What will be the **real value** of this amount that he receives if the average inflation rate is 1.5% per year? [2]

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5 **[Maximum mark: 8]**

Consider the function $f(x)=\dfrac{x+5}{x^2+3x+2}$.

(a) Sketch the graph of $y=f(x)$, labelling clearly any axial intercepts, maximum and minimum points and the equations of any asymptotes. [6]

(b) Given that the equation $f(x)=k$ has no real solutions, find the range of values of k. [2]

TURN OVER

6 **[Maximum mark: 8]**

Consider the three planes defined by the equations

$$\begin{aligned} x+2y+3z&=5 \\ x-z&=b \\ y+az&=2 \end{aligned}$$

where $a, b \in \mathbb{R}$.

(a) If $a=4$ and $b=0$, find the coordinates of the point of intersection of the three planes. [2]

(b) It is given that the three planes meet in a line ℓ.

(i) Find the value of a and of b.

(ii) Find a vector equation of the line ℓ. [6]

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7 [Maximum mark: 8]

With reference to the origin O, the position vectors of points A, B and P are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{p}$ respectively. It is given that P lies on the line segment AB, such that $\mathbf{a}\cdot\mathbf{p}=\mathbf{b}\cdot\mathbf{p}$ and $\angle\text{AOP}$ and $\angle\text{BOP}$ are $30°$ and $60°$ respectively.

(a) Show that AB is perpendicular to OP. [2]

(b) State the geometrical meaning of $\frac{1}{2}|\mathbf{a}\times\mathbf{p}|$. [1]

(c) Show that $|\mathbf{a}\times\hat{\mathbf{p}}|=\text{AP}$, where $\hat{\mathbf{p}}=\frac{\mathbf{p}}{|\mathbf{p}|}$. [1]

(d) Show that $\text{AP}=k|\mathbf{p}|$, where k is an exact value to be found. [2]

(e) Find the ratio $\text{AP}:\text{PB}$. [2]

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SECTION B (55 marks)

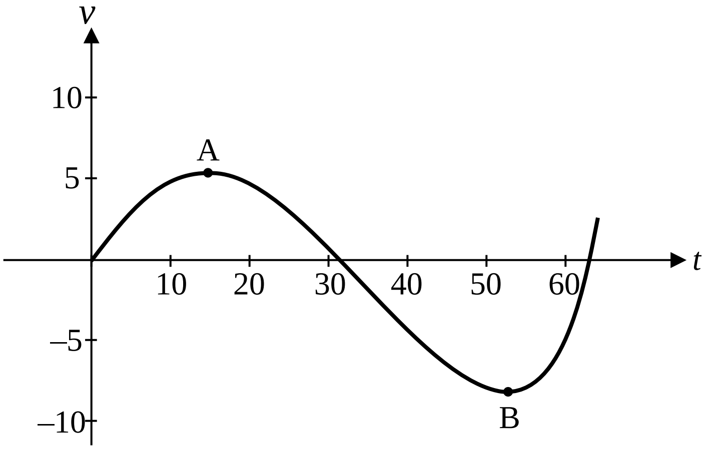
Answer all questions on the foolscap paper provided. ***Please start each question on a new page.***

8 **[Maximum Mark: 13]**

An object starts from rest and moves in a straight line such that its velocity, $v(t)$ $\mathrm{m\,s^{-1}}$, after t seconds is given by

$$v(t) = 5\cos\left(\frac{t}{10} - \frac{\pi}{2}\right)\csc\left(\frac{t}{30} + \frac{\pi}{4}\right) \text{ for } 0 \le t \le 64\,.$$

The following diagram shows the graph of v against t. The point A is a local maximum and the point B is a local minimum.



(a) Find the maximum speed of the object. [3]

(b) The object first comes to rest at $t = t_1$. Find

(i) the value of t_1,

(ii) the acceleration when $t = t_1$. [4]

(c) Find the two values of t when the object is 100 m from the starting point. [3]

(d) Find the total distance travelled in the first 40 seconds. [3]

TURN OVER

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9 **[Maximum Mark: 13]**

Consider the polynomial $P(z) = z^3 - 5z^2 + hz - 4$, where $z \in \mathbb{C}$, for some real number h.

Given that α, β and γ are the three roots of $P(z) = 0$ such that $\alpha = 4$ and $\beta, \gamma \notin \mathbb{R}$.

(a) Find β and γ, leaving your answer in the form $\frac{a \pm \sqrt{b}\mathrm{i}}{c}$ where $a, b, c \in \mathbb{Z}$. [4]

(b) Hence, find

(i) β^3 and γ^3 using de Moivre's theorem.

(ii) the smallest positive integer m for which β^m and γ^m are real.

(iii) the smallest integer $n > 1$ for which β^n and γ^n are also roots of $P(z) = 0$.

[9]

10 **[Maximum Mark: 12]**

It is given that $f(x) = (\arccos x)(\arcsin x)$, where $-1 \leq x \leq 1$.

(a) Show that

(i) $\sqrt{1 - x^2}\, f'(x) = \arccos x - \arcsin x$.

(ii) $(1 - x^2)f''(x) + 2 = x\, f'(x)$ [5]

(b) By considering the graph of $f''(x)$ or otherwise, find the values of $f^{(3)}(0)$ and $f^{(4)}(0)$, giving your answers to 3 significance figures where necessary. [3]

(c) Determine the Maclaurin series for $f(x)$ up to and including the term in x^4. [4]

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11 [Maximum Mark: 17]

(a) **(i)** Find $\int_0^x 2t\ \sin(t^2)\ \mathrm{d}t$, leaving your answer in terms of x.

(ii) Show that $\cos\left(x^2+\frac{\pi}{2}\right) = -\sin(x^2)$.

Hence, show that

$$\int_0^x 2t^3\ \sin\left(t^2+\frac{\pi}{2}\right)\ \mathrm{d}t = \cos(x^2) - 1 + x^2\sin(x^2)$$

[8]

(b) The function $f(x) = 5\arcsin(x^2)$ is defined on its maximal domain.
A bowl is formed by rotating the curve of $y = f(x)$ through π radians about the *y* – axis.

(i) State the maximal domain of *f* and deduce the exact height of the bowl, leaving your answer in terms of π.

Let *h* be the depth of water in the bowl.

(ii) Find the volume of water in the bowl in terms of *h*.

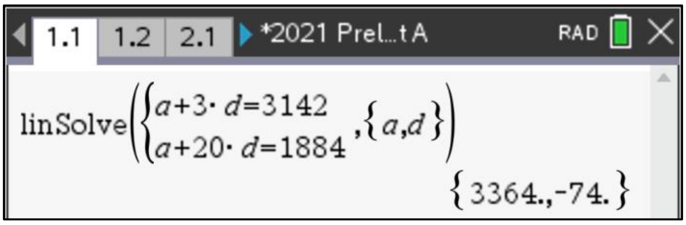
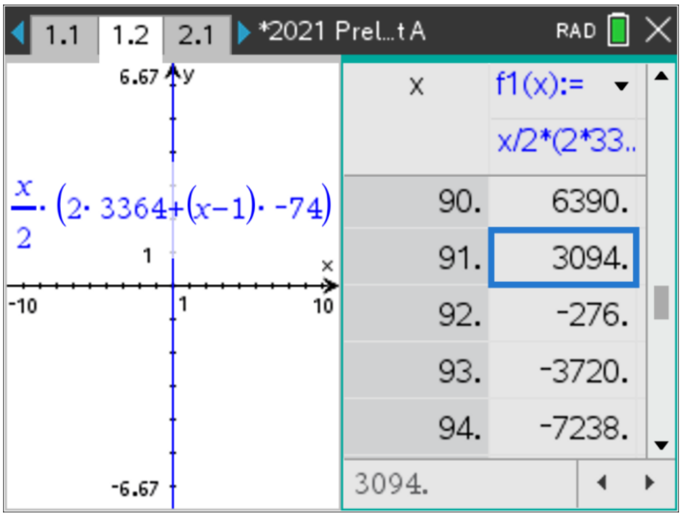
(iii) Water is poured into the bowl at a constant rate of 0.3 unit^3 per seconds. Determine the rate at which the depth of water is increasing when $h = 4$.

[9]

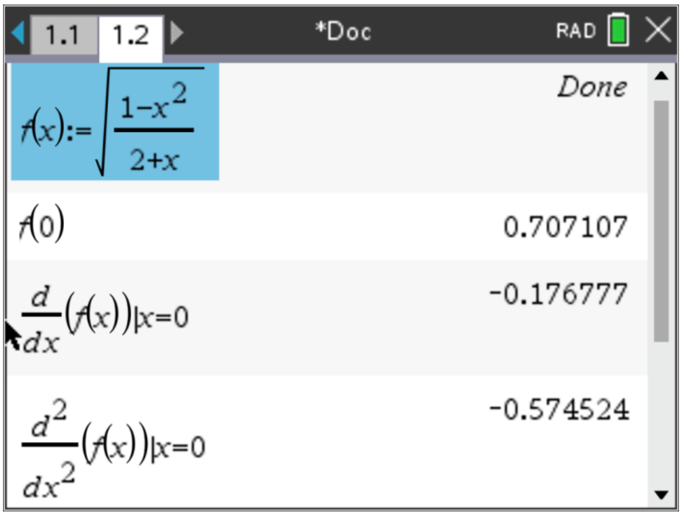
End of Paper

Year 6 HL MAA Preliminary Examination 2021 Paper 2 (Markscheme)

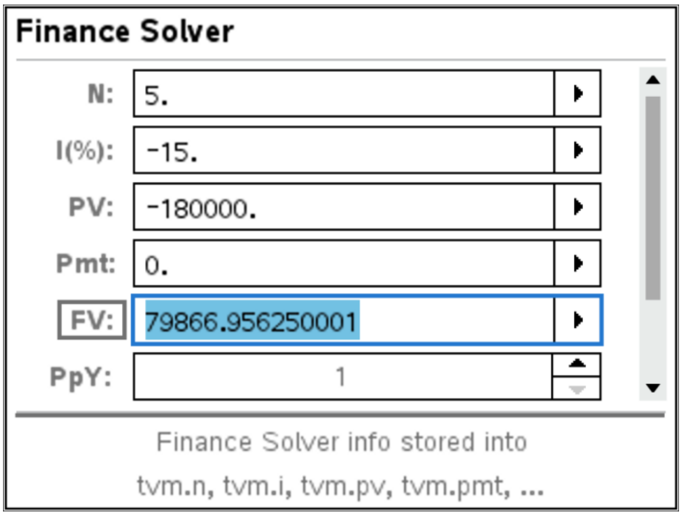
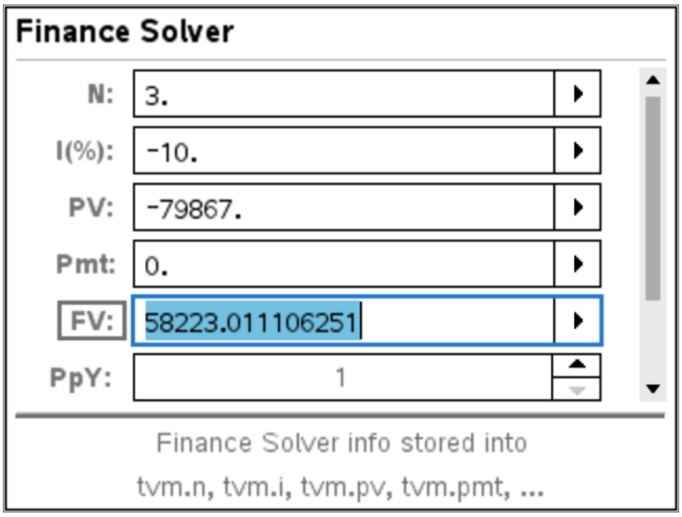
Section A

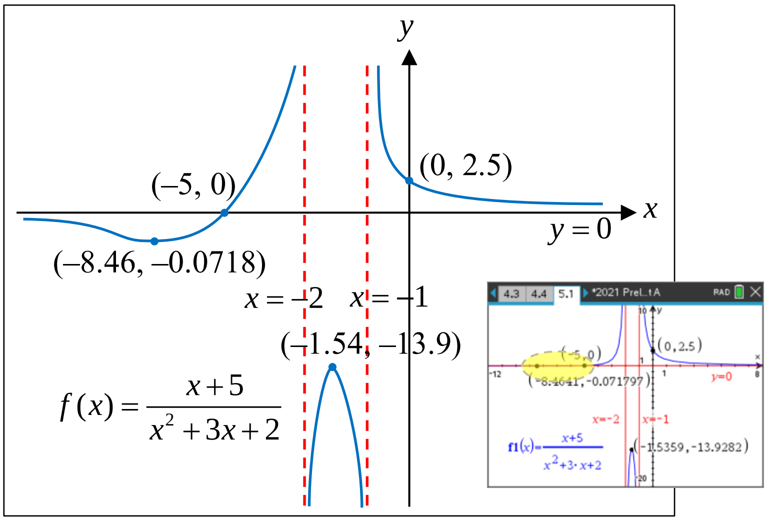
Qn	**Suggested solution**	**Markscheme**
1	***Arithmetic Progression***	**[Marks: 6]**
(a)	$u_4 = a + 3d = 3142$ $u_{21} = a + 20d = 1884$  $\text{linSolve}\left(\begin{cases} a+3\cdot d=3142 \\ a+20\cdot d=1884 \end{cases},\{a,d\}\right)$ $\{3364.,-74.\}$ First term, $a = 3364$ Common difference, $d = -74$	**M1** **A1** **A1**
(b)	$S_n = \frac{n}{2}\left(2(3364) + (n-1)(-74)\right) > 0$  x f1(x):= x/2*(2*33.. 90. 6390. 91. 3094. 92. -276. 93. -3720. 94. -7238. 3094. Largest $n = 91$	**M1** ($S_n > 0$) **M1** (any valid method, incl. graph or solving quadratic inequality) **A1**

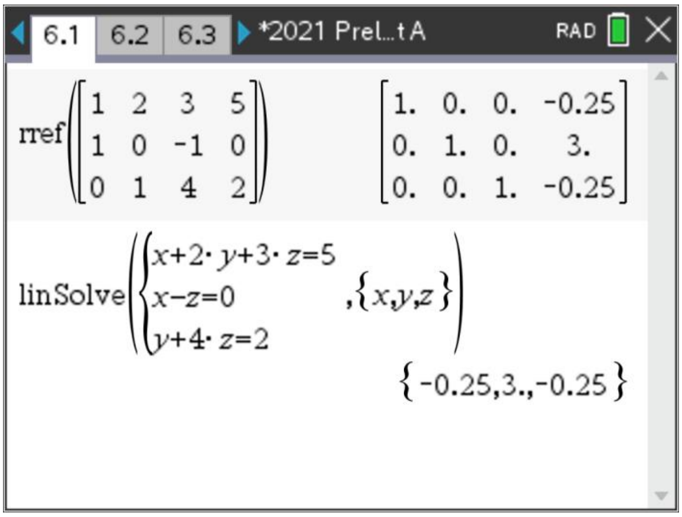
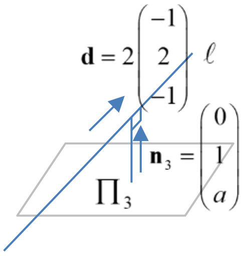
2	***Functions: Domain, Composite, Inverse***	**[Marks: 10]**
(a)	$f(x)=\frac{1}{1-x}$ Let $y=\frac{1}{1-x}$, we have $x=1-\frac{1}{y}$ $\therefore f^{-1}(x)=1-\frac{1}{x},\ x\neq 0,1$ Note: range of f^{-1} excludes $\{0,1\}$, therefore since $f^{-1}(1)=0$, domain of f^{-1} must also exclude $\{1\}$ in addition to the $\{0\}$.	**M1** **A1** (expression) **A1** (domain, also accept if just $x \neq 0$)
(b)	$f^2(x)=f\circ f(x)=f\left(\frac{1}{1-x}\right)$ $=\frac{1}{1-\left(\frac{1}{1-x}\right)}$ $\left(\text{multiply by } \frac{1-x}{1-x}\right)$ $=\frac{1-x}{(1-x)-1}$ $=\frac{1-x}{-x}$ $\therefore f^2(x)=1-\frac{1}{x}$ o.e. (Domain not required, only expression)	**M1** (compositing) **M1** (correct approach to getting 2 layer fraction) **A1**
(c)	Observe that $f^{-1}(x)=f^2(x)$ $f\left(f^{-1}(x)\right)=f\left(f^2(x)\right)$ $\Rightarrow f^3(x)=x$	**M1** (or any valid method) **A1**
(d)	Observe that every 3rd function is an identity function, i.e. $f^3=f^6=f^9=\ldots$ $f^{2021}(x)=f^2\left(f^{2019}(x)\right)$ $=f^2(x)$ $=1-\frac{1}{x}$ $\therefore f^{2021}(2)=\frac{1}{2}$	**(M1)** **A1**

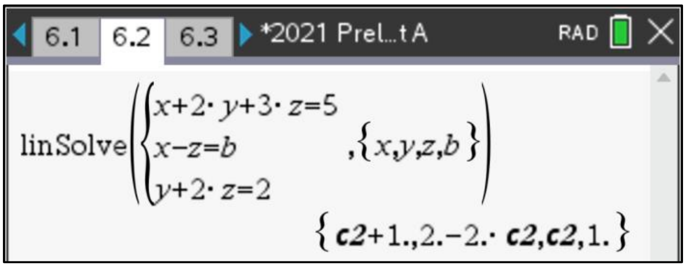
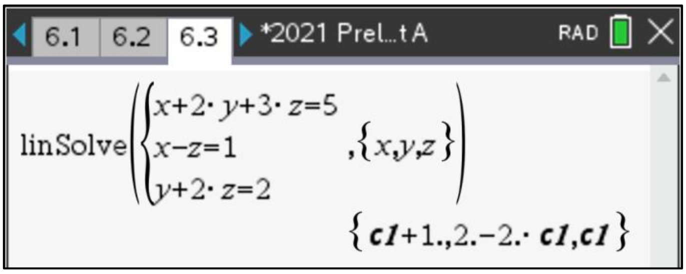
3	***Binomial Theorem and Generalisation Binomial Theorem***	**[Marks: 9]**
(a)	$\sqrt{\frac{1-x^2}{2+x}}=\left(1-x^2\right)^{\frac{1}{2}}\left(2+x\right)^{-\frac{1}{2}}$	**M1**
	Expanding both, we have $\left(1-x^2\right)^{\frac{1}{2}}=1-\frac{1}{2}x^2+...$	**A1**
	$\left(2+x\right)^{-\frac{1}{2}}=2^{-\frac{1}{2}}\left(1+\frac{x}{2}\right)^{-\frac{1}{2}}$ $=\frac{1}{\sqrt{2}}\left(1+{}^{-\frac{1}{2}}C_1\left(\frac{x}{2}\right)+{}^{-\frac{1}{2}}C_2\left(\frac{x}{2}\right)^2+...\right)$	**M1**
	$=\frac{1}{\sqrt{2}}\left(1-\frac{1}{2}\left(\frac{x}{2}\right)+\frac{3}{8}\left(\frac{x^2}{4}\right)-+...\right)$ $=\frac{1}{\sqrt{2}}\left(1-\frac{x}{4}+\frac{3x^2}{32}+...\right)$	**A1**
	Hence, $\sqrt{\frac{1-x^2}{2+x}}=\left(1-\frac{1}{2}x^2+...\right)\frac{1}{\sqrt{2}}\left(1-\frac{x}{4}+\frac{3x^2}{32}+...\right)$	**M1**
	$=\frac{1}{\sqrt{2}}\left(1-\frac{x}{4}+\left(\frac{3x^2}{32}-\frac{x^2}{2}\right)+...\right)$ $=\frac{1}{\sqrt{2}}\left(1-\frac{x}{4}-\frac{13x^2}{32}+...\right)$	**A1**
	Alternatively, use Maclaurin Series with GDC. Note: This cannot be used to continue on to part (b) which requires the estimate to be an exact fraction.	**Alternative**
	 $f(x):=\sqrt{\frac{1-x^2}{2+x}}$ Done $f(0)$ 0.707107 $\frac{d}{dx}(f(x))\|x=0$ -0.176777 $\frac{d^2}{dx^2}(f(x))\|x=0$ -0.574524	**M1** **A1** **A1** **A1**
	$\sqrt{\frac{1-x^2}{2+x}}=0.707-0.177x-\frac{1}{2}\times 0.5745x^2$	**M1** (Maclaurin series)
	$=0.707-0.177x-0.287x^2$	**A1**

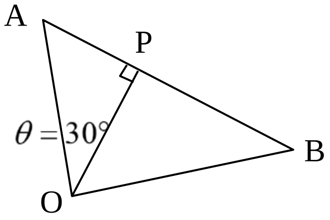
(b)	Substitute $x=\frac{1}{4}$, $\sqrt{\frac{1-\left(\frac{1}{4}\right)^2}{2+\frac{1}{4}}}=\frac{1}{\sqrt{2}}\left(1-\frac{1}{4}\left(\frac{1}{4}\right)-\frac{13}{32}\left(\frac{1}{4}\right)^2+\ldots\right)$ $\sqrt{\frac{15}{16}\times\frac{4}{9}}=\frac{1}{\sqrt{2}}\left(1-\frac{1}{4}\left(\frac{1}{4}\right)-\frac{13}{32}\left(\frac{1}{4}\right)^2+\ldots\right)$ $\sqrt{\frac{5}{12}}=\frac{1}{\sqrt{2}}\left(1-\frac{1}{4}\left(\frac{1}{4}\right)-\frac{13}{32}\left(\frac{1}{4}\right)^2+\ldots\right)$ $\sqrt{\frac{5}{6}}=1-\frac{1}{16}-\frac{13}{32}\left(\frac{1}{16}\right)+\ldots \qquad \left(\times\sqrt{2}\right)$ $\therefore \frac{a}{b}=\frac{467}{512}$ $\therefore a=467,\ b=512$ (not required to write out)	**M1** (obtain $\sqrt{\frac{5}{12}}$ on LHS) **A1 A1**

Qn	**Suggested solution**	**Markscheme**
4	***Financial Mathematics***	**[Marks: 6]**
(a)	Simple interest, $I = \frac{90,000 \times 1.99 \times 8}{100} = 14328$	**M1** (simple interest)
	Total amount paid after 8 years $= \$104,328$	**A1**
(b)	$FV = 180,000\left(1 - \frac{15}{100}\right)^5 \left(1 - \frac{10}{100}\right)^3$	**M1**
	$= \$58,223.01$	**A1**
	<u>Alt. mtd</u>: Finance Solver After 5 years at 15% depreciation  Finance Solver N: 5. I(%): -15. PV: -180000. Pmt: 0. FV: 79866.956250001 PpY: 1 Finance Solver info stored into tvm.n, tvm.i, tvm.pv, tvm.pmt, ...	**M1**
	After another 3 years at 10%, using balance after 5 years  Finance Solver N: 3. I(%): -10. PV: -79867. Pmt: 0. FV: 58223.011106251 PpY: 1 Finance Solver info stored into tvm.n, tvm.i, tvm.pv, tvm.pmt, ...	**A1**

(b)	<u>Method 1:</u> Using Theory of Interest and IB's Method 2 (Compounding the Inflation Rate) $RV=\dfrac{180,000\left(1-\dfrac{15}{100}\right)^5\left(1-\dfrac{10}{100}\right)^3}{\left(1+\dfrac{1.5}{100}\right)^8}$ $=\$51,685.21$ (accept $\$51,685.20$, based on float6) <u>Method 2:</u> Using IB's Method 1 (Substracting Inflation Rate) $RV=180,000\left(1-\dfrac{15}{100}-\dfrac{1.5}{100}\right)^5\left(1-\dfrac{10}{100}-\dfrac{1.5}{100}\right)^3$ $=\$50,644.78$ (accept $\$50,644.80$, based on float6)	**M1** (dividing by $(1.015)^8$) **A1** **M1** **A1** (allow FT)
5	***Sketching of Rational Functions using GDC***	**[Marks: 8]**
(a)	Use exaggerated proportions for sketch 	**A1** y-intercept **A1** x-intercept **A1** min. point and $\lim\limits_{x\to-\infty} f(x)=0^-$ **A1** $\lim\limits_{x\to\infty} f(x)=0^+$ and horizontal asymptote. **A1** vertical asymptotes $x=-2,\ x=-1$ **A1** max point and graph for $-2<x<-1$
(b)	From the graph, for $f(x)=k$ to have no solutions, $-13.9<k<-0.0718$	**A1 A1** (penalize 1m if " $\leq$ " is used)

Qn	**Suggested solution**	**Markscheme**
6	***System of Linear Equations***	**[Marks: 8]**
(a)	Using either RREF or Linear Solve on GDC: Coordinates: $(-0.25,3,-0.25)$	**M1** **A1** (must be in coordinate form)
	 $\text{rref}\left(\begin{bmatrix}1 & 2 & 3 & 5\\ 1 & 0 & -1 & 0\\ 0 & 1 & 4 & 2\end{bmatrix}\right) \quad \begin{bmatrix}1. & 0. & 0. & -0.25\\ 0. & 1. & 0. & 3.\\ 0. & 0. & 1. & -0.25\end{bmatrix}$ $\text{linSolve}\left(\begin{cases}x+2\cdot y+3\cdot z=5\\ x-z=0\\ y+4\cdot z=2\end{cases},\{x,y,z\}\right)$ $\{-0.25,3.,-0.25\}$	
(b) i	<u>Method 1</u>: by row reduction / elementary row operations	<u>Method 1</u>
	For the three planes to meet on a line, we require the system to be consistent **and** <u>NOT</u> have three independent equations. i.e. a full row of zeroes, including the augmented part.	**(R1)**
	$\left(\begin{array}{ccc\|c}1 & 2 & 3 & 5\\ 1 & 0 & -1 & b\\ 0 & 1 & a & 2\end{array}\right) \to R_2 \to R_1 - R_2 \left(\begin{array}{ccc\|c}1 & 2 & 3 & 5\\ 0 & 2 & 4 & 5-b\\ 0 & 1 & a & 2\end{array}\right)$ $\underset{R_3 \to R_2 - 2R_3}{\to} \left(\begin{array}{ccc\|c}1 & 2 & 3 & 5\\ 0 & 2 & 4 & 5-b\\ 0 & 0 & 4-2a & 1-b\end{array}\right)$	**M1** Row reduction
	Hence, $a=2$ and $b=1$.	**A1 A1**
	<u>Method 2</u>: find a vectors concepts (or matrix determinant), then linsolve for x,y,z and b.	<u>Method 1</u>
	$\mathbf{d}=\begin{pmatrix}1\\2\\3\end{pmatrix}\times\begin{pmatrix}1\\0\\-1\end{pmatrix}=\begin{pmatrix}-2\\4\\-2\end{pmatrix}=2\begin{pmatrix}-1\\2\\-1\end{pmatrix}$	**M1**
	 	
	$\ell \parallel \Pi_3 \Rightarrow \begin{pmatrix}-1\\2\\-1\end{pmatrix}\cdot\begin{pmatrix}0\\1\\a\end{pmatrix}=0$ $\Rightarrow 0+2-a=0$ $\Rightarrow a=2$	**A1**

Qn	Suggested solution	Markscheme
	Using $a=2$ and treating b as another variable to solve for linearly:  $\text{linSolve}\left(\begin{cases}x+2\cdot y+3\cdot z=5\\x-z=b\\y+2\cdot z=2\end{cases},\{x,y,z,b\}\right)$ $\{c2+1.,2.-2.\cdot c2,c2,1.\}$ From GDC, $b=1$, and a vector equation of line is $\mathbf{r}=\begin{pmatrix}1\\2\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\-2\\1\end{pmatrix}$, where $\lambda\in\mathbb{R}$	**M1** from (b) (ii) below. **A1** for $b=1$ **A1** from (b) (ii)
(b) ii	Using $a=2$ and $b=1$,  $\text{linSolve}\left(\begin{cases}x+2\cdot y+3\cdot z=5\\x-z=1\\y+2\cdot z=2\end{cases},\{x,y,z\}\right)$ $\{c1+1.,2.-2.\cdot c1,c1\}$ A vector equation of line is $\mathbf{r}=\begin{pmatrix}1\\2\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\-2\\1\end{pmatrix}$, where $\lambda\in\mathbb{R}$	**A1** (any valid fixed point) **A1** (direction vector)
7	***Vectors – Geometrical Interpretations***	**[Marks: 7]**
(a)	Using $\mathbf{a}\cdot\mathbf{p}=\mathbf{b}\cdot\mathbf{p}$, we have $\mathbf{a}\cdot\mathbf{p}-\mathbf{b}\cdot\mathbf{p}=0$ $(\mathbf{a}-\mathbf{b})\cdot\mathbf{p}=0$ (Distributive law) $\Rightarrow \text{BA}\perp\text{OP}$ $\Rightarrow \text{AB}\perp\text{OP}$	**M1** **A1** **AG**
(b)	$\frac{1}{2}\lvert\mathbf{a}\times\mathbf{p}\rvert$ Interpretation: Area of ΔOAP	**A1**

Qn	Suggested solution	Markscheme
(c)	$\left\|\mathbf{a}\times\hat{\mathbf{p}}\right\|$ $\equiv\left\|\mathbf{a}\times\frac{\mathbf{p}}{\|\mathbf{p}\|}\right\|$ $\equiv\|\mathbf{a}\|\sin\theta$ $\equiv \text{AP}$  Note: Reasoning has to be clear. It is not sufficient to say "projection, hence .."	**M1** (sin θ or sin AOP) **AG**
(d)	In terms of $\|\mathbf{p}\|$, $\tan 30° = \frac{\text{AP}}{\|\mathbf{p}\|}$ $\therefore \text{AP} = \|\mathbf{p}\|\tan 30° = \frac{1}{\sqrt{3}}\|\mathbf{p}\|$ $k=\frac{1}{\sqrt{3}}$	**M1** **A1**
(e)	Similarly, $\tan 60° = \frac{\text{BP}}{\|\mathbf{p}\|}$ $\therefore \text{BP} = \|\mathbf{p}\|\tan 60° = \sqrt{3}\|\mathbf{p}\|$ $\frac{\text{AP}}{\text{PB}} = \frac{\frac{1}{\sqrt{3}}\|\boldsymbol{p}\|}{\sqrt{3}\|\boldsymbol{p}\|} = \frac{1}{3}$ $\text{AP}:\text{PB} = 1:3$	**A1** **A1**

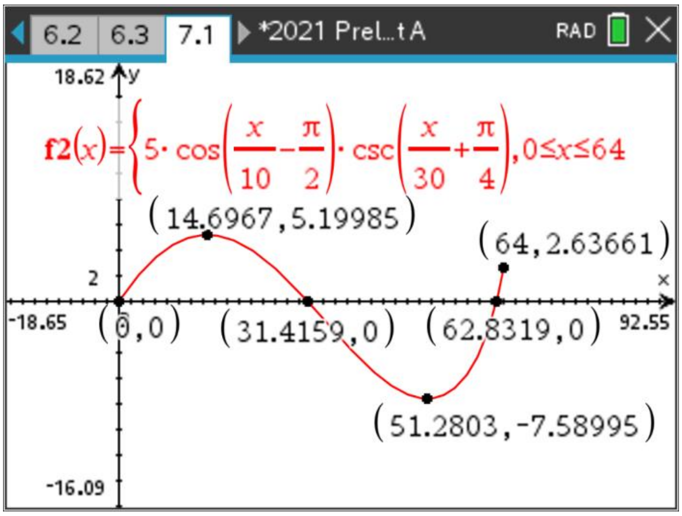
Section B

<table>
<tr><th>Qn</th><th>Suggested Solutions</th><th>Marks</th></tr>
<tr><td>8</td><td><i>Differential graphs and Kinematics</i></td><td>[Marks: 13]</td></tr>
<tr><td>(a)</td><td>

Maximum speed means maximum of $|v|$.

Comparing max/min points (14.7, **5.20**) and (51.3, **–7.59**) ,

Maximum $|v| = 7.59$ m/s



</td><td>

M1 Comparing

A1 Finding either pt

A1

</td></tr>
<tr><td>(b) i</td><td>

Comes to rest $\Rightarrow v = 0$

$t_1 = 31.4\text{s}$

Alternatively,

$$\frac{t_1}{10} - \frac{\pi}{2} = \frac{\pi}{2}$$
$$t_1 = 10\pi\ s$$

</td><td>

(R1)

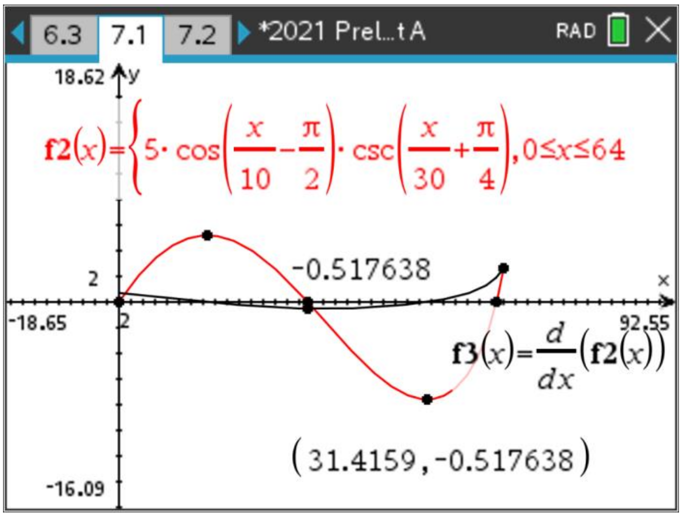
A1

A1

</td></tr>
<tr><td>(b) ii</td><td>

$a(31.4159) = \left.\frac{\mathrm{d}v}{\mathrm{d}t}\right|_{t=31.4159}$

Finding either the coordinate on the graph of $v'(t)$ or the gradient of $v(t)$ when $t = 31.4159$



Acceleration at t_1 is $-0.518\,\text{m}\,\text{s}^{-1}$

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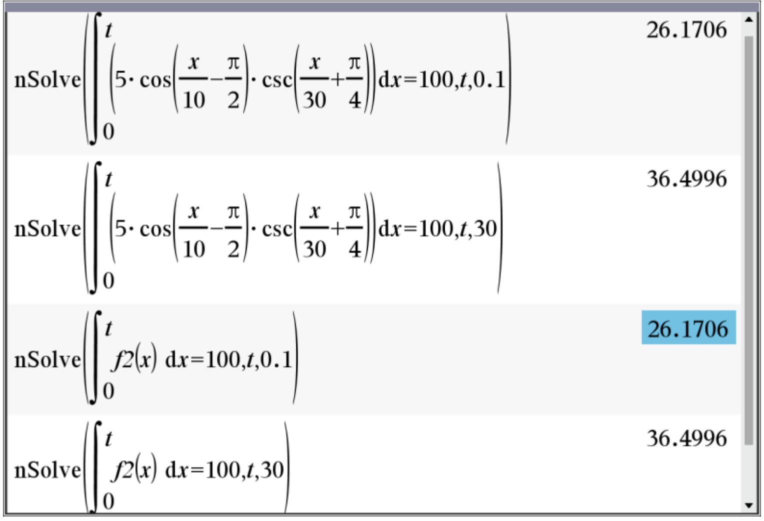
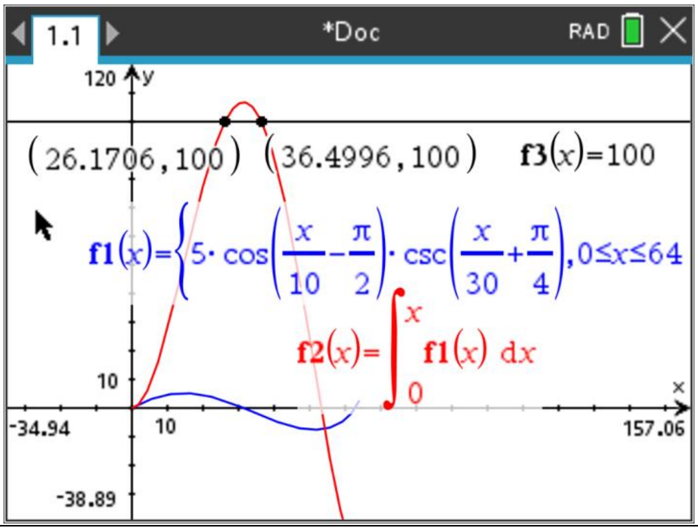
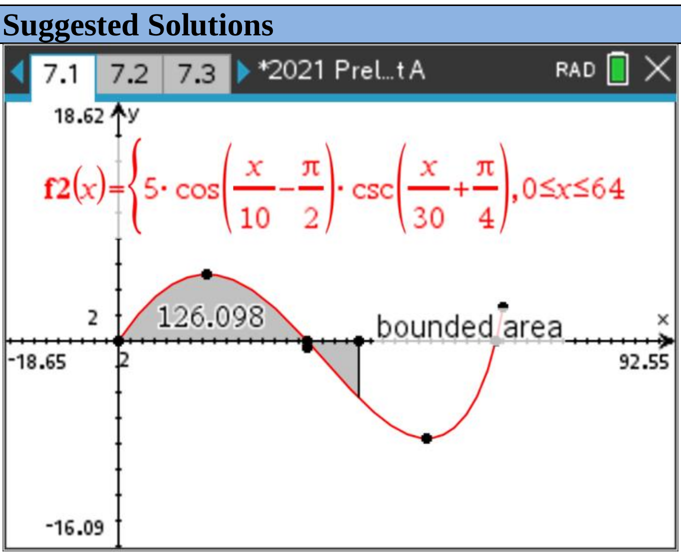
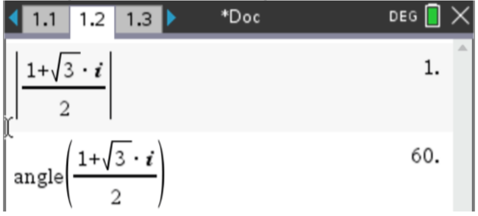
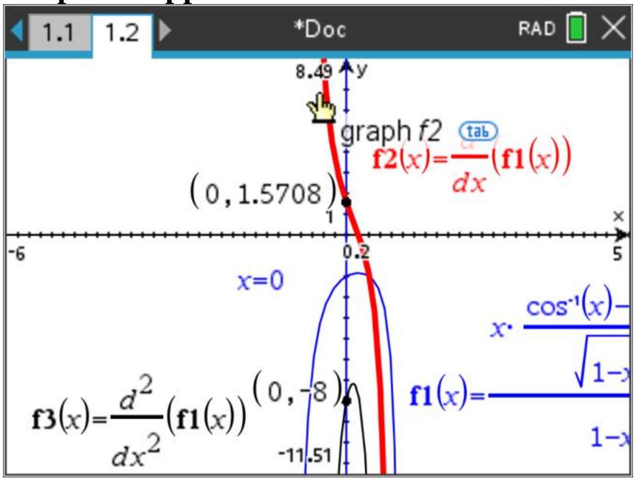
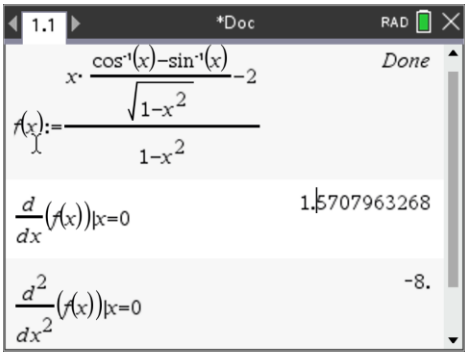
M1

A1

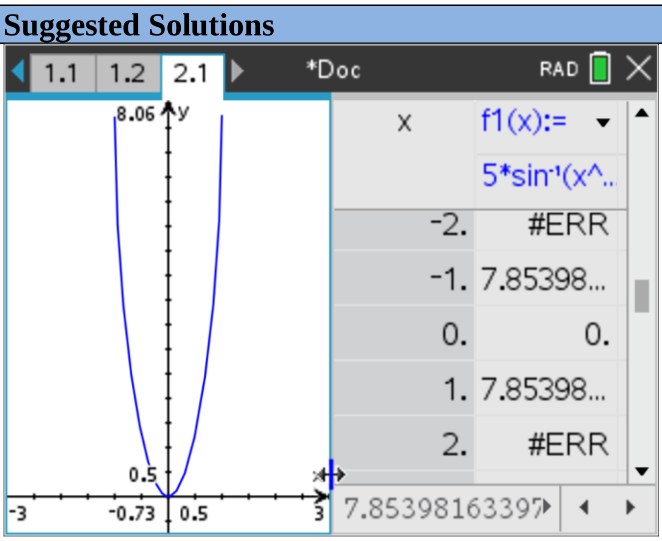
</td></tr>
</table>

Qn	Suggested Solutions	Marks
(c)	$\int_0^t v(x)\,\mathrm{d}x = 100$ Use nsolve to find the two solutions by restricting the range.  $\text{nSolve}\left(\int_0^t \left(5\cdot\cos\left(\frac{x}{10}-\frac{\pi}{2}\right)\cdot\csc\left(\frac{x}{30}+\frac{\pi}{4}\right)\right)\mathrm{d}x=100,t,0.1\right)$ 26.1706 $\text{nSolve}\left(\int_0^t \left(5\cdot\cos\left(\frac{x}{10}-\frac{\pi}{2}\right)\cdot\csc\left(\frac{x}{30}+\frac{\pi}{4}\right)\right)\mathrm{d}x=100,t,30\right)$ 36.4996 $\text{nSolve}\left(\int_0^t f2(x)\,\mathrm{d}x=100,t,0.1\right)$ 26.1706 $\text{nSolve}\left(\int_0^t f2(x)\,\mathrm{d}x=100,t,30\right)$ 36.4996 **Alternative:**  $(26.1706,100)$ $(36.4996,100)$ $f3(x)=100$ $f1(x)=\left\{5\cdot\cos\left(\frac{x}{10}-\frac{\pi}{2}\right)\cdot\csc\left(\frac{x}{30}+\frac{\pi}{4}\right),0\le x\le 64\right.$ $f2(x)=\int_0^x f1(x)\,\mathrm{d}x$	**M1** **A1** **A1** **M1** **A1 A1**
(d)	Total distance travelled in the first 40s $=\int_0^{40}\lvert v(t)\rvert\,\mathrm{d}t$ $=126$ m  $\int_0^{40}\lvert f2(x)\rvert\,\mathrm{d}x$ 126.098	**M1** formula **A1** **M1** GDC work. Either by integration or bounded area (below)

Qn	**Suggested Solutions**	**Marks**
		
9	**Complex numbers, Roots of Polynomials**	**[Maximum mark: 13]**
(a)	$P(z)=(z-4)(z^2-z+1)$ $z=\frac{1\pm\sqrt{1-4}}{2}$ $\beta,\gamma=\frac{1\pm\sqrt{3}i}{2}$	**M1A1** **M1** **A1A0**
(b)i	Using GDC, or using arctan.  $\left\lvert\frac{1+\sqrt{3}\cdot i}{2}\right\rvert$ 1. $\text{angle}\left(\frac{1+\sqrt{3}\cdot i}{2}\right)$ 60. $\beta=e^{i\frac{\pi}{3}},\gamma=e^{-i\frac{\pi}{3}}$ Using De Moivre's Theorem, $\beta^3=\left(e^{i\frac{\pi}{3}}\right)^3=e^{i\pi}=-1$ $\gamma^3=\left(e^{-i\frac{\pi}{3}}\right)^3=e^{-i\pi}=-1$	**(M1)** **A1A1** **A1** **A1**
(b)ii	$\left(e^{\pm\frac{i\pi}{3}}\right)^m=e^{ik\pi}$, where $k\in\mathbb{Z}$ $\Rightarrow m=3k$ For smallest value of *m*, $m=3$	**(M1)** **A1 N2**
(c)iii	$\left(e^{\pm\frac{i\pi}{3}}\right)^n=e^{i\left(2k\pi\pm\frac{\pi}{3}\right)}$, where $k\in\mathbb{Z}$ $\Rightarrow\pm\frac{n}{3}=2k\pm\frac{1}{3}$ Without loss of generality, we have $\frac{n}{3}=2k\pm\frac{1}{3}$. That is, $n=6k\pm1,k\in\mathbb{Z}$ Alt.: Use of Argand diagram for the above / Guess and check Smallest $n=5$	**(M1)** **A1 N2**

Qn	**Suggested Solutions**	**Marks**
10	**Calculus, inverse Trigo, Functions**	**[Maximum mark: 12]**
(a)i	$f(x) = \arccos x \arcsin x$ $f'(x) = \dfrac{\arccos x}{\sqrt{1-x^2}} - \dfrac{\arcsin x}{\sqrt{1-x^2}}$ $\sqrt{1-x^2}\, f'(x) = \arccos x - \arcsin x$	**M1** Product Rule **A1** **AG**
(a)ii	$\sqrt{1-x^2}\, f'(x) = \arccos x - \arcsin x$ $\dfrac{-2x}{2\sqrt{1-x^2}}\, f'(x) + \sqrt{1-x^2} f''(x) = \dfrac{-1}{\sqrt{1-x^2}} - \dfrac{1}{\sqrt{1-x^2}}$ $\dfrac{-2x}{2\sqrt{1-x^2}}\, f'(x) + \sqrt{1-x^2} f''(x) = \dfrac{-2}{\sqrt{1-x^2}}$ $2x\, f'(x) - 2(1-x^2) f''(x) = 4$ $(1-x^2) f''(x) + 2 = x\, f'(x)$	**M1** **A1 A1** **AG**
(b)	**Graphical approach:**  $f^{(3)}(0) = 1.57$ $f^{(4)}(0) = -8$ **Alternative method:**  1.1 *Doc RAD $f(x):=\dfrac{x\cdot\dfrac{\cos^{-1}(x)-\sin^{-1}(x)}{\sqrt{1-x^2}}-2}{1-x^2}$ Done $\dfrac{d}{dx}(f(x))\|x=0$ 1.5707963268 $\dfrac{d^2}{dx^2}(f(x))\|x=0$ -8.	**M1** **A1** **A1** **M1** **A1** **A1**

Qn	Suggested Solutions	Marks
(c)	$f(0)=0;\ f'(0)=\frac{\pi}{2};\ f''(0)=-2,$ $f^{(3)}(0)=1.57\ or\ \frac{\pi}{2},\ f^{(4)}(0)=-8$ $f(x)=\frac{\pi}{2}x-\frac{2x^2}{2!}+\frac{\pi}{2(3!)}x^3-\frac{8}{4!}x^4$ $f(x)=\frac{\pi}{2}x-x^2+\frac{\pi}{12}x^3-\frac{1}{3}x^4$	**A2** (all 3) **A1** (1 wrong) **M1** **A1**
11	**Integration**	**[Maximum mark: 17]**
(a)i	$\int_0^x 2t\sin(t^2)\,dt$ $=\ [-\cos t]_0^{x^2}$ $=1-\cos(x^2)$	**M1** **A1**
(a)ii	$\cos\left(x^2+\frac{\pi}{2}\right)=\cos(x^2)\cos\frac{\pi}{2}-\sin(x^2)\sin\frac{\pi}{2}$ $=-\sin(x^2)$ (shown) $\int_0^x 2t^3\sin\left(t^2+\frac{\pi}{2}\right)dt$ $=\int_0^x t^2[2t\sin\left(t^2+\frac{\pi}{2}\right)]\,dt$ [Integrating by parts, and using results from (a)i] $\left[-t^2\cos\left(t^2+\frac{\pi}{2}\right)\right]_0^x+\int_0^x 2t\cos\left(t^2+\frac{\pi}{2}\right)dt$ $=-x^2\cos\left(x^2+\frac{\pi}{2}\right)-\int_0^x 2t\sin(t^2)\,dt$ $=x^2\sin(x^2)+\cos(x^2)-1$	**M1 A1** **AG** **M1** **A1** (by parts) **A1** **A1 AG**
(b)i	Max $D_f=[-1,1]$	**A1**

Qn	Suggested Solutions	Marks
	 x | f1(x):= 5*sin⁻¹(x^.. -2. | #ERR -1. | 7.85398... 0. | 0. 1. | 7.85398... 2. | #ERR 7.85398163397 Height of bowl = 5 (arcsin 1 − arcsin 0) $= \frac{5\pi}{2}$	**M1** **A1**
(b)ii	$x^2 = \sin\frac{y}{5}$ Volume = $\pi\int_0^h \sin\frac{y}{5}\,dy$ $= -5\pi\left[\cos\left(\frac{y}{5}\right)\right]_0^h$ $= -5\pi\left[\cos\left(\frac{h}{5}\right) - 1\right]$ $= 5\pi\left[1 - \cos\left(\frac{h}{5}\right)\right]$	**M1** **A1** **A1**
iii	$V = 5\pi\left[1 - \cos\left(\frac{h}{5}\right)\right]$ $\frac{dV}{dh} = \pi\sin\left(\frac{h}{5}\right)$ $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$ $0.3 = \pi\sin\frac{h}{5} \times \frac{dh}{dt}$ When $h = 4$, $\frac{dh}{dt} = 0.133$ units/s	**A1** **M1** **A1**

STUDENT NAME: ______________________________

TEACHER NAME: ______________________________

ST. JOSEPH'S INSTITUTION
YEAR 6 PRELIMINARY EXAMINATION 2021

MATHEMATICS: ANALYSIS AND APPROACHES

HIGHER LEVEL

PAPER 3

Friday

9 July 2021

1 hr

0800 – 0900 hrs

INSTRUCTIONS TO CANDIDATES

- Write your name and teacher's name in the spaces provided.
- Do not open this examination paper until instructed to do so.
- Answer all the questions using the writing paper provided.
- The use of a scientific or examination graphical calculator is permitted in this paper.
- TI-Nspire calculators must be in Press-to-Test mode and cleared of all previous data.
- TI-84+ graphical calculators must only have permitted apps and be ram cleared.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **Mathematics: Analysis and Approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.
- This question paper consists of **5** printed pages including the Cover Sheet.

FOR MARKER USE ONLY:

Q1	Q2	TOTAL
		/55

Answer **all** questions on the writing paper provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

……

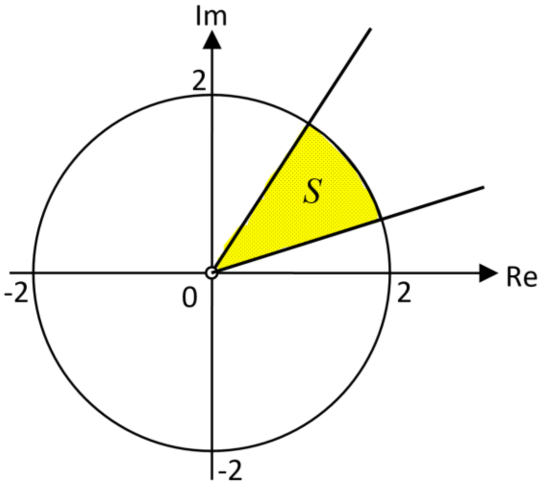
1. [Maximum mark: 24]

This question asks you to investigate some properties of complex numbers in relation to the results in trigonometry and combinations.

The complex number z is given by $z = x + y\mathrm{i}$, where $x, y \in \mathbb{R}$.

(a) Show that $x^2 + y^2 = 4$ if $|z| = 2$. [1]

(b) The point B represents the complex number z on the Argand diagram where $|z| \leq 2$ and $\tan^{-1}\left(\frac{1}{2}\right) \leq \arg z \leq \tan^{-1}(2)$. The diagram below shows the region S where B lies.



(i) Determine if $z = 1 + \mathrm{i}$ lies in S.

(ii) Calculate the area of the region S, expressing your answer in the form $r\tan^{-1}\left(\frac{p}{q}\right)$, where $r, p, q \in \mathbb{Z}^+$.

(iii) The complex number ω lies in S and is a root of the equation $z^3 + 1 = 0$. Show that $\omega^2 - \omega + 1 = 0$. [8]

The complex number v is defined as $v = \cos\theta + \mathrm{i}\sin\theta,$ where $\theta \in \mathbb{R}$.

(c) Show that $\left(1+v\right)^n = \left(2\cos\frac{\theta}{2}\right)^n \left(\cos\frac{n\theta}{2} + \mathrm{i}\sin\frac{n\theta}{2}\right),\ n \in \mathbb{N}.$ [3]

(d) By considering the binomial expansion of $\left(1+\mathrm{e}^{i\theta}\right)^n$, show that

$$1+\binom{n}{1}\cos\theta+\binom{n}{2}\cos 2\theta+\ldots+\binom{n}{r}\cos r\theta+\ldots+\cos n\theta=\left(2\cos\frac{\theta}{2}\right)^n\cos\frac{n\theta}{2},\ n\in\mathbb{N}.$$

[3]

(e) Using the result in part (d),

(i) and by considering a suitable value of θ, show that

$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\binom{n}{3}+\cdots+\binom{n}{n}=2^n,\ n\in\mathbb{N}.$$

(ii) deduce the expression of $\sum_{r=0}^{n} 2\binom{n}{r}\sin^2\left(\frac{r\theta}{2}\right)$, giving your answer in terms of n and θ. [4]

(f) By considering $\sum_{r=0}^{n}\binom{n}{r}\cos\frac{\pi r}{2}=\left(2\cos\frac{\pi}{4}\right)^n\cos\frac{n\pi}{4}$,

(i) write down the value of $1-\binom{4}{2}+\binom{4}{4}$ and $1-\binom{8}{2}+\binom{8}{4}-\binom{8}{6}+\binom{8}{8}$.

(ii) find an expression for $\sum_{r=0}^{4k}(-1)^r\binom{4k}{2r}$ in terms of k when k is odd.

(iii) find an expression for $\sum_{r=0}^{4k}(-1)^r\binom{4k}{2r}$ in terms of k when k is even.

[5]

TURN OVER

2. [Maximum mark: 31]

This question asks you to investigate an approximation for π by using integrals and numerical methods.

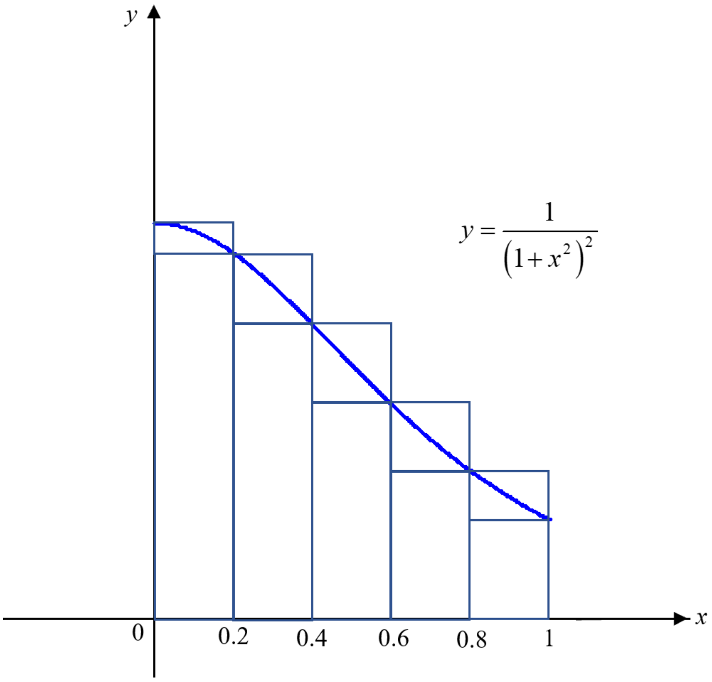
The integral I_n is defined by $I_n = \int_0^1 \frac{1}{\left(1+x^2\right)^n}\,dx,\ n \in \mathbb{Z}^+$.

(a) Find I_1. [2]

(b) Show that $2n\,I_{n+1} = 2^{-n} + \left(2n-1\right)I_n$. [5]

(c) Hence, show that $I_2 = \frac{1}{4} + \frac{\pi}{8}$. [2]

Consider the function $f(x) = \frac{1}{\left(1+x^2\right)^2},\ 0 \le x \le 1$.



The diagram shows part of the graph of $y = \frac{1}{\left(1+x^2\right)^2}$ together with line segments parallel to the coordinate axes.

(d) Using the diagram, show that $\sum_{r=1}^{5} \frac{1}{5} \cdot f\left(\frac{r}{5}\right) < \int_0^1 f(x)\,dx < \sum_{r=0}^{4} \frac{1}{5} \cdot f\left(\frac{r}{5}\right)$ [3]

(e) Use the inequality in part (d) to find a lower and upper bound for π. [3]

(f) Write down a lower and upper bound for π if the area under the curve $f(x)=\dfrac{1}{\left(1+x^2\right)^2}$ between $x=0$ and $x=1$ can be approximated by *n* rectangles with equal width. Hence find the least number of rectangles such that the upper bound and the lower bound differ by less than 0.1. [5]

(g) Consider the differential equation $\left(1+x^2\right)\dfrac{\mathrm{d}y}{\mathrm{d}x}+4xy=0$ where $y=1$ when $x=0$.

(i) Use the Euler's method with step length 0.1 to estimate the value of *y* when $x=0.6$. Show the intermediate steps to four decimal places in a table.

(ii) How can a more accurate answer be obtained using Euler's method?

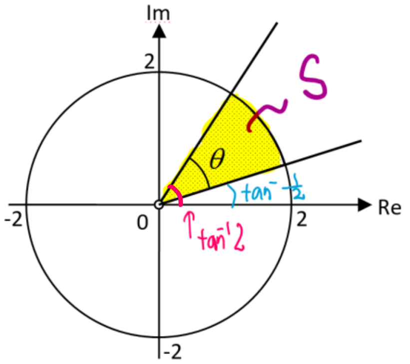
(iii) Solve the differential equation $\left(1+x^2\right)\dfrac{\mathrm{d}y}{\mathrm{d}x}+4xy=0$ given that $y=1$ when $x=0$. Hence find the value of y when $x=0.6$.

(iv) Explain why your approximate value for *y* in part (i) is greater than the actual value of *y*.

[11]

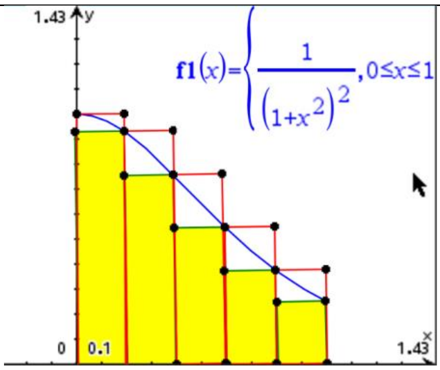
End of Paper

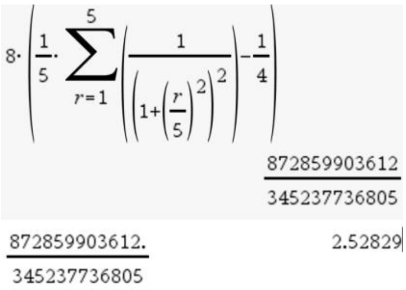
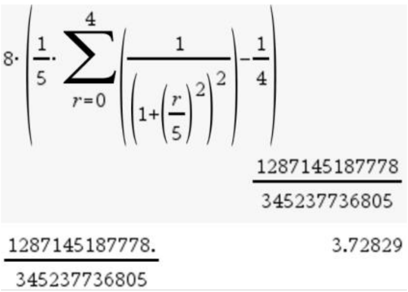
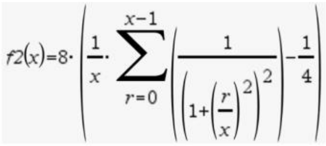
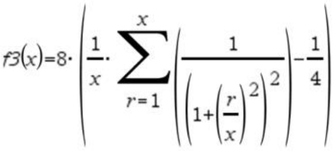
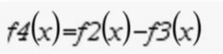
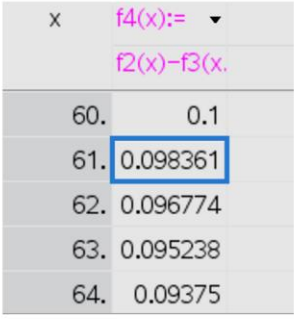
Year 6 HL Math Preliminary Examination 2021 Paper 3 (Mark Scheme)

Qn	Suggested Solutions	Marks
1	**Complex Numbers, Area of sector, nth root of unity, DMT, Trigo, binomial expansion, nCr**	**[Maximum mark: 24]**
(a)	$z = x + yi$ Given that $\lvert z\rvert = 2$, $\Rightarrow \sqrt{x^2 + y^2} = 2 \quad -(*)$ Squaring both sides of (*) $\Rightarrow x^2 + y^2 = 4$	**M1** **AG**
(b)(i)	 $z = 1 + \mathrm{i}$ $\Rightarrow \lvert z\rvert = \sqrt{2}$ and $\arg(z) = \frac{\pi}{4} = \tan^{-1}1$ Since $\lvert z\rvert = \sqrt{2} < 2$ and $\tan^{-1}\frac{1}{2} < \arg(z) = \frac{\pi}{4} = \tan^{-1}1 < \tan^{-1}2,$ z lies in S.	**M1** **R1**
(ii)	$\tan\theta = \tan\left(\tan^{-1}2 - \tan^{-1}\frac{1}{2}\right)$ $= \dfrac{2 - \frac{1}{2}}{1 + 2\left(\frac{1}{2}\right)} = \dfrac{3}{4}$ Exact area of the region S $= \frac{1}{2}r^2\theta$ $= \frac{1}{2}(2)^2\left(\tan^{-1}2 - \tan^{-1}\frac{1}{2}\right)$ $= 2\tan^{-1}\left(\frac{3}{4}\right).$	**M1A1** **M1A1**
(iii)	$z^3 + 1 = 0$ $\Rightarrow (z+1)(z^2 - z + 1) = 0$ Since ω lies in S, $\Rightarrow \omega \neq -1$. $\therefore \omega^2 - \omega + 1 = 0$ (shown)	**M1** **R1** **AG**

Qn	Suggested Solutions	Marks
(c)	$(1+v)^n$ $=(1+\cos\theta+\mathrm{i}\sin\theta)^n$ $=\left(2\cos^2\frac{\theta}{2}+\mathrm{i}2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^n$ $=\left(2\cos\frac{\theta}{2}\right)^n\left(\cos\frac{\theta}{2}+\mathrm{i}\sin\frac{\theta}{2}\right)^n$ $=\left(2\cos\frac{\theta}{2}\right)^n\left(\cos\frac{n\theta}{2}+\mathrm{i}\sin\frac{n\theta}{2}\right)$ by De Moivre's Thm	**M1[half angle formula]** **A1** **R1[DMT]** **AG**
(d)	$\left(1+e^{i\theta}\right)^n$ $=1+\binom{n}{1}e^{i\theta}+\binom{n}{2}e^{i2\theta}+\binom{n}{3}e^{i3\theta}+\ldots+\binom{n}{n}e^{in\theta}$ $=\left[1+\binom{n}{1}\cos\theta+\binom{n}{2}\cos 2\theta+\binom{n}{3}\cos 3\theta+\ldots+\cos n\theta\right]$ $+\mathrm{i}\left[\binom{n}{1}\sin\theta+\binom{n}{2}\sin 2\theta+\binom{n}{3}\sin 3\theta+\ldots+\sin n\theta\right]$ Note that $(1+\cos\theta+\mathrm{i}\sin\theta)^n=\left(1+e^{i\theta}\right)^n$ and $(1+\cos\theta+\mathrm{i}\sin\theta)^n=\left(2\cos\frac{\theta}{2}\right)^n\left(\cos\frac{n\theta}{2}+\mathrm{i}\sin\frac{n\theta}{2}\right)$ Comparing the real parts: $1+\binom{n}{1}\cos\theta+\binom{n}{2}\cos 2\theta+\binom{n}{3}\cos 3\theta+\ldots+\cos n\theta=\left(2\cos\frac{\theta}{2}\right)^n\cos\frac{n\theta}{2}\quad-(\#)$	**M1** **A1** **R1** **AG**
(e)(i)	Subst $\theta=0$ into (#) in part (e), $\Rightarrow 1+\binom{n}{1}+\binom{n}{2}+\binom{n}{3}+\cdots+\binom{n}{n}=2^n$ $\Rightarrow\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\binom{n}{3}+\cdots+\binom{n}{n}=2^n$ (shown)	**M1** **AG**
(ii)	$\sum_{r=0}^{n}\binom{n}{r}\cos r\theta=\left(2\cos\frac{\theta}{2}\right)^n\cos\frac{n\theta}{2}$ $\Rightarrow\sum_{r=0}^{n}\binom{n}{r}\left(1-2\sin^2\frac{r\theta}{2}\right)=\left(2\cos\frac{\theta}{2}\right)^n\cos\frac{n\theta}{2}$ $\Rightarrow\sum_{r=0}^{n}\binom{n}{r}\left(2\sin^2\frac{r\theta}{2}\right)=\sum_{r=0}^{n}\binom{n}{r}-\left(2\cos\frac{\theta}{2}\right)^n\cos\frac{n\theta}{2}$ $=2^n-\left(2\cos\frac{\theta}{2}\right)^n\cos\frac{n\theta}{2}$	**M1[half angle formula]** **A1** **A1**

Qn	Suggested Solutions	Marks
(f)(i)	$\sum_{r=0}^{n}\binom{n}{r}\cos\frac{\pi r}{2}=\left(2\cos\frac{\pi}{4}\right)^{n}\cos\frac{n\pi}{4}$ $\Rightarrow\sum_{r=0}^{n}\binom{n}{r}\cos\frac{\pi r}{2}=\left(\sqrt{2}\right)^{n}\cos\frac{n\pi}{4}$ Since $\cos\frac{\pi}{2}=0,\ \cos\frac{2\pi}{2}=-1,\ \cos\frac{3\pi}{2}=0,\ \cos\frac{4\pi}{2}=1,$ $\binom{n}{0}\cos\frac{\pi(0)}{2}+\binom{n}{1}\cos\frac{\pi}{2}+\binom{n}{2}\cos\frac{2\pi}{2}+\binom{n}{3}\cos\frac{3\pi}{2}+\binom{n}{4}\cos\frac{4\pi}{2}$ $+\binom{n}{5}\cos\frac{5\pi}{2}+\binom{n}{6}\cos\frac{6\pi}{2}+\binom{n}{7}\cos\frac{7\pi}{2}+\binom{n}{8}\cos\frac{8\pi}{2}+\ldots$ $=\left(\sqrt{2}\right)^{n}\cos\frac{n\pi}{4}$ $1+\binom{n}{1}(0)+\binom{n}{2}(-1)+\binom{n}{3}(0)+\binom{n}{4}(1)$ $+\binom{n}{5}(0)+\binom{n}{6}(-1)+\binom{n}{7}(0)+\binom{n}{8}(1)+\ldots$ $=\left(\sqrt{2}\right)^{n}\cos\frac{n\pi}{4}$	**R1**
	when $n=4$, $\Rightarrow\sum_{r=0}^{4}\binom{4}{r}\cos\frac{\pi r}{2}=\left(\sqrt{2}\right)^{4}\cos\pi$ $\Rightarrow 1-\binom{4}{2}+\binom{4}{4}=-4$	**A1**
	when $n=8$, $\Rightarrow\sum_{r=0}^{8}\binom{8}{r}\cos\frac{\pi r}{2}=\left(\sqrt{2}\right)^{8}\cos 2\pi$ $\Rightarrow 1-\binom{8}{2}+\binom{8}{4}-\binom{8}{6}+\binom{8}{8}=16$	**A1**
(ii)	$\sum_{r=0}^{4k}(-1)^{r}\binom{4k}{2r}=\left(\sqrt{2}\right)^{4k}\cos(k\pi)$ $=(-1)^{k}4^{k}$ $=\begin{cases}-4^{k} & , \quad k \text{ is odd}\\ 4^{k} & , \quad k \text{ is even}\end{cases}$	**A1** **A1**

2	**Reduction formula, rectangle rule, approximation of pi, Euler's Method, DE, Maclaurin's Series**	**[Maximum mark: 31]**
(a)	$I_1=\int_0^1\frac{1}{1+x^2}\,\mathrm{d}x=\left[\tan^{-1}x\right]_0^1=\frac{\pi}{4}$	**M1A1**
(b)	$I_n=\int_0^1\frac{1}{\left(1+x^2\right)^n}\,\mathrm{d}x$ $=\left[x\cdot\frac{1}{\left(1+x^2\right)^n}\right]_0^1+2n\int_0^1\frac{x^2}{\left(1+x^2\right)^{n+1}}\,\mathrm{d}x$ $=\frac{1}{2^n}+2n\int_0^1\frac{1+x^2-1}{\left(1+x^2\right)^{n+1}}\,\mathrm{d}x$ $=2^{-n}+2n\int_0^1\frac{1+x^2}{\left(1+x^2\right)^{n+1}}\,\mathrm{d}x-2n\int_0^1\frac{1}{\left(1+x^2\right)^{n+1}}\,\mathrm{d}x$ $=2^{-n}+2nI_n-2nI_{n+1}$ $\Rightarrow 2nI_{n+1}=2^{-n}+\left(2n-1\right)I_n$ (shown)	**M1[by parts]** **A1** **M1** **A1** **A1** **AG**
(c)	Let $n=1$, $\Rightarrow 2I_2=2^{-1}+I_1$ $\Rightarrow 2I_2=\frac{1}{2}+\frac{\pi}{4}$ $\Rightarrow I_2=\frac{1}{4}+\frac{\pi}{8}$ (shown)	**M1** **A1**
(d)	 The area under the curve is sandwiched between the sum of the areas of the lower rectangles and the upper rectangles. $\int_0^1 f(x)\,\mathrm{d}x=$ Area under the graph bounded by the x-axis, $x=0, x=1$. Sum of Areas of the lower rectangles $=\frac{1}{5}f\left(\frac{1}{5}\right)+\frac{1}{5}f\left(\frac{2}{5}\right)+\ldots+\frac{1}{5}f\left(1\right)=\sum_{r=1}^{5}\frac{1}{5}f\left(\frac{r}{5}\right)$ Sum of Areas of the upper rectangles $=\frac{1}{5}f\left(\frac{0}{5}\right)+\frac{1}{5}f\left(\frac{1}{5}\right)+\ldots+\frac{1}{5}f\left(\frac{4}{5}\right)=\sum_{r=0}^{4}\frac{1}{5}f\left(\frac{r}{5}\right)$ $\therefore\sum_{r=1}^{5}\frac{1}{5}f\left(\frac{r}{5}\right)<\int_0^1 f(x)\,\mathrm{d}x<\sum_{r=0}^{4}\frac{1}{5}f\left(\frac{r}{5}\right)$	**R1** **M1[for lower rectangles]** **M1[for upper rectangles]** **AG**

(e)	$\frac{1}{5}\sum_{r=1}^{5} f\left(\frac{r}{5}\right) < \int_0^1 f(x)\,\mathrm{d}x < \frac{1}{5}\sum_{r=0}^{4} f\left(\frac{r}{5}\right)$ $\Rightarrow \frac{1}{5}\sum_{r=1}^{5} f\left(\frac{r}{5}\right) < \frac{1}{4}+\frac{\pi}{8} < \frac{1}{5}\sum_{r=0}^{4} f\left(\frac{r}{5}\right)$	
	$\Rightarrow 8\left(\frac{1}{5}\sum_{r=1}^{5} f\left(\frac{r}{5}\right)-\frac{1}{4}\right) < \pi < 8\left(\frac{1}{5}\sum_{r=0}^{4} f\left(\frac{r}{5}\right)-\frac{1}{4}\right)$	**M1**
	 $8\cdot\left(\frac{1}{5}\cdot\sum_{r=1}^{5}\left(\frac{1}{\left(1+\left(\frac{r}{5}\right)^2\right)^2}\right)-\frac{1}{4}\right)$ $\frac{872859903612}{345237736805}$ $\frac{872859903612.}{345237736805}$ 2.52829  $8\cdot\left(\frac{1}{5}\cdot\sum_{r=0}^{4}\left(\frac{1}{\left(1+\left(\frac{r}{5}\right)^2\right)^2}\right)-\frac{1}{4}\right)$ $\frac{1287145187778}{345237736805}$ $\frac{1287145187778.}{345237736805}$ 3.72829	
	Lower bound for $\pi = 2.53$ (3sf)	**A1**
	Upper bound for $\pi = 3.73$ (3sf)	**A1**
(f)	$8\left(\frac{1}{n}\sum_{r=1}^{n} f\left(\frac{r}{n}\right)-\frac{1}{4}\right) < \pi < 8\left(\frac{1}{n}\sum_{r=0}^{n-1} f\left(\frac{r}{n}\right)-\frac{1}{4}\right)$	**A1A1**
	$\therefore 8\left(\frac{1}{n}\sum_{r=0}^{n-1} f\left(\frac{r}{n}\right)-\frac{1}{4}\right)-8\left(\frac{1}{n}\sum_{r=1}^{n} f\left(\frac{r}{n}\right)-\frac{1}{4}\right) < 0.1$	**M1**
	 $f2(x)=8\cdot\left(\frac{1}{x}\cdot\sum_{r=0}^{x-1}\left(\frac{1}{\left(1+\left(\frac{r}{x}\right)^2\right)^2}\right)-\frac{1}{4}\right)$  $f3(x)=8\cdot\left(\frac{1}{x}\cdot\sum_{r=1}^{x}\left(\frac{1}{\left(1+\left(\frac{r}{x}\right)^2\right)^2}\right)-\frac{1}{4}\right)$  $f4(x)=f2(x)-f3(x)$	
	 x f4(x):= f2(x)−f3(x. 60. 0.1 61. 0.098361 62. 0.096774 63. 0.095238 64. 0.09375	**M1**
	Least n is 61.	**A1**

(g)(i)

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-4xy}{1+x^2}$$

Euler's formula: $y_{n+1}=y_n+0.1\left(\frac{-4x_ny_n}{1+x_n{}^2}\right)$ **(M1)**

M1

A1

x_n	y_n
0	1
0.1	1
0.2	0.9604
0.3	0.8865
0.4	0.7889
0.5	0.6801
0.6	**0.5713**

By Euler's method, when $x=0.6,\ \ y=0.571$ **A1**

(ii) Decrease the step size. **A1**

(iii)

$$\left(1+x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x}+4xy=0$$

$$\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-4xy}{1+x^2}$$

M1 (separable variables)

$$\Rightarrow\int\frac{1}{y}\,\mathrm{d}y=\int\frac{-4x}{1+x^2}\ \mathrm{d}x$$

$$\Rightarrow\ln|y|=-2\ln\left|1+x^2\right|+C$$ **A1**

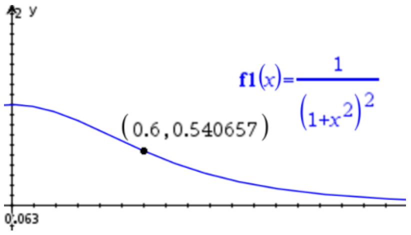
$$\Rightarrow|y|=\mathrm{e}^{-2\ln\left|1+x^2\right|+C}=\mathrm{e}^{-2\ln\left|1+x^2\right|}\cdot\mathrm{e}^C$$

$$\Rightarrow|y|=\frac{1}{\left(1+x^2\right)^2}\cdot\mathrm{e}^C$$

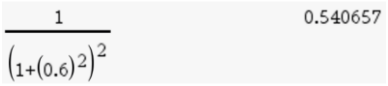
$$\Rightarrow y=\frac{A}{\left(1+x^2\right)^2},\text{ where }A=\pm\mathrm{e}^C$$

$y=1$ when $x=0\Rightarrow A=1$ **M1**

$$\Rightarrow y=\frac{1}{\left(1+x^2\right)^2}\qquad-(\wedge)$$ **A1**



Using $(\wedge)$, when $x=0.6,\ \ y=0.541$ (actual value of y) **A1**



$\frac{1}{\left(1+(0.6)^2\right)^2}$ 0.540657

[Note: By Euler's method, when $x=0.6,\ \ y=0.571$]

(iv) Approximate value of y in part (i) is **greater** than actual value of y because it is assumed that $\frac{\mathrm{d}y}{\mathrm{d}x}$ remains constant throughout each interval when it is actually a decreasing function. [Note that there is a point of inflexion at $(0.577,0.75)$] **R1**

STUDENT NAME: ______________________________

TEACHER NAME: ______________________________

CANDIDATE SESSION NUMBER

0	2	5	0	1	2				

EXAMINATION CODE

8	8	2	2	-	7	1	0	1

ST JOSEPH'S INSTITUTION
YEAR 6 PRELIMINARY EXAMINATION 2022

MATHEMATICS: ANALYSIS AND APPROACHES
HIGHER LEVEL
PAPER 1
Monday

15 August 2022
2 hours
1400 – 1600 hrs

INSTRUCTIONS TO CANDIDATES

- Write your session number in the boxes above.
- Write your name and your teacher's name in the spaces provided.
- Do not open this examination paper until instructed to do so.
- **Section A**: Answer all questions showing working and answers in the spaces provided in the exam paper.
- **Section B**: Answer all questions using the writing paper provided.
- The use of calculators is **not** permitted in this paper.
- A clean copy of the **Mathematics: Analysis and Approaches formula booklet** is required for this paper.
- Unless otherwise stated in the question, all numerical answers are to be given exactly or correct to three significant figures.
- The maximum mark for this examination paper is **[110 marks]**.
- This question paper consists of **12** printed pages including the Cover Sheet.
- Sections A and B are to be submitted **separately**.

FOR MARKER USE ONLY:

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	TOTAL
										/110

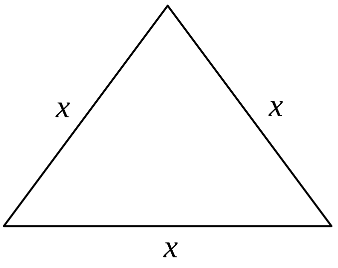
Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are advised to show all working.

SECTION A (55 marks)

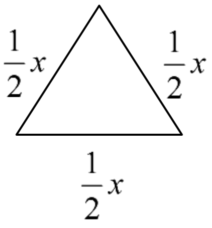
Answer **all** questions in the spaces provided.

1 **[Maximum mark: 5]**

Consider a sequence of equilateral triangles, where the length of a side of each triangle is half that of the previous triangle. The length of a side of Triangle 1 is x cm, and Triangles 1 and 2 are shown below.



Triangle 1



Triangle 2

(a) Find the area of Triangle 1 in terms of x. [2]

(b) The sequence of triangles continues to infinity. Given that the **total** area of all the triangles is $27\sqrt{3}\ \text{cm}^2$, find the value of x. [3]

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2 **[Maximum mark: 7]**

Anthony cycles on the weekends for leisure. Anthony has an equal chance of cycling or not on any given Saturday. If Anthony cycles on Saturday, then he has a probability of 0.4 of not cycling on Sunday. If he does not cycle on Saturday, then he has a probability of 0.2 of not cycling on Sunday as well.

(a) Find the probability that Anthony cycles on exactly one day in a typical weekend. [2]

(b) Find the probability that Anthony cycled on exactly one day per weekend over all the four weekends in June. (You do not need to evaluate your answer.) [2]

(c) On a typical weekend, given that Anthony cycled on Sunday, find the probability that he also cycled on Saturday. [3]

3 **[Maximum mark: 6]**

(a) Find $\frac{\mathrm{d}}{\mathrm{d}x}\left(\cos^2 x\right)$. [2]

(b) Find the area bounded by the curve $y = \frac{\sin 2x}{\sqrt{2-\cos^2 x}}$, the x-axis, and the lines $x = \frac{\pi}{4}$ and $x = \frac{\pi}{2}$. [4]

TURN OVER

4 **[Maximum mark: 7]**

(a) Find the coefficient of x^4 in the expansion of $\left(x^2-1\right)^5$. [2]

(b) Hence or otherwise, find the term in x^4 in the expansion of $\left(x^2-1\right)^5\sqrt{1-x^2}$. [4]

(c) Write down the range of values of *x* for which the expansion in **(b)** is valid. [1]

TURN OVER

5 **[Maximum mark: 11]**

Consider the function $f\left(x\right)=\dfrac{x^2}{1+\ln x}$, defined on its maximal domain.

(a) Find the maximal domain of f. [3]

(b) Find the equation of the tangent to the curve $y=f\left(x\right)$ at the point where $x=\mathrm{e}$. [5]

(c) Find the range of values of x for which f is increasing. [3]

MORE SPACE IS AVAILABLE ON THE NEXT PAGE

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6 **[Maximum mark: 12]**

Let $f_k(x) = kx^2 + x + k,\ x \in \mathbb{R}$, where *k* is a real constant.

(a) Find the range of values of *k* for which $f_k(x)$ is always positive. [4]

Let $k = 2$.

(b) Find the coordinates of the turning point of the graph of $y = f_2(x)$. [3]

(c) Sketch the graph of $y = \dfrac{1}{f_2(x) - 2}$, labelling clearly the coordinates of all turning point(s) and the equations of the asymptote(s). [5]

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7 **[Maximum mark: 7]**

Consider two skew lines $l_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 5 \\ 1 \end{pmatrix},\ \lambda \in \mathbb{R}$ and $l_2 : \mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix},\ \mu \in \mathbb{R}$.

(a) Find a vector that is perpendicular to both lines. [3]

(b) Find the shortest distance between the two skew lines. [4]

TURN OVER

*Do **NOT** write solutions on this page.*

SECTION B (55 marks)

*Answer **all** questions on the writing paper provided. **Please start each question on a new page.***

8 [Maximum mark: 20]

Consider the equation $z^4 + 2z^2 - 4z + 8 = 0$, where $z \in \mathbb{C}$.

(a) Given that $z_1 = -1 + \sqrt{3}\mathrm{i}$ is a root, find all other roots of the equation. [5]

(b) Let z_2 be the root for which $\arg(z_2) = \dfrac{\pi}{4}$.

(i) Find $\dfrac{z_1}{z_2}$, leaving your answer in the form $x + \mathrm{i}y$, where $x, y \in \mathbb{R}$.

(ii) Consider the modulus and argument of z_1 and z_2 to find $\left|\dfrac{z_1}{z_2}\right|$ and show that

$$\arg\left(\frac{z_1}{z_2}\right) = \frac{5\pi}{12}.$$

(iii) With the results in (i) and (ii), determine the exact value of $\cos\left(\dfrac{5\pi}{12}\right)$. [11]

(c) Complex numbers z_1 and z_2 are represented by points A and B respectively on an Argand diagram.

(i) Sketch the Argand diagram with A and B, showing clearly the moduli and arguments of the complex numbers represented.

(ii) Show that the **square** of length AB is $8 - 2\sqrt{3}$ units^2. [4]

TURN OVER

*Do **NOT** write solutions on this page.*

9 **[Maximum mark: 15]**

(a) **(i)** Show that $\sum_{r=1}^{n}\left((r+1)^3-r^3\right)=(n+1)^3-1$.

(ii) Hence, show that $\sum_{r=1}^{n}r^2=\frac{n(n+1)(2n+1)}{6}$. [7]

(b) A deck of cards is labelled with the numbers 1, 2, 3, … , *n* such that there are *r* cards labelled with the number *r*. That is, there is one card with the number 1, two cards with the number 2 and so on. A card is drawn randomly from the deck and *X* represents the number labelled on the card drawn.

(i) Show that $\mathrm{P}(X=r)=\frac{2r}{n(n+1)}$, where $1\le r\le n$.

(ii) Find E(*X*) in terms of *n*.

(iii) Given that $n=4$, find Var(*X*). [8]

10 **[Maximum mark: 20]**

Suppose $f_k(x)=x\mathrm{e}^{kx}$, where *k* is a non-zero real constant.

(a) **(i)** Find $f_k{}'(x)$.

(ii) Find the value of *x* for which $f_k{}'(x)=0$.

(iii) Given that the stationary point of each of the graphs of $y=f_k(x)$, for all *k*, lies on the same straight line $y=mx+c$, determine the value of *m* and of *c*. [8]

(b) The *n*th derivative of $f_k(x)$ is denoted by $f_k{}^{(n)}(x)$.

Prove by mathematical induction $f_k{}^{(n)}(x)=nk^{n-1}\mathrm{e}^{kx}+k^n x\mathrm{e}^{kx}$ for all $n\in\mathbb{Z}^+$. [7]

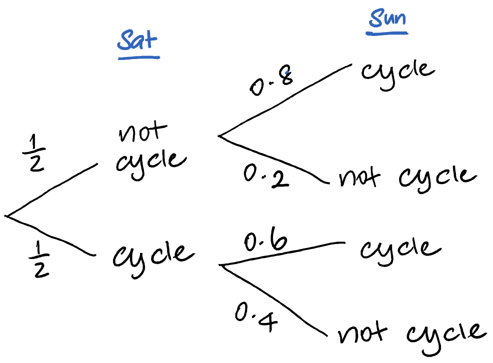
(c) Find the value of *k* if the coefficient of x^3 is equal to the coefficient of x^4 in the Maclaurin expansion of $f_k(x)$. [3]

(d) Describe geometrically an ordered sequence of transformations that map the graph of $y=f_1(x)$ to the graph of $y=f_2(x)$. [2]

End of Paper

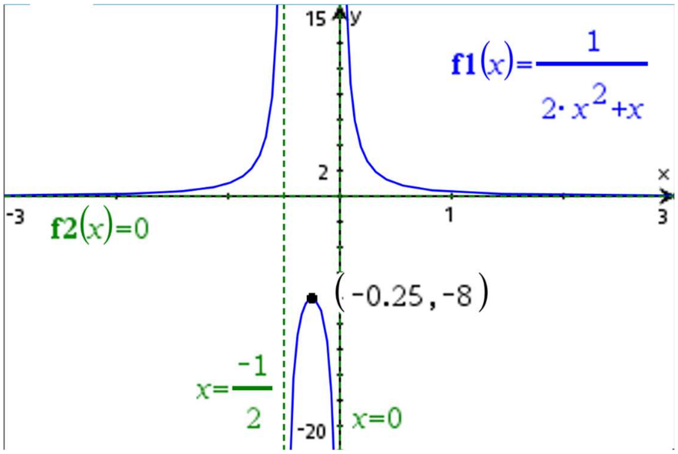
Year 6 HL MAA Preliminary Examination 2022 Paper 1 (Markscheme)

Section A

Qn	Suggested solution	Markscheme
1	***Area of Triangle & Infinite geometric series***	**[Marks: 5]**
(a)	Area of Triangle 1 $=\frac{1}{2}x^2\sin\frac{\pi}{3}$ $=\frac{\sqrt{3}}{4}x^2$	**M1** **A1**
(b)	Total area of all triangles $=\dfrac{\frac{\sqrt{3}}{4}x^2}{1-\frac{1}{4}}$ $\frac{4}{3}\left(\frac{\sqrt{3}}{4}\right)x^2=27\sqrt{3}$ $x^2=81$ Since $x>0$, $x=9$ cm	**M1** for S_∞ **A1** for $r=\frac{1}{4}$ **A1**
2	***Probability & Conditional Probability***	**[Marks: 7]**
		
(a)	$\text{P}(\text{cycle exactly one day})=\frac{1}{2}\left(\frac{2}{5}\right)+\frac{1}{2}\left(\frac{4}{5}\right)=\frac{3}{5}$ **OR** $0.5(0.4)+0.5(0.8)=0.6$	**M1 A1**
(b)	Probability $=\left(\frac{3}{5}\right)^4$ or $(0.6)^4$ $\left[=\frac{81}{625}\text{ or }0.1296\text{ (exact from }0.36\times0.36)\right]$	**M1** – power 4 **A1**
(c)	$\text{P}(\text{cycled on Sat} \mid \text{cycled on Sun})$ $=\dfrac{\frac{1}{2}\left(\frac{3}{5}\right)}{\frac{1}{2}\left(\frac{3}{5}\right)+\frac{1}{2}\left(\frac{4}{5}\right)}$ OR $\dfrac{0.5(0.6)}{0.5(0.6)+0.5(0.8)}$ $=\frac{3}{7}$	**M1** – conditional prob **A1** **A1**

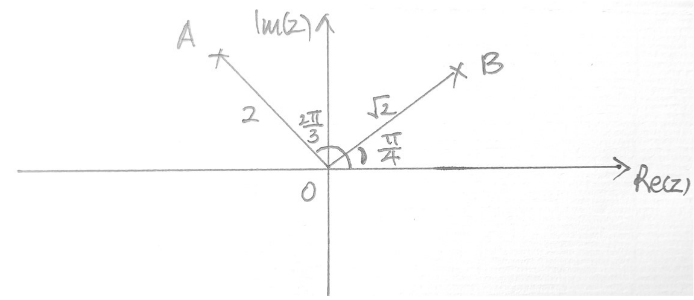
Qn	**Suggested solution**	**Markscheme**
3	***Integration – Bounded Area with Trigo***	**[Marks: 6]**
(a)	$\frac{\mathrm{d}}{\mathrm{d}x}\left(\cos^2 x\right)=2\cos x\left(-\sin x\right)$ $=-\sin 2x$	**M1** – chain rule **A1**
(b)	Note: For $\frac{\pi}{4}\le x\le\frac{\pi}{2}$, $\frac{\sin 2x}{\sqrt{2-\cos^2 x}}\ge 0$ Bounded area $=\int_{\pi/4}^{\pi/2}\frac{\sin 2x}{\sqrt{2-\cos^2 x}}\,\mathrm{d}x$ $=\int_{\pi/4}^{\pi/2}\frac{-\left(-\sin 2x\right)}{\sqrt{2-\cos^2 x}}\,\mathrm{d}x$ $=\left[\frac{\sqrt{2-\cos^2 x}}{1/2}\right]_{\pi/4}^{\pi/2}$ $=2\left(\sqrt{2-\cos^2\frac{\pi}{2}}-\sqrt{2-\cos^2\frac{\pi}{4}}\right)$ $=2\left(\sqrt{2}-\sqrt{\frac{3}{2}}\right)=\sqrt{2}\left(2-\sqrt{3}\right)$ units^2	**M1** **M1** $\int\left[f(x)\right]^{-1/2}f'(x)\,\mathrm{d}x$ **A1** **A1** – any answer on this line o.e.
4	***Binomial Theorem***	**[Marks: 7]**
(a)	General term $=\binom{5}{r}\left(x^2\right)^r\left(-1\right)^{5-r}$ For term in x^4, $r=2$ Coefficient of $x^4=\binom{5}{2}\left(-1\right)^3=-10$	**(M1)** **A1**
(b)	**<u>Hence:</u>** $\sqrt{1-x^2}=1+\frac{1}{2}\left(-x^2\right)+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}\left(-x^2\right)^2+\ldots$ $=1-\frac{1}{2}x^2-\frac{1}{8}x^4+\ldots$ Term in $x^4=-10x^4\left(1\right)+\binom{5}{1}\left(-1\right)^4x^2.\left(-\frac{1}{2}x^2\right)-\left(-\frac{1}{8}x^4\right)$ $=\left(-10-\frac{5}{2}+\frac{1}{8}\right)x^4$ $=-\frac{99}{8}x^4$ **<u>Otherwise:</u>** $\left(x^2-1\right)^5\sqrt{1-x^2}=-\left(1-x^2\right)^5\sqrt{1-x^2}$ $=-\left(1-x^2\right)^{11/2}$	**M1** – product of terms **A1** 2nd term **A1** 3rd term **A1** **A1**

Qn	Suggested solution	Markscheme
	Term in $x^4 = -\dfrac{\frac{11}{2}\left(\frac{9}{2}\right)}{2!}\left(-x^2\right)^2$ $= -\dfrac{99}{8}x^4$	**M1 A1** **A1**
(c)	$\left\|x^2\right\|<1 \quad \therefore \ \left\|x\right\|<1$ or $-1<x<1$	**A1**
5	***Maximal domain & Differentiation***	**[Marks: 11]**
(a)	f is defined for $x>0$ except when $1+\ln x=0$ i.e. $x=\dfrac{1}{\mathrm{e}}$ Maximal domain of f: $\left\{x\in\mathbb{R}: x>0, x\neq\dfrac{1}{\mathrm{e}}\right\}$ OR $\left]0,\dfrac{1}{\mathrm{e}}\right[\cup\left]\dfrac{1}{\mathrm{e}},\infty\right[$ o.e.	**(M1)** **A1** – for $x>0$ **A1** – for $x\neq\dfrac{1}{\mathrm{e}}$
(b)	$f'(x)=\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{x^2}{1+\ln x}\right)$ $=\dfrac{(1+\ln x)(2x)-\frac{x^2}{x}}{(1+\ln x)^2}$ $=\dfrac{x(1+2\ln x)}{(1+\ln x)^2}$ $f'(\mathrm{e})=\dfrac{3\mathrm{e}}{4}$ and $f(\mathrm{e})=\dfrac{\mathrm{e}^2}{2}$ Equation of tangent at $x=\mathrm{e}$ is: $y-\dfrac{\mathrm{e}^2}{2}=\dfrac{3\mathrm{e}}{4}(x-\mathrm{e})$ $y=\dfrac{3\mathrm{e}}{4}x-\dfrac{\mathrm{e}^2}{4}$ or $(3\mathrm{e})x-4y=\mathrm{e}^2$ o.e.	**M1** – quotient rule **A1** **A1** for $f'(\mathrm{e})=\dfrac{3\mathrm{e}}{4}$ **M1** **A1**
(c)	Set $f'(x)>0$ Since $x>0$ and $(1+\ln x)^2>0 \ \forall x\in D_f$, it suffices for $1+2\ln x>0$ $\therefore \ln x>-\dfrac{1}{2}$ $x>\dfrac{1}{\sqrt{\mathrm{e}}}$	**M1** **(R1)** **A1**

Qn	**Suggested solution**	**Markscheme**
6	***Quadratic Function & Graph Transformation***	**[Marks: 12]**
(a)	Leading coefficient $k > 0$ **and** $1-4k^2 < 0 \Rightarrow (2k-1)(2k+1) > 0$ $k < -\frac{1}{2}$ or $k > \frac{1}{2}$ Hence, $k > \frac{1}{2}$	**A1** **M1** – $\Delta < 0$ **A1** – both **A1**
(b)	**Method 1** $y = 2x^2 + x + 2$ $= 2\left[\left(x+\frac{1}{4}\right)^2 - \frac{1}{16}\right] + 2$ $= 2\left(x+\frac{1}{4}\right)^2 + \frac{15}{8}$ Coordinates of turning point are $\left(-\frac{1}{4}, \frac{15}{8}\right)$ **Method 2** $y = 2x^2 + x + 2$ Turning point occurs when $x = -\frac{1}{2(2)} = -\frac{1}{4}$ $\therefore$ Coordinates of turning point are $\left(-\frac{1}{4}, \frac{15}{8}\right)$	**M1** – completing square **A1** **A1** **(M1) A1** **A1**
(c)		**(M1)** – downward translation then reciprocal **A1** – 3 branches **A1** – max pt $\left(-\frac{1}{4}, -8\right)$ **A1** – $y = 0$ (H.A.) **A1** – $x = 0$ and $x = -\frac{1}{2}$ (both V.A.)

Qn	**Suggested solution**	**Markscheme**
7	***Vector Product & Skew Lines***	**[Marks: 7]**
(a)	$\begin{pmatrix}0\\5\\1\end{pmatrix}\times\begin{pmatrix}1\\2\\1\end{pmatrix}$ or $\begin{pmatrix}1\\2\\1\end{pmatrix}\times\begin{pmatrix}0\\5\\1\end{pmatrix}$	**M1** – vector product **A1** – use of **d₁** and **d₂**
	$=\begin{pmatrix}3\\1\\-5\end{pmatrix}$ $=\begin{pmatrix}-3\\-1\\5\end{pmatrix}$	**A1** (accept scalar multiples)
(b)	**<u>Method 1 (distance between parallel planes):</u>** $\prod_{l_1}: 3x+y-5z=0$	**A1**
	$\prod_{l_2}: 3x+y-5z=4$	**A1**
	Distance between l_1 and l_2 = distance betw two parallel planes containing l_1 and l_2 $=\left\lvert\dfrac{0-4}{\sqrt{3^2+1^2+(-5)^2}}\right\rvert$	**M1**
	$=\dfrac{4}{\sqrt{35}}$ or $\dfrac{4\sqrt{35}}{35}$ units	**A1**
	<u>Method 2 (scalar projection):</u> Let $\overrightarrow{AB}=\begin{pmatrix}1\\2\\1\end{pmatrix}-\begin{pmatrix}3\\0\\1\end{pmatrix}=\begin{pmatrix}-2\\2\\1\end{pmatrix}$ and $\mathbf{n}=\begin{pmatrix}3\\1\\-5\end{pmatrix}$	**A1** for $\overrightarrow{AB}$ or $\overrightarrow{BA}$
	Distance between l_1 and l_2 $=\left\lvert\overrightarrow{AB}\cdot\hat{\mathbf{n}}\right\rvert$ $=\left\lvert\begin{pmatrix}-2\\2\\0\end{pmatrix}\bullet\dfrac{1}{\sqrt{35}}\begin{pmatrix}3\\1\\-5\end{pmatrix}\right\rvert$	**M1 A1** – use of $\hat{\mathbf{n}}$
	$=\dfrac{4}{\sqrt{35}}$ or $\dfrac{4\sqrt{35}}{35}$ units	**A1**

Section B

Qn	**Suggested solution**	**Markscheme**
8	***Complex numbers & Trigo***	**[marks: 20]**
(a)	Since it is a real polynomial, $-1-\sqrt{3}\mathrm{i}$ is also a root $\left(-1+\sqrt{3}\mathrm{i}\right)+\left(-1-\sqrt{3}\mathrm{i}\right)=-2;\left(-1+\sqrt{3}\mathrm{i}\right)\left(-1-\sqrt{3}\mathrm{i}\right)=4$ $z^4+2z^2-4z+8=\left(z^2+2z+4\right)\left(z^2+az+b\right)\Rightarrow a=-2,b=2$ $z^2-2z+2=0 \quad \Rightarrow \quad z=\dfrac{2\pm\sqrt{4-8}}{2}=1\pm\mathrm{i}$	**A1** (reason not required) **M1** **M1 A1 A1**
(b) (i)	$z_2=1+\mathrm{i}$ $\dfrac{z_1}{z_2}=\dfrac{-1+\sqrt{3}\mathrm{i}}{1+\mathrm{i}}\times\dfrac{1-\mathrm{i}}{1-\mathrm{i}}$ $\dfrac{z_1}{z_2}=\dfrac{-1+\mathrm{i}+\sqrt{3}\mathrm{i}+\sqrt{3}}{2}$ $\dfrac{z_1}{z_2}=\dfrac{\sqrt{3}-1}{2}+\dfrac{\sqrt{3}+1}{2}\mathrm{i}$	**M1** **A1** **A1**
(b) (ii)	$\lvert z_1\rvert=\sqrt{(-1)^2+\left(\sqrt{3}\right)^2}=2; \qquad \lvert z_2\rvert=\sqrt{(1)^2+(1)^2}=\sqrt{2}$ $\left\lvert\dfrac{z_1}{z_2}\right\rvert=\dfrac{\lvert z_1\rvert}{\lvert z_2\rvert}=\dfrac{2}{\sqrt{2}}=\sqrt{2}$ $\arg\left(z_1\right)=\dfrac{2\pi}{3}$ $\arg\left(\dfrac{z_1}{z_2}\right)=\arg\left(z_1\right)-\arg\left(z_2\right)=\dfrac{2\pi}{3}-\dfrac{\pi}{4}=\dfrac{5\pi}{12}$	**A1** – for $\lvert z_1\rvert$ **A1** – for $\lvert z_2\rvert$ **M1 A1** **A1** – $\arg\left(z_1\right)$ **M1 AG**
(b) (iii)	$\dfrac{\sqrt{3}-1}{2}+\dfrac{\sqrt{3}+1}{2}\mathrm{i}=\sqrt{2}\mathrm{cis}\left(\dfrac{5\pi}{12}\right)\Rightarrow\sqrt{2}\cos\left(\dfrac{5\pi}{12}\right)=\dfrac{\sqrt{3}-1}{2}$ $\cos\left(\dfrac{5\pi}{12}\right)=\dfrac{\sqrt{3}-1}{2\sqrt{2}}$	**M1** **A1**
(c) (i)		**A1** **A1**

Qn	Suggested solution	Markscheme
(c) (ii)	$[\text{AB}]^2 = 2^2+\left(\sqrt{2}\right)^2-2(2)\left(\sqrt{2}\right)\cos\left(\frac{5\pi}{12}\right)$ $= 4+2-4\sqrt{2}\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)=6-2\left(\sqrt{3}-1\right)=8-2\sqrt{3}$ Alternatively, $z_1-z_2=-1+\sqrt{3}\text{i}-(1+\text{i})$ (or z_2-z_1) $z_1-z_2=-2+\left(\sqrt{3}+1\right)\text{i}$ $[\text{AB}]^2=(-2)^2+\left(\sqrt{3}-1\right)^2$ $[\text{AB}]^2=4+\left(3-2\sqrt{3}+1\right)=8-2\sqrt{3}$	**M1** – cosine rule **A1 AG** **M1** **A1** **AG**
9	***Summation & AP with D.R.V.***	**[marks: 15]**
(a) (i)	$\sum_{r=1}^{n}\left((r+1)^3-r^3\right)=\left(2^3-1^3\right)+\left(3^3-2^3\right)...+\left((n+1)^3-n^3\right)$ $=(n+1)^3-1$	**M1** – sub in values **A1** – cancellation of terms **AG**
(a) (ii)	$\sum_{r=1}^{n}\left((r+1)^3-r^3\right)=\sum_{r=1}^{n}\left(3r^2+3r+1\right)$ $=\sum_{r=1}^{n}3r^2+\sum_{r=1}^{n}3r+\sum_{r=1}^{n}1$ $=\sum_{r=1}^{n}3r^2+3\left(\frac{n(n+1)}{2}\right)+n(1)$ From (a)(i), $(n+1)^3-1=3\sum_{r=1}^{n}r^2+\frac{3n(n+1)}{2}+n$ $3\sum_{r=1}^{n}r^2=(n+1)^3-1-\frac{3n(n+1)}{2}-n$ $3\sum_{r=1}^{n}r^2=\frac{(n+1)}{2}\left[2(n+1)^2-3n-2\right]$ $\sum_{r=1}^{n}r^2=\frac{(n+1)}{6}\left(2n^2+4n+2-3n-2\right)=\frac{(n+1)}{6}\left(2n^2+n\right)$ $\sum_{r=1}^{n}r^2=\frac{n(n+1)(2n+1)}{6}$	**M1** – expansion & simplification **A1** – $\sum_{r=1}^{n}1$ **A1** – $\sum_{r=1}^{n}r$ **M1** – equate and re-arrange **A1** **AG**

Qn	Suggested solution	Markscheme
(b) (i)	Total number of cards in the deck $=1+2+...+n=\frac{n(n+1)}{2}$ $\mathrm{P}(X=r)=\frac{r}{\left(\frac{n(n+1)}{2}\right)}=\frac{2r}{n(n+1)}$	**M1** **A1 AG**
(b) (ii)	$\mathrm{E}(X)=\sum_{r=1}^{n} r\cdot\mathrm{P}(X=r)=\sum_{r=1}^{n} r\left(\frac{2r}{n(n+1)}\right)$ $\mathrm{E}(X)=\frac{2}{n(n+1)}\sum_{r=1}^{n} r^2$ $\mathrm{E}(X)=\frac{2}{n(n+1)}\left(\frac{n(n+1)(2n+1)}{6}\right)$ $\mathrm{E}(X)=\frac{(2n+1)}{3}$	**M1** – formula for E(X) **M1** – use result of $\sum_{r=1}^{n} r^2$ **A1**
(b) (iii)	$\mathrm{Var}(X)=\mathrm{E}\left(X^2\right)-\left[\mathrm{E}(X)\right]^2$ $\mathrm{Var}(X)=\left[\sum_{r=1}^{4} r^2\cdot\mathrm{P}(X=r)\right]-\left[\frac{2(4)+1}{3}\right]^2$ $\mathrm{Var}(X)=\left[\sum_{r=1}^{4} r^2\cdot\left(\frac{r}{10}\right)\right]-(3)^2$ $\mathrm{Var}(X)=\left(\frac{1^3}{10}+\frac{2^3}{10}+\frac{3^3}{10}+\frac{4^3}{10}\right)-9=\left(\frac{1+8+27+64}{10}\right)-9=1$	**M1** – formula for Var(X) **M1 A1**
10	***Differentiation (product rule with exponent) with Maclaurin Expansion and mathematical induction***	**[marks: 20]**
(a) (i)	$f_k\,'(x)=(1)\mathrm{e}^{kx}+(x)k\mathrm{e}^{kx}=\mathrm{e}^{kx}+kx\mathrm{e}^{kx}$ or $\mathrm{e}^{kx}(1+kx)$	**M1** – product rule **A1**
(a) (ii)	$\mathrm{e}^{kx}+kx\mathrm{e}^{kx}=0$ $\mathrm{e}^{kx}(1+kx)=0$ $1+kx=0$ (since $\mathrm{e}^{kx}\neq 0$) $x=-\frac{1}{k}$	**M1** – solving **A1 –** allow f.t.
(a) (iii)	$f_k\left(-\frac{1}{k}\right)=\left(-\frac{1}{k}\right)\mathrm{e}^{k\left(-\frac{1}{k}\right)}=-\frac{1}{k}\mathrm{e}^{-1}$ or $-\frac{1}{k\mathrm{e}}$ $\left(-\frac{1}{k},-\frac{1}{k\mathrm{e}}\right)$ is the stationary point of $y=f_k(x)$ Let $y=-\frac{1}{k}\mathrm{e}^{-1}$ $\Rightarrow y=\mathrm{e}^{-1}x$ for $x=-\frac{1}{k}$	**A1 –** allow f.t. **M1**

Qn	**Suggested solution**	**Markscheme**
	$m=\frac{1}{\mathrm{e}}$ and $c = 0$	**A1 A1 –** allow f.t.
(b)	Prove $f_k^{(n)}(x)=nk^{n-1}\mathrm{e}^{kx}+k^n x\mathrm{e}^{kx}$, $n\in\mathbb{Z}^+$ Let $n=1$, LHS $= f_k^{(1)}(x)=\mathrm{e}^{kx}+kx\mathrm{e}^{kx}$ [from (a)(i)] RHS $=(1)k^{1-1}\mathrm{e}^{kx}+k^1 x\mathrm{e}^{kx}=\mathrm{e}^{kx}+kx\mathrm{e}^{kx}=$ LHS so true for $n=1$.	**A1**
	Assume proposition true for $n=m$, for some $m\in\mathbb{Z}^+$ i.e. $f_k^{(m)}(x)=mk^{m-1}\mathrm{e}^{kx}+k^m x\mathrm{e}^{kx}$	**M1**
	To show $f_k^{(m+1)}(x)=(m+1)k^m\mathrm{e}^{kx}+k^{m+1}x\mathrm{e}^{kx}$ LHS $= f_k^{(m+1)}(x)$ $=\frac{\mathrm{d}}{\mathrm{d}x}\left[f_k^{(m)}(x)\right]$	**M1** – rewrite $f_k^{(m+1)}(x)$ as $\frac{\mathrm{d}}{\mathrm{d}x}\left[f_k^{(m)}(x)\right]$
	$=\frac{\mathrm{d}}{\mathrm{d}x}\left[mk^{m-1}\mathrm{e}^{kx}+k^m x\mathrm{e}^{kx}\right]$	**M1** – use of assumption
	$=mk^{m-1}\left(k\mathrm{e}^{kx}\right)+k^m\left(\mathrm{e}^{kx}+x\cdot k\mathrm{e}^{kx}\right)$	**A1**
	$=mk^m\mathrm{e}^{kx}+k^m\mathrm{e}^{kx}+k^m\left(kx\mathrm{e}^{kx}\right)$ $=(m+1)k^m\mathrm{e}^{kx}+k^{m+1}x\mathrm{e}^{kx}$ = RHS	**A1**
	Since true for $n=1$ and true for $n=m+1$ if true for $n=m$. Therefore true for all $n\in\mathbb{Z}^+$.	**R1**
(c)	$f_k(x)=x\left(1+kx+\frac{(kx)^2}{2!}+\frac{(kx)^3}{3!}+...\right)$	**M1**
	$f_k(x)=x+kx^2+\frac{k^2x^3}{2}+\frac{k^3x^4}{6}+...$	**A1**
	Given $\frac{k^2}{2}=\frac{k^3}{6}\Rightarrow k=3$ or $k=0$ (rej $k\neq 0$)	**A1**
(d)	$y=x\mathrm{e}^x \xrightarrow[\text{by } 2x]{\text{replace } x} y=2x\mathrm{e}^{2x} \xrightarrow[\text{by } 2y]{\text{replace } y} 2y=2x\mathrm{e}^{2x}\Rightarrow y=x\mathrm{e}^{2x}$ Stretch horizontally by scale factor ½. Stretch vertically by scale factor ½. *either order accepted*	**A1** **A1**

STUDENT NAME: ______________________________

TEACHER NAME: ______________________________

CANDIDATE SESSION NUMBER

0	2	5	0	1	2				

EXAMINATION CODE

8	8	2	2	-	7	1	0	2

ST. JOSEPH'S INSTITUTION
YEAR 6 PRELIMINARY EXAMINATION 2022

MATHEMATICS: ANALYSIS AND APPROACHES
HIGHER LEVEL
PAPER 2
Friday

19 August 2022
2 hours
1400 – 1600 hrs

INSTRUCTIONS TO CANDIDATES

- Write your session number in the boxes above.
- Write your name and teacher's name in the spaces provided.
- Do not open this examination paper until instructed to do so.
  - **Section A:** Answer all questions showing working and answers in the spaces provided in the exam paper.
  - **Section B:** Answer all questions using the foolscap paper provided.
- The use of a scientific or examination graphical calculator is permitted in this paper.
- TI-Nspire calculators must be in Press-to-Test mode and cleared of all previous data.
- TI-84+ graphical calculators must only have permitted apps and be ram cleared.
- A clean copy of the **Mathematics: Analysis and Approaches formula booklet** is required for this paper.
- Unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.
- The maximum mark for this examination paper is **[110 marks]**.
- Number of printed pages = 14.
- Sections A and B are to be submitted **separately**.

FOR MARKER USE ONLY:

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	TOTAL
										/110

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, for example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are advised to *show all working.*

SECTION A (53 marks)

Answer **all** questions in the **spaces** provided.

1 **[Maximum mark: 6]**

The graph of the function $h(x)=\log_7(x-a)+b$ passes through the points $(0,1)$ and $\left(\mathrm{e}^3, 1+\log_7 2\right)$.

Find the value of *a* and the value of *b*.

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2 **[Maximum mark: 7]**

For $\frac{\pi}{4} < x < \frac{3\pi}{4}$, find the value of x for which the sum to infinity

$$\frac{1}{\tan x} + \frac{1}{\tan^2 x} + \frac{1}{\tan^3 x} + \cdots$$

equals to $\tan x$.

TURN OVER

3 [Maximum mark: 7]

The functions f and g are defined by

$$f(x) = \sqrt{-x^2 - 2x + 15}, \text{ for } -5 \le x \le -1$$

$$g(x) = -x + c, \text{ for } x, c \in \mathbb{R}.$$

(a) Find $f^{-1}(2)$ [4]

(b) Find the range of f. [1]

(c) Given that $(g \circ f)(x) \ge 0$ for $-5 \le x \le -1$, determine the set of possible values for c. [2]

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4 **[Maximum mark: 6]**

Alex and Bobby have each been given \$1000 to save for university.

Alex invests his money in an account that pays a nominal annual interest rate of 2.35% **compounded half-yearly.**

(a) Calculate the amount Alex will have in his account after 5 years. Give your answer correct to 2 decimal places. [3]

Bobby wants to invest his money in an account such that his investment will increase to 2 times the initial amount in 5 years. Assume the account pays a nominal annual interest of p% **compounded quarterly**.

(b) Determine the value of p. [3]

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5 **[Maximum mark: 5]**

Six vectors $\mathbf{v_1}, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6$ are each chosen to be either $\begin{pmatrix}1\\1\end{pmatrix}$ or $\begin{pmatrix}3\\2\end{pmatrix}$ with equal probability, with each choice made independently. Find the probability that the sum $\mathbf{v_1} + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 + \mathbf{v}_5 + \mathbf{v}_6$ is equal to the vector $\begin{pmatrix}10\\8\end{pmatrix}$.

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6 [Maximum mark: 11]

A particle P starts from point A and moves along a straight line so that its velocity, $v\,\text{ms}^{-1}$, after t seconds is given by $v(t) = 2\sin t + 1 - \text{e}^{\sin t}$ for $0 \le t \le 6$.

(a) Find the values of t when P is at rest. [3]

(b) Find the acceleration of P when it changes direction. [2]

(c) Write down the number of times that the acceleration of P is $0\,\text{ms}^{-2}$. [1]

(d) Find the total distance travelled by P from $t = 0$ to $t = 6$. [2]

(e) Explain why P passes through A again. [3]

TURN OVER

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7 [Maximum mark: 11]

(a) Show that $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\ln x}{x}\right)=\frac{1}{x^2}-\frac{\ln x}{x^2}$. [2]

Hence, find $\int \frac{\ln x}{x^2}\,\mathrm{d}x$. [2]

(b) The region bounded by the curve $y=\frac{\ln x}{x}$ and the lines $x=1, x=\mathrm{e}, y=0$ is rotated through 2π radians about the x-axis.

Find the **exact** volume of the solid generated. [7]

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*Do **NOT** write solutions on this page.*

SECTION B (57 marks)

Answer all questions on the foolscap paper provided. ***Please start each question on a new page.***

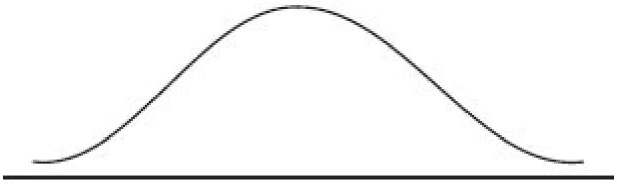
8 [Maximum Mark: 18]

A shop sells apples and pears. The masses, in grams, of apples and pears are normally distributed with means and standard deviations as shown in the following table.

	Mean	**Standard deviation**
Apple	204	6
Pear	150	σ

(a) Find the probability that a randomly chosen **apple** weighs between 186 grams and 222 grams. [2]

(b) Copy and complete the following normal distribution diagram, to represent the probability that the masses of the **apples** lie within three standard deviations of the mean, and shade the appropriate region. [2]



(c) Tom bought n **apples**. Find the least value of n such that the probability that more than 7 apples will each weigh between 198 grams and 210 grams is at least 0.85. [4]

(d) Three **pears** are chosen at random. Find the probability that one of them weighs less than the mean mass and each of the other two pears has a mass within one standard deviation of the mean mass. [4]

TURN OVER

Do ***NOT*** *write solutions on this page.*

The probability of a randomly chosen **pear** weighing between t and 158 grams, and the probability that a randomly chosen **pear** weighs less than t grams are each 0.35.

(e) Find the value of σ and of t. [4]

(f) In a random sample of 50 pears, find the most probable number of pears that each weighs less than t grams. [2]

9 **[Maximum Mark:17]**

Consider the planes

$$\Pi_1: \ 2x+7y+5z=24$$
$$\Pi_2: \ 3x-4y+\lambda z=\mu$$

and the line l passing through the point $\mathrm{A}(5,2,4)$ and point $\mathrm{B}(5,-1,3)$.

(a) Find a vector equation of line l. [1]

(b) The point C lies on l such that the foot of perpendicular of C onto Π_1 has coordinates $(3, 1, 1)$. Find the coordinates of C. [5]

The line l does not intersect Π_2.

(c) Show that $\lambda=12$ and find the possible values of μ. [4]

(d) Find the possible values of μ if the distance between Π_2 and l is 2 units. [5]

(e) Find the acute angle between the planes Π_1 and Π_2. [2]

TURN OVER

*Do **NOT** write solutions on this page.*

10 [Maximum Mark: 22]

The population, P (in thousands), of a colony of ants on a small island can be modelled by the differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{N}\right)$$

where t is the time measured in days and k and N are positive constants.

The constant N represents the maximum population of this colony of ants that the island can sustain indefinitely.

(a) In the context of the population model, interpret the meaning of $\frac{\mathrm{d}P}{\mathrm{d}t}$. [1]

(b) Show that $\frac{\mathrm{d}^2P}{\mathrm{d}t^2} = k^2P\left(1 - \frac{P}{N}\right)\left(1 - \frac{2P}{N}\right)$. [4]

(c) Hence show that that the colony of ants grows the fastest when $P = \frac{N}{2}$. [6]

(d) Hence determine the corresponding fastest growth rate. [1]

Let P_0 being the initial population of the colony of ants.

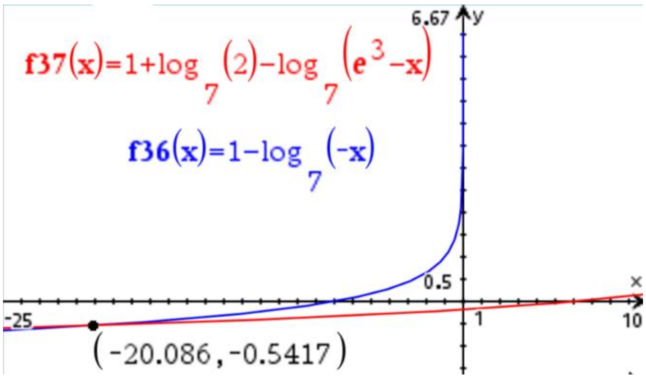
(e) Show that the solution to the differential equation can be written in the form

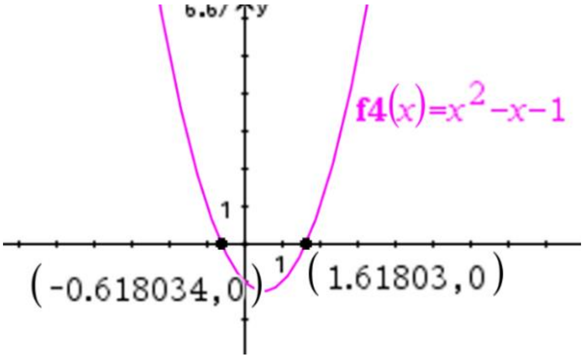
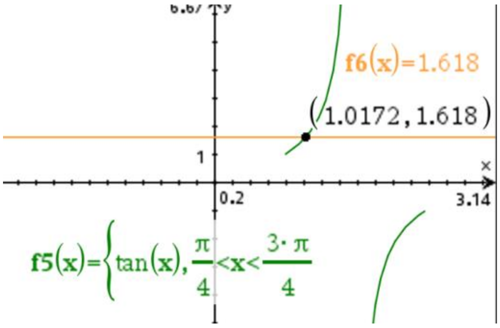
$$kt = \ln\frac{P}{P_0}\left(\frac{N - P_0}{N - P}\right). \quad [8]$$

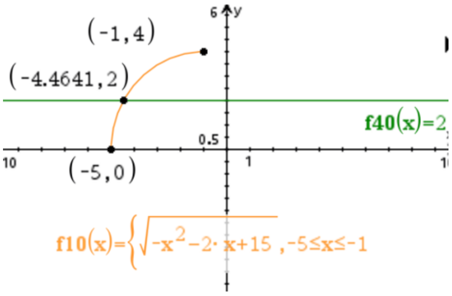
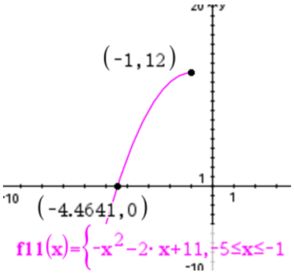
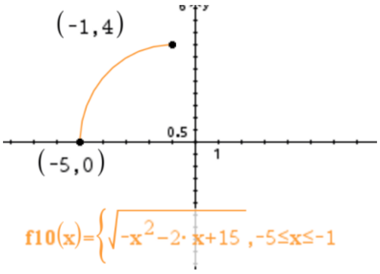
After 10 days the population is $5P_0$.

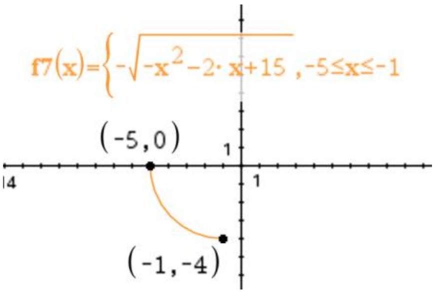
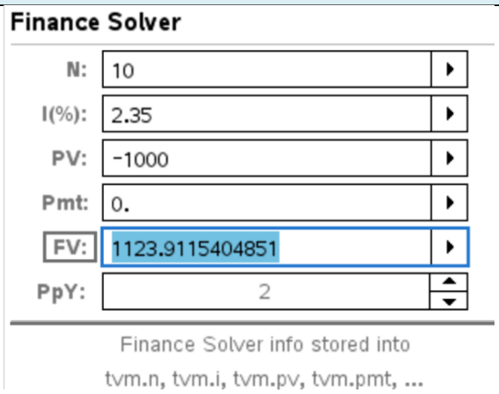
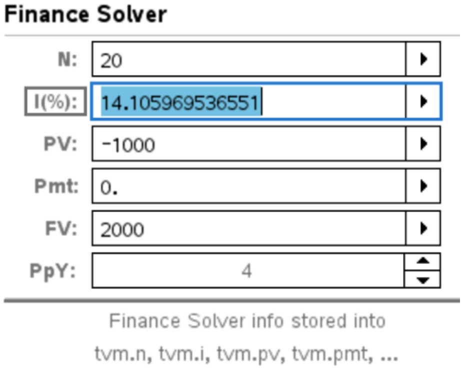
(f) Show that $k = \frac{1}{10}\ln(5.1)$ if $N = 205P_0$. [2]

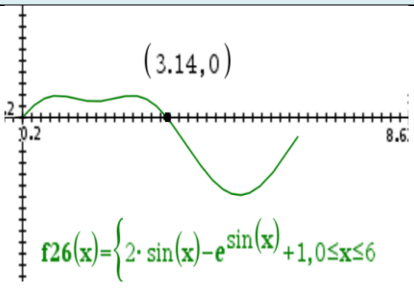
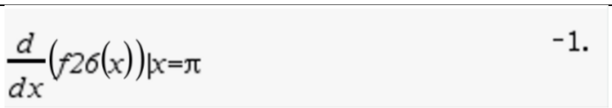
End of Paper

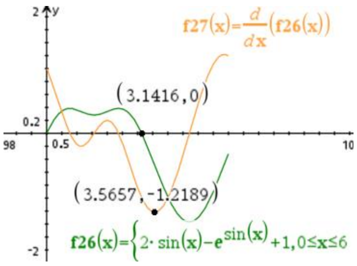
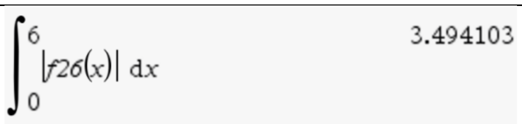
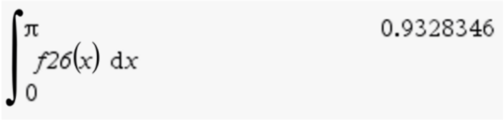
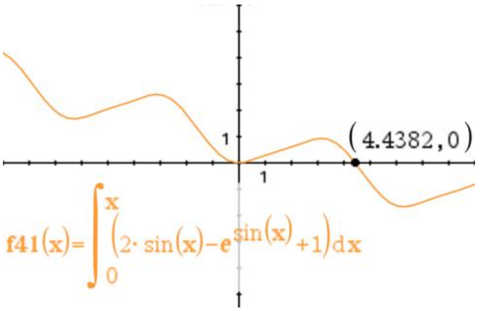
Qn	Suggested Solutions	Marks
1	**Log + simultaneous eqs**	**[Maximum mark: 6]**
	Subst $(0,1)$ and $\left(\mathrm{e}^3,1+\log_7 2\right)$ into $h(x)=\log_7(x-a)+b$,	**(M1)**
	$\log_7(-a)+b=1 \qquad -(1)$	**A1**
	$\log_7\left(\mathrm{e}^3-a\right)+b=1+\log_7 2 \quad -(2)$	**A1**
	$(2)-(1)$,	
	$\log_7\left(\mathrm{e}^3-a\right)-\log_7(-a)=\log_7 2$	
	$\Rightarrow \log_7\left(\dfrac{\mathrm{e}^3-a}{-a}\right)=\log_7 2$	**(M1)**
	$\Rightarrow \dfrac{\mathrm{e}^3-a}{-a}=2$	
	$\Rightarrow \mathrm{e}^3-a=-2a$	
	$\Rightarrow a=-\mathrm{e}^3$	**A1**
	$\Rightarrow b=1-3\log_7 \mathrm{e} \ \left(\text{or } 1-\log_7 \mathrm{e}^3 \text{ or } -0.542 \text{ (3sf)}\right)$	**A1**
	Alternatively,	
	Subst $(0,1)$ and $\left(\mathrm{e}^3,1+\log_7 2\right)$ into $h(x)=\log_7(x-a)+b$,	**(M1)**
	$\log_7(-a)+b=1 \qquad -(1)$	**A1**
	$\log_7\left(\mathrm{e}^3-a\right)+b=1+\log_7 2 \quad -(2)$	**A1**
	$b=1-\log_7(-a) \qquad -(1)$	**(M1)**
	$b=1+\log_7 2-\log_7\left(\mathrm{e}^3-a\right) \quad -(2)$	
		
	By GDC,	
	$a=-20.1$(3sf)	**A1**
	$b=-0.542$ (3sf)	**A1**

2	**GP Sum to infinity/existence + quadratic + trigo**	**[Maximum mark: 7]**
(a)	$\frac{1}{\tan x}+\frac{1}{\tan^2 x}+\frac{1}{\tan^3 x}+\cdots=\tan x$	
	$\Rightarrow \dfrac{\frac{1}{\tan x}}{1-\frac{1}{\tan x}}=\tan x$	**M1** **A1**
	$\Rightarrow \frac{1}{\tan x}=\tan x-1$	
	$\Rightarrow \tan^2 x-\tan x-1=0$	**A1**
(b)		**M1**
	Since $\left\lvert\frac{1}{\tan x}\right\rvert<1\Rightarrow\lvert\tan x\rvert>1$.	**R1**
	$\therefore \tan x=1.6180=1.62$ (3sf).	**A1**
		
	tan⁻¹(1.618) 1.017213	**A1**
	$x=1.0172=1.02$ rad (3sf)	

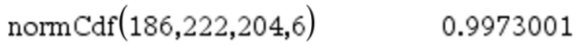
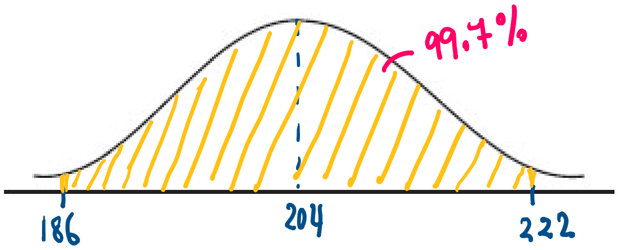
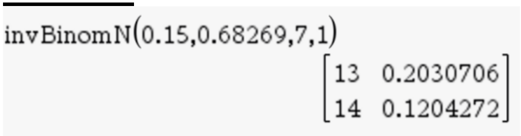
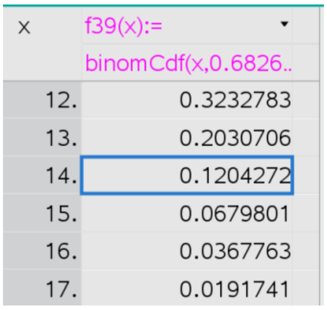
3	**Functions**	**[Maximum mark: 7]**
(a)	**<u>Method 1</u>** $f^{-1}(2)=a$ $\Rightarrow f(a)=2$ $\Rightarrow \sqrt{-a^2-2a+15}=2$  $\Rightarrow a=-4.46\,(3\text{s f})\quad (\text{rej } a=2.46 \because\ -5\le x\le -1)$ **<u>Method 2</u>** $f^{-1}(2)=a$ $\Rightarrow f(a)=2$ $\Rightarrow \sqrt{-a^2-2a+15}=2$ $\Rightarrow -a^2-2a+11=0$ $\Rightarrow a=-4.46\,(3\text{s f})\quad (\text{rej } a=2.46 \because\ -5\le x\le -1)$  **<u>Method 3</u>** $f^{-1}(2)=a$ $\Rightarrow f(a)=2$ $\Rightarrow \sqrt{-a^2-2a+15}=2$ $\Rightarrow -a^2-2a+11=0$ $\Rightarrow a=-1\pm\sqrt{12}=-1-\sqrt{12}\quad (\because -5\le x\le -1)$	**M1** **A1** **M1** **A1** **M1** **A1** **M1A1** **M1** **A1** **M1** **A1**
(b)	$R_f=[0,4]$	**A1**
(c)	$(g\circ f)(x)\ge 0$ $\Rightarrow -\sqrt{-x^2-2x+15}+c\ge 0$ $\Rightarrow \sqrt{-x^2-2x+15}\le c$  $c\ge 4$	**M1** **A1**

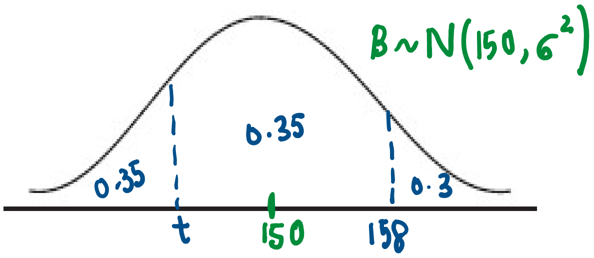
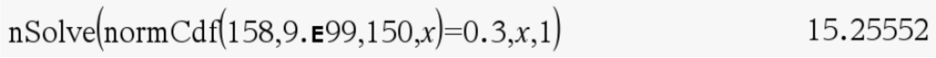
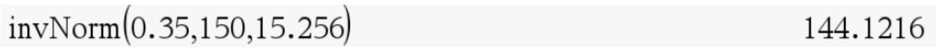
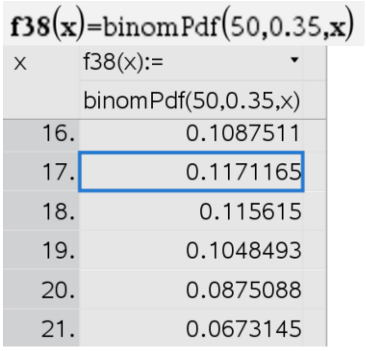
	Alternatively, $(g\circ f)(x)\geq 0$ $-\sqrt{-x^2-2x+15}+c\geq 0$  Reflection of $f(x)=\sqrt{-x^2-2x+15}$ in x axis and translation by $\begin{pmatrix}0\\4\end{pmatrix}$. $c\geq 4$	**M1** **A1**
4	**Financial Math**	**[Maximum mark: 6]**
(a)	 **Finance Solver** N: 10 I(%): 2.35 PV: -1000 Pmt: 0. FV: 1123.9115404851 PpY: 2 Finance Solver info stored into tvm.n, tvm.i, tvm.pv, tvm.pmt, ... The amount Alex will have in his account after 5 years is $1123.91.	**(M1)** – use financial app in GDC **(A1)** – correct entries in GDC **A1**
(b)	 **Finance Solver** N: 20 I(%): 14.105969536551 PV: -1000 Pmt: 0. FV: 2000 PpY: 4 Finance Solver info stored into tvm.n, tvm.i, tvm.pv, tvm.pmt, ... $p=14.1$ (3sf)	**(M1)** – use financial app in GDC **(A1)** – correct entries in GDC; FV and PV must have opposite signs. **A1**

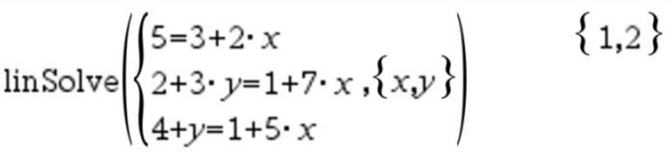
5	**P&C**	**[Maximum mark: 5]**
	Let n be the no. of times $\begin{pmatrix}1\\1\end{pmatrix}$ is selected and m be the no of times $\begin{pmatrix}3\\2\end{pmatrix}$ is selected. $n\begin{pmatrix}1\\1\end{pmatrix}+m\begin{pmatrix}3\\2\end{pmatrix}=\begin{pmatrix}10\\8\end{pmatrix}$ Since there are 6 vectors, $n+m=6$. Solving the simultaneous equations $n+m=6$ $n+3m=10$ $n+2m=8$ $\Rightarrow n=4, m=2$	**M1** **A1**
	We want exactly 2 of the 6 vectors to be $\begin{pmatrix}3\\2\end{pmatrix}$ i.e. there are ${}^6C_2=15$ ways.	**A1**
	Each vector can be either $\begin{pmatrix}1\\1\end{pmatrix}$ or $\begin{pmatrix}3\\2\end{pmatrix}$ i.e. total no of ways = 2^6.	**A1**
	$\therefore$ The probability that the sum $\mathbf{v_1}+\mathbf{v}_2+\mathbf{v}_3+\mathbf{v}_4+\mathbf{v}_5+\mathbf{v}_6$ is equal to the vector $\begin{pmatrix}10\\8\end{pmatrix}=\frac{{}^6C_2}{2^6}=\frac{15}{64}$.	**A1**
6	**Kinematics**	**[Maximum mark: 11]**
(a)	 $t=0,\ \pi\ \left(3.14\ \ (3\text{sf})\right)$	**M1** **A1 A1**
(b)	$\frac{d}{dx}(f26(x))\|x=\pi$ -1. Acceleration = -1	**M1** **A1**

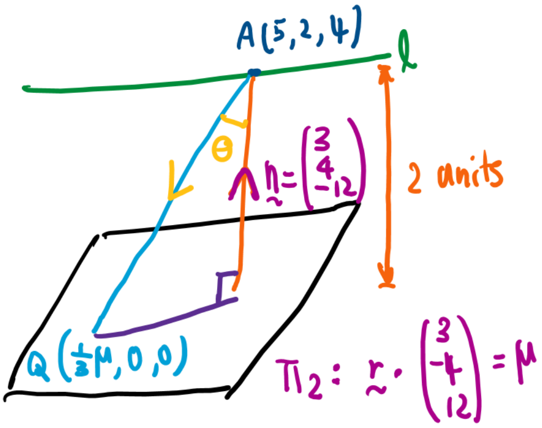
(c)	 The number of times that the acceleration of P is $0\,\text{ms}^{-2}$ = 4	**A1**
(d)	 $\int_0^6 \lvert f26(x)\rvert\,dx$ 3.494103 Total distance = 3.49 (3sf)	**M1** **A1**
(e)	**<u>Method 1</u>**  $\int_0^{\pi} f26(x)\,dx$ 0.9328346  $\int_{\pi}^{6} f26(x)\,dx$ -2.561268 valid explanation comparing **their** distances *eg* $2.56 > 0.932$, distance moving back is more than distance moving forward, hence P passes through A again.	**A1 for 0.932** **A1 for -2.56** **R1**
	<u>Method 2</u> For $0 \le t \le 6$, $v(t) = 2\sin t + 1 - \mathrm{e}^{\sin t}$ $s(t) = \int_0^t v(x)\,\mathrm{d}x$  From GDC, since $s(4.4382) = 0$, i.e. displacement is zero at $t = 4.4382\,\text{s}$ hence P passes through A again.	**M1** [find t for $s(t)=0$] **A1** [$t = 4.4382\,\text{s}$] **R1**

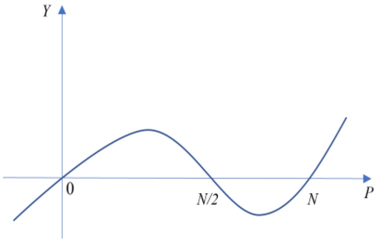
7	**Differentiation + Application of Integration (Volume)**	**[Maximum mark:11]**
(a)	By quotient rule, $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\ln x}{x}\right)=\frac{x\left(\frac{1}{x}\right)-1(\ln x)}{x^2}=\frac{1}{x^2}-\frac{\ln x}{x^2}$ $\Rightarrow \int \frac{\ln x}{x^2}\mathrm{d}x=\int \frac{1}{x^2}\mathrm{d}x-\left(\frac{\ln x}{x}\right)$ $=\frac{-1}{x}-\frac{\ln x}{x}+c$	**M1A1** **M1** **A1**
(b)	$V=\pi\int_1^{\mathrm{e}} y^2\mathrm{d}x$ $=\pi\int_1^{\mathrm{e}}\left(\frac{\ln x}{x}\right)^2\mathrm{d}x$ $=\pi\int_1^{\mathrm{e}}\left(\frac{1}{x^2}\right)(\ln x)^2\,\mathrm{d}x$ $=\pi\left[\left[(\ln x)^2\left(\frac{-1}{x}\right)\right]_1^{\mathrm{e}}-\int_1^{\mathrm{e}}\left(\frac{-1}{x}\right)\left(\frac{2\ln x}{x}\right)\mathrm{d}x\right]$ $=\pi\left[\frac{-1}{\mathrm{e}}+2\int_1^{\mathrm{e}}\left(\frac{\ln x}{x^2}\right)\mathrm{d}x\right]$ $=\pi\left[\frac{-1}{\mathrm{e}}-2\left[\frac{1}{x}+\frac{\ln x}{x}\right]_1^{\mathrm{e}}\right]$ $=\pi\left[\frac{-1}{\mathrm{e}}-2\left(\frac{2}{\mathrm{e}}-1\right)\right]$ $=\pi\left(2-\frac{5}{\mathrm{e}}\right)$	**M1**(vol formula) **M1**(by parts) **A1** **M1** (use (a)) **A1** **M1** (subst) **A1**

8	**Normal Distribution + Binomial Dist**	**[Maximum mark: 18]**
(a)	Let A be the mass of an apple. $A \sim N\left(204, 6^2\right)$ $P(186 < A < 222)$ $= 0.99730$ $= 0.997\,(3\text{sf})$  normCdf(186,222,204,6) 0.9973001	**M1** **A1**
(b)		**A1-Shading -** (*Note: 99.7% of the data will lie within 3 standard deviations, thus the shading should take up almost the whole area*) **A1** - $204 \pm 3(6) = [186, 222]$
(c)	Let A be the mass of an apple. $A \sim N\left(204, 6^2\right)$ $P(198 < A < 210)$ $= 0.68269$ $= 0.683\,(3\text{sf})$	A1
	Let X be the no. of apples out of n that weigh between 198 grams and 210 grams. $X \sim B(n, 0.68269)$ $P(X > 7) \geq 0.85$ $\Rightarrow P(X \leq 7) \leq 0.15$ **<u>Method 1</u>**  invBinomN(0.15,0.68269,7,1) $\begin{bmatrix} 13 & 0.2030706 \\ 14 & 0.1204272 \end{bmatrix}$ Least n is 14. **<u>Method 2</u>** $P(X > 7) \geq 0.85$ $\Rightarrow P(X \leq 7) \leq 0.15$  x f39(x):= binomCdf(x,0.6826.. 12. 0.3232783 13. 0.2030706 14. 0.1204272 15. 0.0679801 16. 0.0367763 17. 0.0191741 Least n is 14.	A1 M1 A1 A1 M1 A1

(d)	Let B be the mass of a pear. $B \sim N\left(150, \sigma^2\right)$ $P(B<150)=0.5$ $P(150-\sigma \le B \le 150+\sigma)=P(-1 \le Z \le 1)=0.68269$ Required probability $=3 \times P(B<150) \times \left[P(150-\sigma \le B \le 150+\sigma)\right]^2$ $=3 \times 0.5 \times 0.68269^2$ $=0.699$ (3sf)	A1 –3 A1– 0.5 A1 – 0.68269 A1
(e)	 1−2·0.35 0.3 $P(B>158)=0.3$ By GDC nsolve, $\sigma=15.256=15.3$ (3sf). nSolve(normCdf(158,9.E99,150,x)=0.3,x,1) 15.25552 $P(B<t)=0.35$ By GDC invNorm, $t=144$ (3sf). invNorm(0.35,150,15.256) 144.1216	M1A1 M1A1
(f)	Let Y be the no of pears, out of 50, that weighs less that t grams. $Y \sim B(50, 0.35)$ f38(x)=binomPdf(50,0.35,x) x f38(x):= binomPdf(50,0.35,x) 16. 0.1087511 17. 0.1171165 18. 0.115615 19. 0.1048493 20. 0.0875088 21. 0.0673145 $P(Y=16)=0.10875$ (5sf) $P(Y=17)=0.11711$ (5sf) ← the one with the highest probability $P(Y=18)=0.11562$ (5sf) From GDC, most probable value = Mode = 17 Note: **Do NOT accept** if students use $E(Y)=50(0.35)=17.5$.	M1 A1

9	**Vectors: Line + planes**	**[Maximum mark: 17]**
(a)	$l:\mathbf{r}=\begin{pmatrix}5\\2\\4\end{pmatrix}+\beta\begin{pmatrix}0\\3\\1\end{pmatrix},\beta\in\mathbb{R}$	**A1**
(b)	Line *m* perpendicular to $\prod_1$ passing through (3, 1, 1) is $m:\ \mathbf{r}=\begin{pmatrix}3\\1\\1\end{pmatrix}+\alpha\begin{pmatrix}2\\7\\5\end{pmatrix},\alpha\in\mathbb{R}$ The 2 lines will intersect at *C*. $\begin{pmatrix}5\\2+3\beta\\4+\beta\end{pmatrix}=\begin{pmatrix}3+2\alpha\\1+7\alpha\\1+5\alpha\end{pmatrix}$ $5=3+2\alpha$ $2+3\beta=1+7\alpha$ $4+\beta=1+5\alpha$  $\text{linSolve}\left(\begin{cases}5=3+2\cdot x\\2+3\cdot y=1+7\cdot x\\4+y=1+5\cdot x\end{cases},\{x,y\}\right)$ $\{1,2\}$ $\Rightarrow\beta=2,\alpha=1$ Point *C* has coordinates $(5,8,6)$.	**A1** **M1** **M1**(solve simultaneous eqs) **A1** **A1**
(c)	Since *l* does not intersect $\prod_2$, $\begin{pmatrix}0\\3\\1\end{pmatrix}\cdot\begin{pmatrix}3\\-4\\\lambda\end{pmatrix}=0$ $\Rightarrow -12+\lambda=0$ $\Rightarrow\lambda=12$ (shown) $\mu\neq\begin{pmatrix}5\\2\\4\end{pmatrix}\cdot\begin{pmatrix}3\\-4\\12\end{pmatrix}=15-8+48=55$ $\Rightarrow\mu\neq55$	**M1** **A1** **AG** **M1** **A1**

(d)	From (**c**), $\lambda = 12$ Pick a point on $\prod_2$, say $Q\left(\frac{1}{3}\mu,\ 0,\ 0\right)$. $\left\|\overrightarrow{AQ}\cdot\hat{\mathbf{n}}\right\| = 2$ $\Rightarrow \dfrac{\left\|\begin{pmatrix}5-\frac{1}{3}\mu\\ 2\\ 4\end{pmatrix}\cdot\begin{pmatrix}3\\ -4\\ 12\end{pmatrix}\right\|}{\left\|\begin{pmatrix}3\\ -4\\ 12\end{pmatrix}\right\|} = 2$  $\Rightarrow \dfrac{\|55-\mu\|}{13} = 2$ $\Rightarrow 55-\mu = 26$ or $55-\mu = -26$ $\Rightarrow \mu = 29$ or 81	 **A1** **M1** **A1** **A1A1**
(e)	$\cos\theta = \dfrac{\begin{pmatrix}2\\ 7\\ 5\end{pmatrix}\bullet\begin{pmatrix}3\\ -4\\ 12\end{pmatrix}}{\sqrt{78}\,(13)}$ $\cos\theta = \dfrac{38}{\sqrt{78}\,(13)}$ $\theta = 70.7^{\circ}$ *or* $1.23\,\text{rad}$	**M1** **A1**

10	**DE - Population model /implicit differentiation**	**[Maximum mark: 22]**
(a)	$\frac{\mathrm{d}P}{\mathrm{d}t}$ represents the rate of population growth over time.	**A1**
(b)	$\frac{\mathrm{d}P}{\mathrm{d}t}=kP\left(1-\frac{P}{N}\right)=k\left(P-\frac{P^2}{N}\right)$ $\frac{\mathrm{d}^2P}{\mathrm{d}t^2}=k\left(\frac{dP}{dt}-\frac{2P}{N}\cdot\frac{dP}{dt}\right)$ $=k\frac{dP}{dt}\left(1-\frac{2P}{N}\right)$ $=k\cdot kP\left(1-\frac{P}{N}\right)\left(1-\frac{2P}{N}\right)$ $=k^2P\left(1-\frac{P}{N}\right)\left[1-\frac{2P}{N}\right]$ (shown)	**M1A1** **M1** (subst $\frac{\mathrm{d}P}{\mathrm{d}t}=kP\left(1-\frac{P}{N}\right)$) **A1** **AG**
(c)	$\frac{\mathrm{d}^2P}{\mathrm{d}t^2}=\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}P}{\mathrm{d}t}\right)$ is the rate of change of $\frac{\mathrm{d}P}{\mathrm{d}t}$. So $\frac{\mathrm{d}^2P}{\mathrm{d}t^2}=0\Rightarrow k^2P\left(1-\frac{P}{N}\right)\left(1-\frac{2P}{N}\right)=0\Rightarrow P=0,N,\frac{N}{2}$ Let $Y=\frac{\mathrm{d}^2P}{\mathrm{d}t^2}=\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}P}{\mathrm{d}t}\right)$  Using first derivative test on $P=0,N,\frac{N}{2}$ From the sketch, $P=0,N$ will give min point for $\frac{\mathrm{d}P}{\mathrm{d}t}$. Only when $P=\frac{N}{2}$, $\frac{\mathrm{d}P}{\mathrm{d}t}$ is maximum (+ve, 0, -ve). [Shown]	**M1** $[\frac{\mathrm{d}^2P}{\mathrm{d}t^2}=0]$ **A1** $[P=0,N,\frac{N}{2}]$ **M1** **A1**(graph) **R1** **R1** **AG**
(d)	The corresponding fastest growth rate at $P=\frac{N}{2}$: $\frac{\mathrm{d}P}{\mathrm{d}t}=k\left(\frac{N}{2}\right)\left(1-\frac{N/2}{N}\right)=\frac{kN}{4}$	**A1**

(e)	$\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1-\frac{P}{N}\right)$ $\Rightarrow \int k\,\mathrm{d}t = \int \frac{1}{P\left(1-\frac{P}{N}\right)}\mathrm{d}P$ $\Rightarrow \int k\,\mathrm{d}t = \int \frac{N}{P(N-P)}\mathrm{d}P = \int\left(\frac{1}{P}+\frac{1}{N-P}\right)\mathrm{d}P$ $\therefore kt = \ln\lvert P\rvert - \ln\lvert N-P\rvert + c$ $\Rightarrow kt = \ln\left\lvert\frac{P}{N-P}\right\rvert + C$ $\Rightarrow kt = \ln\left(\frac{P}{N-P}\right) + C \quad (\because N,\ P\ and\ (N-P)\ > 0)$	**M1** (Sep Variable) **M1** (partial fraction) **A1** $\frac{1}{P}+\frac{1}{N-P}$ **M1** **A1** **A1**
	When t = 0 $P = P_0$ $\Rightarrow C = -\ln\left(\frac{P_0}{N-P_0}\right)$ $\therefore kt = \ln\left(\frac{P}{N-P}\right) - \ln\left(\frac{P_0}{N-P_0}\right)$ $\Rightarrow kt = \ln\frac{P}{P_0}\left(\frac{N-P_0}{N-P}\right)$ (shown)	**A1** **A1** **AG**
(f)	When $t = 10, P = 5P_0, N = 205P_0$ $10k = \ln 5\left(\frac{204}{200}\right) \Rightarrow k = \frac{1}{10}\ln(5.1)$	**M1A1**

STUDENT NAME: ______________________________

TEACHER NAME: ______________________________

CANDIDATE SESSION NUMBER

0	2	5	0	1	2				

EXAMINATION CODE

8	8	2	2	-	7	1	0	3

ST. JOSEPH'S INSTITUTION
YEAR 6 PRELIMINARY EXAMINATION 2022

MATHEMATICS: ANALYSIS AND APPROACHES
HIGHER LEVEL
PAPER 3
Monday

22 August 2022
1 hour
0800 – 0900 hrs

INSTRUCTIONS TO CANDIDATES

- Write your session number in the boxes above.
- Write your name and teacher's name in the spaces provided.
- Do not open this examination paper until instructed to do so.
- Answer all the questions using the foolscap paper provided.
- The use of a scientific or examination graphical calculator is permitted in this paper.
- TI-Nspire calculators must be in Press-to-Test mode and cleared of all previous data.
- TI-84+ graphical calculators must only have permitted apps and be ram cleared.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **Mathematics: Analysis and Approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.
- This question paper consists of **4** printed pages including the cover sheet.

FOR MARKER USE ONLY:

Q1	Q2	TOTAL
		/55

Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

……………………………………………………………………………………………

1. [Maximum mark: 25]

In this question you will investigate the Gamma function and some of its properties using calculus and methods of proofs.

The Gamma function, $\Gamma(n)$, is defined as

$$\Gamma(n) = \int_0^\infty t^{n-1}\mathrm{e}^{-t}\mathrm{d}t,$$

for any number $n \in \mathbb{R}$ except non-positive integers. (E.g., $n \neq 0, -1, -2, \ldots$)

To approximate $\Gamma(n)$ using technology for some value of n, replace "∞" by "999999".

(a) Find the value of $\Gamma(1)$, of $\Gamma(2)$, of $\Gamma(3)$, of $\Gamma(4)$, of $\Gamma(5)$ and of $\Gamma(6)$. [3]

(b) Using a factorial, formulate a conjecture on the value of $\Gamma(n)$ **for any positive integer n.** [1]

Consider $f(x) = x^n\mathrm{e}^{-x}$, **for any $n > 0$.**

(c) By sketching the graph of $y = f(x)$, deduce what happens to $f(x)$ as $x \to \infty$. [2]

(d) Using integration by parts, show that $\Gamma(n+1) = n\,\Gamma(n)$, **for any $n > 0$.** [3]

(e) Hence, using mathematical induction, prove your conjecture in (b). [5]

Consider $\Gamma\left(\frac{1}{2}\right)$.

(f) By using the substitution $t = \frac{1}{2}y^2$, express $\Gamma\left(\frac{1}{2}\right)$ as an integral in terms of y. [5]

The probability density function of the random variable $Z \sim \mathrm{N}(0,1)$ is given by

$$f(z) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}z^2}.$$

(g) Deduce, with justification, the value of $\int_0^\infty f(z)\mathrm{d}z$. [2]

(h) Hence, show that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. [2]

(i) Hence, find the exact value of $\Gamma\left(\frac{3}{2}\right)$. [2]

2. [Maximum mark: 30]

In this question you will investigate the geometrical properties of complex numbers and roots of unity.

Consider $z = 1 + \sqrt{2} + \mathrm{i}$, where $\mathrm{i}^2 = -1$.

(a) Find the exact value of z^2, leaving your answer in the form $x + \mathrm{i}y$, where $x, y \in \mathbb{R}$. [2]

(b) Find the exact value of $\arg z^2$. [2]

(c) Find the exact value of $\arg z$. [2]

(d) Hence, find the exact value of $\tan\frac{\pi}{8}$. [2]

(e) For any two complex numbers $z_1 = x_1 + \mathrm{i}y_1$ and $z_2 = x_2 + \mathrm{i}y_2$,

(i) show that $|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

(ii) Hence, interpret $|z_1 - z_2|$ **geometrically**. [2]

Let $\omega = \frac{z}{|z|}$.

(f) Show that $\left|\omega^k\right| = 1$, for any integer k. [1]

(g) Sketch the 16 points representing the complex numbers ω^k, for $k = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 7, 8$, in an Argand diagram. [2]

(h) The points in the Argand diagram represented by the complex numbers ω^{-5}, ω^{-1}, ω^3 and ω^7 form a square. Find the area of this square. [2]

(i) Find the exact value of $\left|\omega - \frac{1}{\omega}\right|$ and of $\left|\omega^2 - \frac{1}{\omega^2}\right|$. [5]

(j) Show that $\left|\omega - \frac{1}{\omega}\right|\left|\omega^3 - \frac{1}{\omega^3}\right| = \sqrt{2}$. [4]

(This question continues on the following page)

(Question 2 continued)

Define

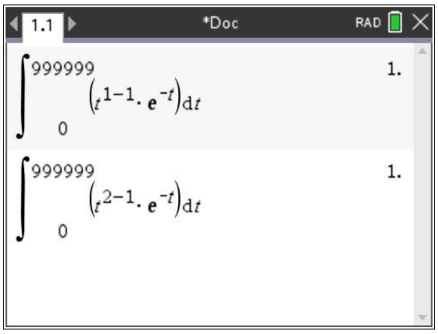
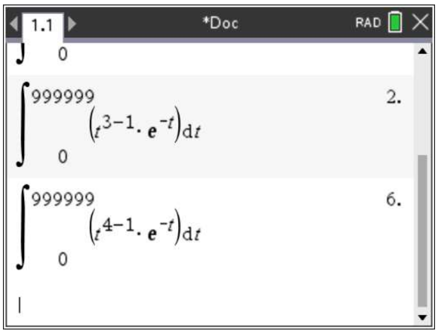
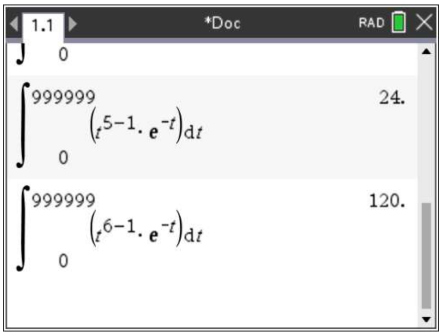
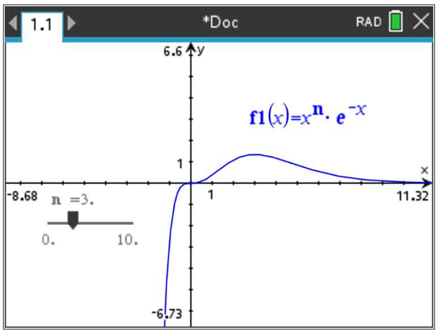
$$\prod_{k=1}^{n} a_k = a_1 \times a_2 \times \cdots \times a_n.$$

(k) Copy and complete the table below: [6]

n	4	7	10
$\prod_{k=1}^{n} \left\lvert \omega^k - \frac{1}{\omega^k} \right\rvert =$			

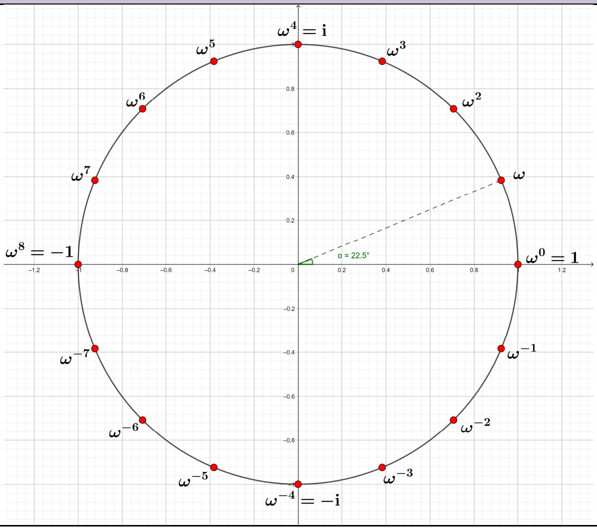
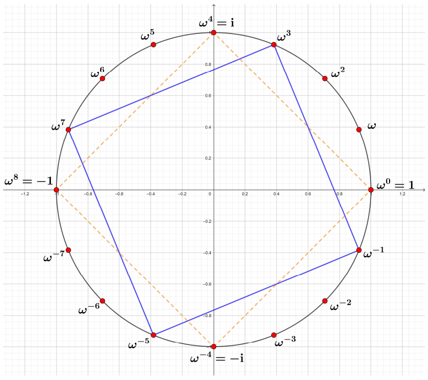
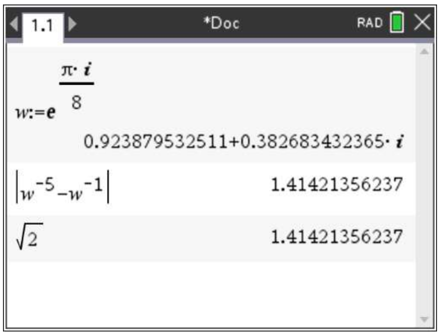
End of Paper

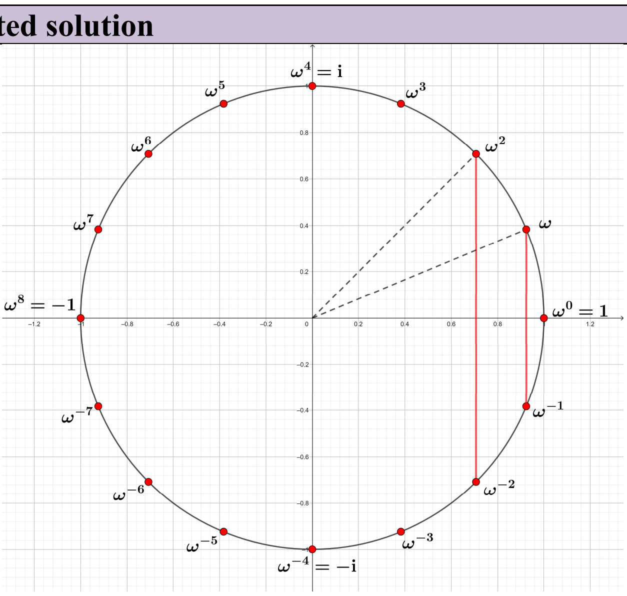
Year 6 HL MAA Preliminary Examination 2022 Paper 3 (Markscheme)

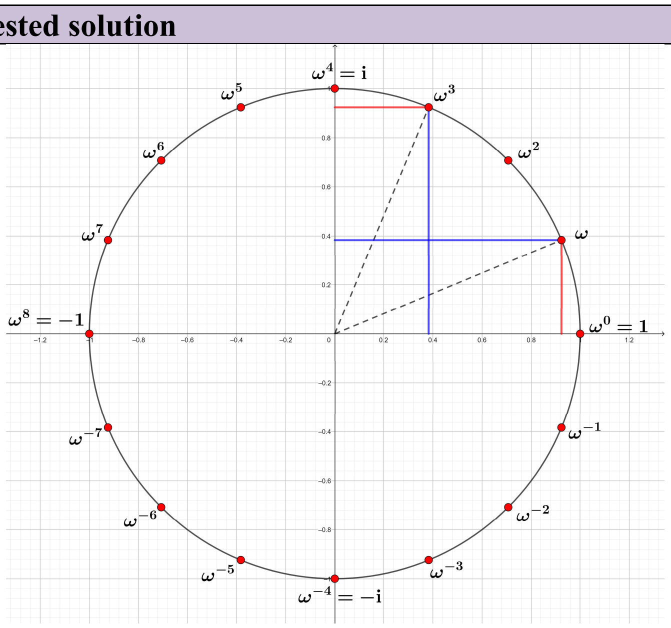
Qn	**Suggested solution**	**Markscheme**
1	***Calculus and Proofs***	**[Marks: 25]**
(a)	 $\int_0^{999999} (t^{1-1}\cdot e^{-t})dt$ 1. $\int_0^{999999} (t^{2-1}\cdot e^{-t})dt$ 1.	**A1** – 2 correct values
	 $\int_0^{999999} (t^{3-1}\cdot e^{-t})dt$ 2. $\int_0^{999999} (t^{4-1}\cdot e^{-t})dt$ 6.	**A1** – 2 correct values
	 $\int_0^{999999} (t^{5-1}\cdot e^{-t})dt$ 24. $\int_0^{999999} (t^{6-1}\cdot e^{-t})dt$ 120.	**A1** – 2 correct values
(b)	$\Gamma(n) = (n-1)!$	**A1**
(c)		**M1**
	$f(x) \to 0$ as $x \to \infty$	**A1**

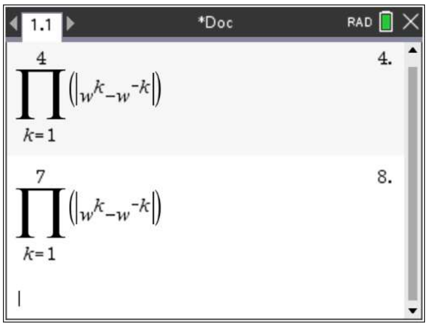
Qn	Suggested solution	Markscheme
(d)	$\Gamma(n+1) = \int_0^\infty \underbrace{t^n}_{u}\,\underbrace{\mathrm{e}^{-t}}_{dv}\,\mathrm{d}t$ $u = t^n \Longrightarrow \mathrm{d}u = nt^{n-1}$ $\mathrm{d}v = \mathrm{e}^{-t}\mathrm{d}t \Longrightarrow v = -e^{-t}$ $= -t^n e^{-t}\vert_0^\infty - \int_0^\infty -nt^{n-1}\mathrm{e}^{-t}\mathrm{d}t$ $= 0 + n\int_0^\infty t^{n-1}\mathrm{e}^{-t}\mathrm{d}t$ $= n\Gamma(n)$	**(M1)** **A1** – $-t^n e^{-t}\vert_0^\infty$ **A1** – $n\int_0^\infty t^{n-1}\mathrm{e}^{-t}\mathrm{d}t$ **AG**
(e)	Let $P(n)$ be the statement $\Gamma(n) = (n-1)!$ $\Gamma(1) = 1 = 0! = (1-1)!$, thus, $P(1)$ is true. Assume $P(k)$ is true for some positive integer k, i.e. $\Gamma(k) = (k-1)!$ From **(b)**, $\Gamma(k+1) = k\Gamma(k) = k(k-1)! = k!$. Thus, $P(k+1)$ is also true. Therefore, since $P(1)$ is true and $P(k+1)$is true whenever $P(k)$ is assumed true, by mathematical induction, $P(n)$ is true for all positive integer n.	**A1** **M1** **M1** – use of **(b)** **A1** – use of inductive assumption **R1** – only if M1M1A1 have been awarded prior
(f)	$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{-1/2}\mathrm{e}^{-t}\mathrm{d}t$ $t = \frac{1}{2}y^2 \Longrightarrow \mathrm{d}t = y\,\mathrm{d}y$ $\Gamma\left(\frac{1}{2}\right) = \int_0^\infty \left(\frac{1}{2}y^2\right)^{-1/2} \mathrm{e}^{-\frac{1}{2}y^2} y\,\mathrm{d}y$ $= \int_0^\infty \frac{\sqrt{2}}{y}\mathrm{e}^{-\frac{1}{2}y^2} y\,\mathrm{d}y$ $= \sqrt{2}\int_0^\infty \mathrm{e}^{-\frac{1}{2}y^2}\mathrm{d}y$	**A1** – correct $n = \frac{1}{2}$ **(M1)** – differentiation **M1** – substitution **A1** – condone $\sqrt{y^2} = y$ **A1**
(g)	Since the total probability is 1 and through symmetry, we deduce that $\int_0^\infty \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}z^2}\mathrm{d}z = \frac{1}{2}$	**R1** **A1**

Qn	Suggested solution	Markscheme
(h)	Hence, $\int_0^\infty \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}y^2}\,\mathrm{d}y = \frac{1}{2} \Rightarrow \int_0^\infty \mathrm{e}^{-\frac{1}{2}y^2}\,\mathrm{d}y = \frac{\sqrt{2\pi}}{2} = \frac{\sqrt{\pi}}{\sqrt{2}}$ and so $\Gamma\left(\frac{1}{2}\right) = \sqrt{2}\int_0^\infty \mathrm{e}^{-\frac{1}{2}y^2}\,\mathrm{d}y \quad = \quad \sqrt{2} \times \frac{\sqrt{\pi}}{\sqrt{2}}$ $= \quad \sqrt{\pi}$	**(A1)** **M1** **AG**
(i)	Hence (and from (c)), $\Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$	**M1** – using (c) **A1**
2	***Complex Numbers & Roots of Unity***	**[Marks: 30]**
(a)	$\left(\left(1+\sqrt{2}\right)+\mathrm{i}\right)^2 = \left(1+\sqrt{2}\right)^2 + 2\left(1+\sqrt{2}\right)\mathrm{i} + \mathrm{i}^2$ $= 1 + 2\sqrt{2} + 2 + 2\mathrm{i} + 2\sqrt{2}\mathrm{i} - 1$ $= \left(2+2\sqrt{2}\right) + \left(2+2\sqrt{2}\right)\mathrm{i}$	**M1** **A1**
(b)	$\arg z^2 = \arctan\left(\frac{2+2\sqrt{2}}{2+2\sqrt{2}}\right) = \arctan 1 = \frac{\pi}{4}$	**M1A1**
(c)	$\arg z = \frac{1}{2}\arg z^2 = \frac{\pi}{8}$	**M1A1**
(d)	Hence, $\tan\frac{\pi}{8} = \frac{\mathrm{Im}(z)}{\mathrm{Re}(z)} = \frac{1}{1+\sqrt{2}}$ OR $= \frac{1-\sqrt{2}}{1-2} = \sqrt{2}-1$	**M1A1**
(e)	(i) $\lvert z_1 - z_2\rvert = \lvert x_1 - x_2 + \mathrm{i}(y_1 - y_2)\rvert$ $= \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$ (ii) $\lvert z_1 - z_2\rvert = \mathrm{dist}(z_1, z_2) =$ the distance between z_1 and z_2 in the Argand diagram	**A1** **A1**
(f)	$\lvert\omega^k\rvert = \lvert\omega\rvert^k = \left\lvert\frac{z}{\lvert z\rvert}\right\rvert^k = \left(\frac{\lvert z\rvert}{\lvert z\rvert}\right)^k = 1^k = 1$	**A1**

Qn	Suggested solution	Markscheme
(g)		**A1** – $\omega^{-4}, \omega^0, \omega^4, \omega^8$ **A1** – all other powers
(h)		
	The square formed by $\omega^{-5}, \omega^{-1}, \omega^3, \omega^7$ is congruent to the square formed by $\omega^{-4}, \omega^0, \omega^4, \omega^8$ which clearly has side length $\sqrt{2}$.	**(M1)**
	Thus, the area is $\sqrt{2} \times \sqrt{2} = 2$	**A1**
	OR via GDC:	
	 $w:=e^{\frac{\pi \cdot i}{8}}$ $0.923879532511+0.382683432365 \cdot i$ $\lvert w^{-5}-w^{-1} \rvert$ 1.41421356237 $\sqrt{2}$ 1.41421356237	**(M1)A1**
	OR	
	Area $= 4 \times \left(\frac{1}{2}(1)(1)\sin 90°\right) = 2$	**M1A1**

Qn	**Suggested solution**	**Markscheme**
(i)		
	From (e), $$\left\lvert \omega - \frac{1}{\omega} \right\rvert = \mathrm{dist}(\omega, \omega^{-1}) = 2\mathrm{Im}(\omega) = \frac{2}{\sqrt{4 + 2\sqrt{2}}}$$	**(M1)** – any valid approach, e.g. use of (e) or cosine rule **A1**
	since $$\omega = \frac{z}{\lvert z \rvert} = \frac{1 + \sqrt{2} + \mathrm{i}}{\sqrt{\left(1 + \sqrt{2}\right)^2 + 1^2}} = \frac{1 + \sqrt{2} + \mathrm{i}}{\sqrt{4 + 2\sqrt{2}}}$$	**(A1)**
	OR	
	Consider the triangle $\mathcal{O}\omega\omega^{-1}$, where $\mathcal{O}$ is the origin. By cosine rule, $\left\lvert \omega - \frac{1}{\omega} \right\rvert = \sqrt{1^2 + 1^2 - 2(1)(1)\cos 45°} = \sqrt{2 - \sqrt{2}}$	**(M1)A1A1**
	From the sketch, it follows that $$\omega^2 = \frac{1}{\sqrt{2}} + \frac{\mathrm{i}}{\sqrt{2}}$$	**(A1)**
	Thus, $$\left\lvert \omega^2 - \frac{1}{\omega^2} \right\rvert = \mathrm{dist}(\omega^2, \omega^{-2}) = 2\mathrm{Im}(\omega^2) = \frac{2}{\sqrt{2}} = \sqrt{2}$$	**A1**

Qn	Suggested solution	Markscheme
(j)	 By symmetry, (as compared to ω)	
	$\omega^3 = \frac{1 + \mathrm{i}\left(1+\sqrt{2}\right)}{\sqrt{4+2\sqrt{2}}}$	**(A1)**
	Thus, $\left\lvert \omega^3 - \frac{1}{\omega^3} \right\rvert = 2\mathrm{Im}(\omega^3) = \frac{2+2\sqrt{2}}{\sqrt{4+2\sqrt{2}}}$	**A1**
	And so, $\left\lvert \omega - \frac{1}{\omega} \right\rvert \left\lvert \omega^3 - \frac{1}{\omega^3} \right\rvert = \frac{2}{\sqrt{4+2\sqrt{2}}} \times \frac{2+2\sqrt{2}}{\sqrt{4+2\sqrt{2}}} = \frac{2+2\sqrt{2}}{4+2\sqrt{2}}$	**M1**
	$\frac{2+2\sqrt{2}}{4+2\sqrt{2}} = \frac{1+\sqrt{2}}{2+\sqrt{2}} \times \frac{2-\sqrt{2}}{2-\sqrt{2}} = \frac{4+4\sqrt{2}-2\sqrt{2}-4}{4-2} = \sqrt{2}$	**M1** – rationalization of the denominator **AG**
	OR	
	$\left\lvert \omega^3 - \frac{1}{\omega^3} \right\rvert = \sqrt{1^2+1^2-2(1)(1)\cos 135^\circ} = \sqrt{2+\sqrt{2}}$	**(M1)A1**
	And so, $\left\lvert \omega - \frac{1}{\omega} \right\rvert \left\lvert \omega^3 - \frac{1}{\omega^3} \right\rvert = \sqrt{2-\sqrt{2}}\sqrt{2+\sqrt{2}}$	**M1**
	$= \sqrt{2^2-2}$	**A1**
	$= \sqrt{2}$	**AG**

Qn	**Suggested solution**	**Markscheme**
(k)	$\prod_{k=1}^{4}\left\lvert\omega^k-\frac{1}{\omega^k}\right\rvert$ $= \left\lvert\omega^1-\frac{1}{\omega^1}\right\rvert\left\lvert\omega^2-\frac{1}{\omega^2}\right\rvert\left\lvert\omega^3-\frac{1}{\omega^3}\right\rvert\left\lvert\omega^4-\frac{1}{\omega^4}\right\rvert$ $= \underbrace{\left\lvert\omega^1-\frac{1}{\omega^1}\right\rvert\left\lvert\omega^3-\frac{1}{\omega^3}\right\rvert}_{=\sqrt{2}}\underbrace{\left\lvert\omega^2-\frac{1}{\omega^2}\right\rvert}_{=\sqrt{2}}\underbrace{\left\lvert\omega^4-\frac{1}{\omega^4}\right\rvert}_{=2}$ $= \sqrt{2}\times\sqrt{2}\times 2$ $= 4$	**(M1)A1**
	$\prod_{k=1}^{7}\left\lvert\omega^k-\frac{1}{\omega^k}\right\rvert$ $= \prod_{k=1}^{4}\left\lvert\omega^k-\frac{1}{\omega^k}\right\rvert\times\underbrace{\left\lvert\omega^5-\frac{1}{\omega^5}\right\rvert}_{=\left\lvert\omega^3-\frac{1}{\omega^3}\right\rvert}\underbrace{\left\lvert\omega^6-\frac{1}{\omega^6}\right\rvert}_{=\left\lvert\omega^2-\frac{1}{\omega^2}\right\rvert}\underbrace{\left\lvert\omega^7-\frac{1}{\omega^7}\right\rvert}_{=\left\lvert\omega^1-\frac{1}{\omega^1}\right\rvert}$ $= 4\times\sqrt{2}\times\sqrt{2}$ $= 8$	**(M1)A1**
	$\prod_{k=1}^{10}\left\lvert\omega^k-\frac{1}{\omega^k}\right\rvert = 0$ because $\left\lvert\omega^8-\frac{1}{\omega^8}\right\rvert = 0$	**A1(R1)**
	OR via GDC:  $\prod_{k=1}^{4}\left(\left\lvert w^k-w^{-k}\right\rvert\right)$ 4. $\prod_{k=1}^{7}\left(\left\lvert w^k-w^{-k}\right\rvert\right)$ 8.	