H

TEMASEK JUNIOR COLLEGE,
SINGAPORE
JC 2
Preliminary Examination 2019

CANDIDATE NAME | | CG | |

MATHEMATICS

9758/01

Higher 2

30 Aug 2019

Paper 1

3 hours

Candidates answer on the Question Paper.
Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your Civics group and name on all the work that you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
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You are reminded of the need for clear presentation in your answers.
The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

For Examiner's Use	
Question Number	Marks Obtained
1	
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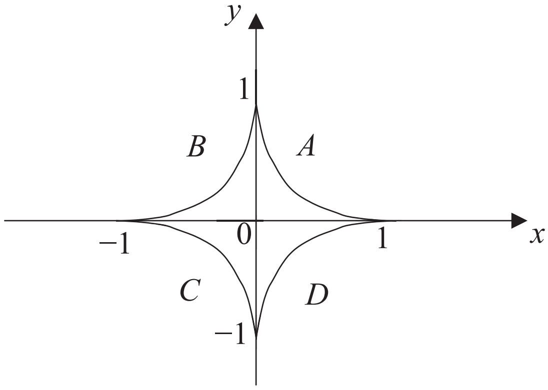
11	
Total Marks	

TEMASEK JUNIOR COLLEGE, SINGAPORE
PASSION PURPOSE DRIVE

1 The diagram below shows a shape which is symmetrical about the x- and y-axes. The shape is made up of four curves, A, B, C and D.



The curve A has equation $\sqrt{x}+\sqrt{y}=1$ for $0 \le x \le 1$ and $0 \le y \le 1$.

(i) State the equations and the range of values of x and y for curves B and C. [3]

(ii) The curves A and B are scaled by a factor $\frac{1}{2}$ parallel to the x-axis and the curves C and D are scaled by a factor 2 parallel to the y-axis. Sketch the resulting shape. [2]

2 The position vectors of A, B, C and D are $\begin{pmatrix}\alpha\\1\\5\end{pmatrix}$, $\begin{pmatrix}1\\-2\\1\end{pmatrix}$, $\begin{pmatrix}-2\\3\\1\end{pmatrix}$ and $\begin{pmatrix}1\\\beta\\7\end{pmatrix}$ respectively, where α and β are real numbers. Given that BD is a perpendicular bisector of AC, find the values of α and β. [5]

3 The hyperbola C passes through the point (2, 0) and has oblique asymptotes $y=-2x$ and $y=2x$.

(i) Sketch C, showing the relevant features of the curve. [2]

(ii) Write down the equation of C. [1]

(iii) By adding a suitable curve to your sketch in part **(i)**, solve the inequality

$$\sqrt{\frac{x^2}{4}-1} < \sqrt{x-1}\,.$$ [3]

4 A curve C has equation $y = \dfrac{e^x}{x-k}$, $x \neq k$, where k is a positive real number.

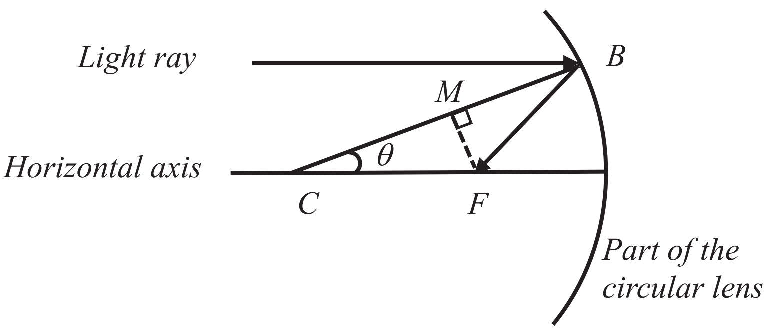
Show algebraically that C has exactly one stationary point, and show that the stationary point lies in the first quadrant. [3]

Sketch C for $x > k$, indicating clearly the equation of the asymptote and the coordinates of the stationary point. [2]

Deduce that $\displaystyle\int_{k+\frac{1}{2}}^{k+\frac{3}{2}} \frac{e^x}{x-k}\,dx < \frac{1}{3}\left(3e^{k+\frac{1}{2}} + e^{k+\frac{3}{2}}\right)$ for all positive real values of k. [2]

5 In geometric optics, the paraxial approximation is a small-angle approximation used in Gaussian optics and ray tracing of light through an optical system such as a lens.

In the diagram below, a light ray parallel to the horizontal axis is reflected at point B on the circular lens centred at point C and has radius r cm. Let $\angle BCF = \theta$ radians. FM is the perpendicular bisector of CB.



(i) Show that $CF = \dfrac{r}{k\cos\theta}$, where k is a real constant to be determined. [1]

(ii) Hence find the series expansion for CF if θ is sufficiently small for θ^3 and terms in higher powers of θ to be neglected. [2]

Suppose that the source of the light ray is now repositioned such that $\angle BCF = \left(\theta + \dfrac{\pi}{6}\right)$ radians.

(iii) Find the corresponding series expansion for CF, up to and including the term in θ^2. [4]

6 An arithmetic sequence has first term a and common difference d, where a and d are non-zero. The ninth, tenth and thirteenth terms of the arithmetic sequence are the first three terms of a geometric sequence.

(i) Show that $a = -\frac{15}{2}d$. [3]

(ii) The sum of the first n terms of the arithmetic sequence is denoted by S_n. Find the value of S_{16}. [2]

(iii) Given that the k^{th} term of the arithmetic sequence is the fourth term of the geometric sequence, find the value of k. [3]

7 A curve C has parametric equations

$$x = t^2, \qquad y = t\,e^{t^2}, \qquad \text{for } t \geq 0.$$

(i) Find the equation of the tangent to C at the point P with coordinates $\left(p^2, p\,e^{p^2}\right)$, where $p \neq 0$. Hence, or otherwise, find the exact equation of the tangent L to C which passes through the origin. [5]

(ii) **(a)** Find the cartesian equation of C. [1]

(b) Find the exact volume of the solid formed when the region bounded by C and L is rotated through 2π radians about the x-axis. [5]

8 **Do not use a calculator in answering this question.**

The complex numbers z and w are given by $z = \frac{(1+i)^4}{(1-i)^2}$ and $w = \frac{8}{\left(\sqrt{3}+i\right)^2}$.

(i) Express z and w in polar form $r(\cos\theta + i\sin\theta)$, where $r > 0$ and $-\pi < \theta \leq \pi$. Give r and θ in exact form. [4]

(ii) Given that z^2, w and w^* are the roots of the equation $x^3 + bx^2 + cx + d = 0$ where b, c and d are real values, find the equation. [3]

(iii) Sketch on an Argand diagram with origin O, the points P, Q and R representing the complex numbers z, w and $z+w$ respectively. [2]

(iv) By considering the quadrilateral $OPRQ$ and the argument of $z+w$, deduce that

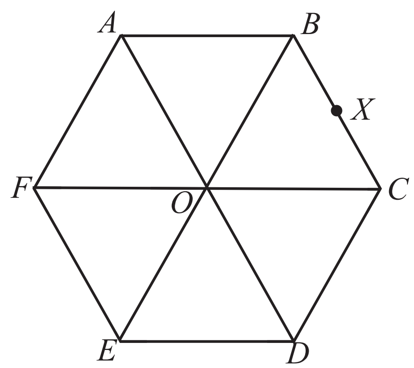
$$\tan\frac{5\pi}{12} = 2 + \sqrt{3}\,. \qquad [3]$$

9 **(a)** Vectors **u** and **v** are such that $\mathbf{u}\cdot\mathbf{v}=-1$ and $(\mathbf{u}\times\mathbf{v})+\mathbf{u}$ is perpendicular to $(\mathbf{u}\times\mathbf{v})+\mathbf{v}$.

Show that $|\mathbf{u}\times\mathbf{v}|=1$. [3]

Hence find the angle between **u** and **v**. [3]

(b) The figure shows a regular hexagon *ABCDEF* with *O* at the centre of the hexagon. *X* is the midpoint of *BC*.



Given that $\overrightarrow{OA}=\mathbf{a}$ and $\overrightarrow{OB}=\mathbf{b}$, find $\overrightarrow{OF}$ and $\overrightarrow{OX}$ in terms of **a** and **b**. [2]

Line segments *AC* and *FX* intersect at the point *Y*. Determine the ratio *AY* : *YC*. [4]

10 Mr Ng wants to hang a decoration on the vertical wall above his bookshelf. He needs a ladder to climb up.

The rectangle $ABCD$ is the side-view of the bookshelf and HK is the side-view of the ladder where $AB = 24$ cm and $BC = 192$ cm (see **Figure 1**). The ladder touches the wall at H, the edge of the top of the bookshelf at B and the floor at K.

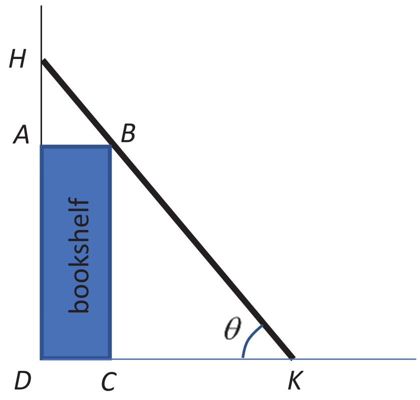


Figure 1

(i) Given that $\angle HKD = \theta$, show that the length , L cm of the ladder is given by

$$L = \frac{24}{\cos\theta} + \frac{192}{\sin\theta} .$$ [1]

(ii) Use differentiation to find the exact value of the shortest length of the ladder as θ varies. [4]
[You do not need to verify that this length of the ladder is the shortest.]

Take L to be 270 for the rest of this question.

The ladder starts to slide such that H moves away from the wall and K moves towards E (see **Figure 2**). The ladder maintains contact with the bookshelf at B.

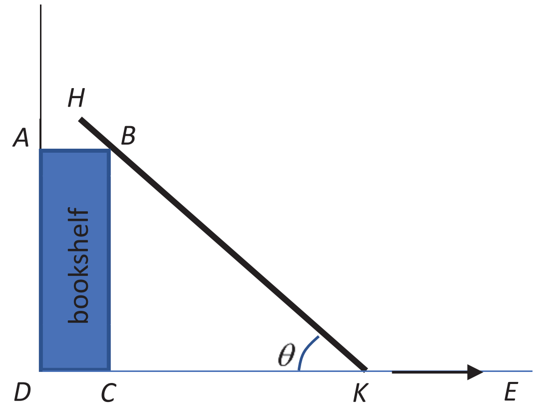


Figure 2

The horizontal distances from the wall to H and from the wall to K are x cm and y cm respectively.

(iii) By expressing $y - x$ in terms of θ, determine whether the rate of change of y is greater than the rate of change of x. [3]

(iv) Given that the rate of change of θ is -0.1 rad s^{-1} when $CK = 160$ cm, find the rate of change of x at this instant. [5]

11 The daily food calories, L, taken in by a human body are partly used to fulfill the daily requirements of the body. The daily requirements is proportional to the body mass, M kg, with a constant of proportionality p. The rate of change in body mass is proportional to the remaining calories.

It is given that the body mass, M kg, at time t days satisfies the differential equation

$$\frac{dM}{dt} = k\left(L - pM\right),$$

where k and L are constants.

John's initial body mass is 100 kg. Find, in terms of p, the daily food calories needed to keep his body mass constant at 100 kg. [1]

To lose weight, John decides to start on a diet where his daily food calorie intake is 75% of the daily calories needed to keep his body mass constant at 100 kg.

(i) Show that $M = 75 + 25e^{-pkt}$. [4]

(ii) John attained a body mass of 90 kg after 50 days on this diet. If it takes him n more days to lose at least another 10 kg, find the smallest integer value of n. [5]

(iii) John's goal with this diet plan is to achieve a body mass of 70 kg. With the aid of a graph, explain why he can never achieve his goal. [2]

(iv) By considering $\frac{d^2M}{dt^2}$, comment on his rate of body mass loss as time passes. [2]

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JC 2
Preliminary Examination 2019

CANDIDATE NAME ______________________ CG ________

MATHEMATICS

9758/02

Higher 2

16 Sep 2019

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3 hours

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TEMASEK JUNIOR COLLEGE, SINGAPORE
PASSION PURPOSE DRIVE

[Turn over

1 The function f is defined by

$$\mathrm{f}: x \mapsto \frac{x^2}{2-x},\ x \in \mathbb{R},\ 0 \le x < 2\ .$$

(i) Find $\mathrm{f}^{-1}(x)$ and write down the domain of f^{-1}. [4]

It is given that

$$\mathrm{g}: x \mapsto \frac{1}{1+\mathrm{e}^{-x}},\ x \in \mathbb{R},\ x \ge 0\ .$$

(ii) Show that fg exists. [2]

(iii) Find the range of fg. [2]

2 Express $\dfrac{6r+7}{r(r+1)}$ as partial fractions. [1]

(i) Hence use method of differences to find $\displaystyle\sum_{r=1}^{N}\left(\left(\frac{1}{7}\right)^{r+1}\frac{6r+7}{r(r+1)}\right)$ in terms of N.

(There is no need to express your answer as a single algebraic fraction.) [3]

(ii) Give a reason why the series $\displaystyle\sum_{r=1}^{\infty}\left(\left(\frac{1}{7}\right)^{r+1}\frac{6r+7}{r(r+1)}\right)$ converges, and write down its value. [2]

(iii) Use your answer in part **(i)** to find $\displaystyle\sum_{r=1}^{N}\left(\left(\frac{1}{7}\right)^{r+1}\frac{6r+13}{(r+1)(r+2)}\right)$. [3]

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3 A curve C with equation $y = \mathrm{f}(x)$ satisfies the equation

$$\left(x^2 + 2x + 2\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 2$$

and passes through the point $(0, \pi)$.

(i) By further differentiation, find the Maclaurin expansion of $\mathrm{f}(x)$ in ascending powers of x up to and including the term x^3. [5]

(ii) Solve the differential equation $\left(x^2 + 2x + 2\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 2$, given that $y = \pi$ when $x = 0$, leaving y in terms of x.

Hence show that

$$\tan^{-1}(x+1) \approx \frac{\pi}{4} + \frac{1}{2}x - \frac{1}{4}x^2 + \frac{1}{12}x^3$$

for small values of x. [4]

(iii) With the aid of a sketch, explain why $\int_0^2 \frac{\pi}{4} + \frac{1}{2}x - \frac{1}{4}x^2 + \frac{1}{12}x^3 \, \mathrm{d}x$ gives a more accurate approximation of $\int_0^2 \tan^{-1}(x+1) \, \mathrm{d}x$ than $\int_0^2 \frac{\pi}{4} + \frac{1}{2}x - \frac{1}{4}x^2 \, \mathrm{d}x$. [2]

4 The points A, B, C and D have coordinates $(1, 0, 3)$, $(-1, 0, 1)$, $(1, 1, 3)$ and $(1, k, 0)$ respectively, where k is a positive real number. The plane p_1 contains A, B and C while the plane p_2 contains A, B and D.

Given that p_1 makes an angle of $\frac{\pi}{3}$ with p_2, show that $k = \frac{\sqrt{6}}{2}$. [5]

The point X lies on p_2 such that the vector $\overrightarrow{XC}$ is perpendicular to p_1. Find $\overrightarrow{XC}$. [5]

Hence find the exact area of the triangle AXC. [2]

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Section B: Probability and Statistics [60 marks]

5 Anand, Beng, Charlie, Dayanah and 6 other people attend a banquet dinner, and are to sit at a round table.

(i) Dayanah will only sit next to Anand, Beng or Charlie (and no one else), and Anand, Beng and Charlie do not want to sit next to each other. Find the number of ways the 10 people can seat themselves around the table. [4]

(ii) As part of dinner entertainment, 4 people from the table are chosen to participate in a game.

Among Anand, Beng and Charlie, if any one of them is chosen, the other two will refuse to participate in the game. Furthermore, Dayanah refuses to participate unless at least one of Anand, Beng or Charlie is also chosen.

Find the number of ways the 4 people can be chosen for the game. [3]

6 A bag contains four identical counters labelled with the digits 0, 1, 2, and 3. In a game, Amira chooses one counter randomly from the bag and then tosses a fair coin. If the coin shows a Head, her score in the game is the digit labelled on the counter chosen. If the coin shows a Tail, her score in the game is the negative of the digit labelled on the counter chosen. T denotes the score in a game.

(i) Find the probability distribution of T. [2]

(ii) Amira tosses the coin and it shows a Tail. Find the probability that $T < -1$. [3]

(iii) Amira plays the game twice. Find the probability that the sum of her two scores is positive. [3]

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7 It is generally accepted that a person's diet and cardiorespiratory fitness affects his cholesterol levels. The results of a study on the relationship between the cholesterol levels, C mmol/L, and cardiorespiratory fitness, F, measured in suitable units, on 8 individuals with similar diets are given in the following table.

Cardiorespiratory Fitness (F units)	55.0	50.7	45.3	40.2	34.7	31.9	27.9	26.0
Cholesterol (C mmol/L)	4.70	4.98	5.30	5.64	6.04	6.30	6.99	6.79

(i) Draw a scatter diagram of these data. Suppose that the relationship between F and C is modelled by an equation of the form $\ln C = aF + b$, where a and b are constants. Use your diagram to explain whether a is positive or negative. [4]

(ii) Find the product moment correlation coefficient between ln C and F, and the constants a and b for the model in part **(i)**. [3]

(iii) Bronz is a fitness instructor. His cardiorespiratory fitness is 52.0 units. Estimate Bronz's cholesterol level using the model in **(i)** and the values of a and b in part **(ii)**. Comment on the reliability of the estimate. [2]

(iv) Bronz then had a medical checkup and found his actual cholesterol level to be 6.2 mmol/L. Assuming his cholesterol level is measured accurately, explain why there is a great difference between Bronz's cholesterol level and the estimated value in **(iii)**. [1]

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8 A research laboratory uses a data probe to collect data for its experiments. There is a probability of 0.04 that the probe will give an incorrect reading. In a particular experiment, the probe is used to take 80 readings, and X denotes the number of times the probe gives an incorrect reading.

(i) State, in context, two assumptions necessary for X to be well modelled by a binomial distribution. [2]

(ii) Find the probability that between 5 and 10 (inclusive) incorrect readings are obtained in the experiment. [3]

When the probe gives an incorrect reading, it will give a reading that is 5% greater than the actual value.

(iii) Suppose the 80 readings are multiplied together to obtain a Calculated Value. Find the probability that the Calculated Value is at least 50% more than the product of the 80 actual values. [5]

9 A Wheel Set refers to a set of wheel rim and tyre. The three types of wheel sets are the Clincher Bike Wheel Set, Tubular Bike Wheel Set and Mountain Bike Wheel Set.
The weight of a rim of a Clincher Bike Wheel Set follows a normal distribution with mean 1.5 kg and standard deviation 0.01 kg. The weight of its tyre follows a normal distribution with mean 110 g and standard deviation 5 g.

(i) Let C be the total weight in grams of a randomly chosen Clincher Bike Wheel Set in grams. Find $\mathrm{P}(C > 1620)$. [3]

(ii) State, in the context of the question, an assumption required in your calculation in **(i)**. [1]

Let T be the total weight in grams of a Tubular Bike Wheel Set, where $T \sim \mathrm{N}(\mu,\ 15^2)$.

(iii) The probability that the weight of a randomly chosen Clincher Bike Wheel Set exceeds a randomly chosen Tubular Bike Wheel Set by more than 150 g is smaller than 0.70351 correct to 5 decimal places. Find the range of values that μ can take. [5]

Let M be the total weight in grams of a randomly chosen Mountain Bike Wheel Set with mean 1800 g and standard deviation 20 g.

(iv) Find the probability that the mean weight of 50 randomly chosen Mountain Bike Wheel Sets is more than 1795 g. [3]

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10 **(a)** Two random samples of different sample sizes of households in the town of Aimek were taken to find out the mean number of computers per household there. The first sample of 50 households gave the following results.

Number of computers	0	1	2	3	4
Number of households	5	12	18	10	5

The results of the second sample of 60 households were summarised as follows.

$$\sum y = 118 \qquad \sum y^2 = 314,$$

where y is the number of computers in a household.

(i) By combining the two samples, find unbiased estimates of the population mean and variance of the number of computers per household in the town. [4]

(ii) Describe what you understand by ‘population’ in the context of this question. [1]

(b) Past data has shown that the working hours of teachers in a city are normally distributed with mean 48 hours per week. In a recent study, a large random sample of n teachers in the city was surveyed and the number of working hours per week was recorded. The sample mean was 46 hours and the sample variance was 131.1 hours^2. A hypothesis test is carried out to determine whether the mean working hours per week of teachers has been reduced.

(i) State appropriate hypotheses for the test. [1]

The calculated value of the test statistic is $z = -1.78133$ correct to 5 decimal places.

(ii) Deduce the conclusion of the test at the 2.5 % level of significance. [2]

(iii) Find the value of n. [3]

(iv) In another test, using the same sample, there is significant evidence at the $\alpha\%$ level that there is a change in the mean working hours per week of teachers in that city. Find the smallest possible integral value of α . [2]

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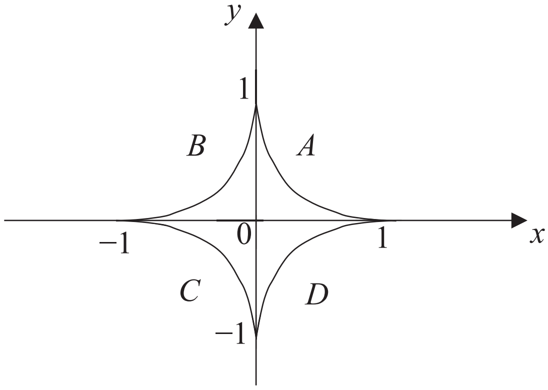
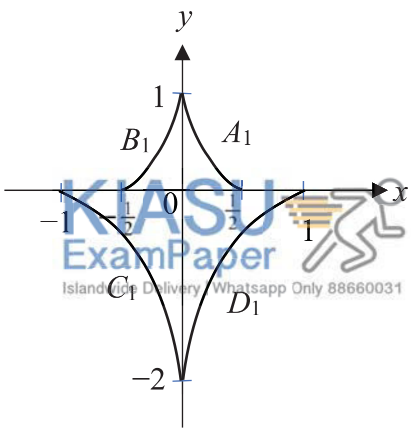


The curve A has equation $\sqrt{x}+\sqrt{y}=1$ for $0 \le x \le 1$ and $0 \le y \le 1$.

(i) State the equations and the range of values of x and y for curves B and C. [3]

(ii) The curves A and B are scaled by a factor $\frac{1}{2}$ parallel to the x-axis and the curves C and D are scaled by a factor 2 parallel to the y-axis. Sketch the resulting shape. [2]

Solution:

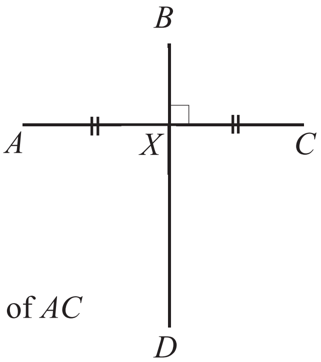
(i) B is the reflection of A in the y-axis. So replace x with $-x$: Equation of B: $\sqrt{-x}+\sqrt{y}=1,\quad -1 \le x \le 0,\ 0 \le y \le 1$ C is the reflection of B in the x-axis. So replace y with $-y$: Equation of C: $\sqrt{-x}+\sqrt{-y}=1,\ -1 \le x \le 0,\ -1 \le y \le 0$	A common mistake is the re-writing of the equation $\sqrt{x}+\sqrt{y}=1$ as $y=\left(1-\sqrt{x}\right)^2$. Note that $y=\left(1-\sqrt{x}\right)^2 \Rightarrow 1-\sqrt{x}=\pm\sqrt{y}$ which is the equation of 2 curves $1-\sqrt{x}=\sqrt{y}$ and $1-\sqrt{x}=-\sqrt{y}$.
(ii) 	Curves A and B scaled by a factor $\frac{1}{2}$ parallel to the x-axis: Multiply $\frac{1}{2}$ to the x-coordinate with y-coordinate invariant. Curves C and D scaled by a factor 2 parallel to the y-axis: Multiply 2 to the y-coordinate with x-coordinate invariant. <u>Note</u> that the **symmetry** (stated in question) must be **clearly seen**.

2 The position vectors of A, B, C and D are $\begin{pmatrix}\alpha\\1\\5\end{pmatrix}$, $\begin{pmatrix}1\\-2\\1\end{pmatrix}$, $\begin{pmatrix}-2\\3\\1\end{pmatrix}$ and $\begin{pmatrix}1\\\beta\\7\end{pmatrix}$ respectively, where α and β are real numbers. Given that BD is a perpendicular bisector of AC, find the values of α and β. [5]

Solution:

$$\overrightarrow{BD}=\begin{pmatrix}1\\\beta\\7\end{pmatrix}-\begin{pmatrix}1\\-2\\1\end{pmatrix}=\begin{pmatrix}0\\\beta+2\\6\end{pmatrix}$$

$$\overrightarrow{AC}=\begin{pmatrix}-2\\3\\1\end{pmatrix}-\begin{pmatrix}\alpha\\1\\5\end{pmatrix}=\begin{pmatrix}-2-\alpha\\2\\-4\end{pmatrix}$$



BD is a perpendicular bisector of AC

$\Rightarrow \overrightarrow{BD}\cdot\overrightarrow{AC}=0$

$\Rightarrow \begin{pmatrix}0\\\beta+2\\6\end{pmatrix}\cdot\begin{pmatrix}-2-\alpha\\2\\-4\end{pmatrix}=0$

$\Rightarrow 2(\beta+2)-24=0$

$\Rightarrow 2\beta=20 \Rightarrow \beta=10$

Let X be the midpoint of AC.

$$\overrightarrow{OX}=\frac{\overrightarrow{OA}+\overrightarrow{OC}}{2}=\frac{\begin{pmatrix}\alpha\\1\\5\end{pmatrix}+\begin{pmatrix}-2\\3\\1\end{pmatrix}}{2}=\begin{pmatrix}\frac{\alpha-2}{2}\\2\\3\end{pmatrix}$$

Line BD: $\underset{\sim}{r}=\begin{pmatrix}1\\-2\\1\end{pmatrix}+\lambda\begin{pmatrix}0\\12\\6\end{pmatrix}$

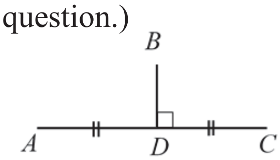
X lies on line BD: $\begin{pmatrix}\frac{\alpha-2}{2}\\2\\3\end{pmatrix}=\begin{pmatrix}1\\-2\\1\end{pmatrix}+\lambda\begin{pmatrix}0\\12\\6\end{pmatrix}$

1st row: $\dfrac{\alpha-2}{2}=1 \Rightarrow \alpha=4$

Many students still do not use appropriate notations for vectors solutions.

<u>Common **mis**-interpretations of question:</u>

1) $BX:XD\neq 1:1$ (BD bisects AC; AC does not bisect BD.)
2) **Neither** B **nor** D is the midpoint of AC. (Do not assume what is not given in the question.)



<u>Common mistake:</u>

$$\overrightarrow{AX}=\begin{pmatrix}1\\-2\\1\end{pmatrix}+\lambda\begin{pmatrix}0\\12\\6\end{pmatrix}$$

This shows poor understanding of the equation of a line.

The $\underset{\sim}{r}$ represents the **position vector** of **a point** on the line and not a "direction vector".

Alternative solution Let X be the midpoint of AC. $\overrightarrow{OX}=\dfrac{\overrightarrow{OA}+\overrightarrow{OC}}{2}=\dfrac{\begin{pmatrix}\alpha\\1\\5\end{pmatrix}+\begin{pmatrix}-2\\3\\1\end{pmatrix}}{2}=\begin{pmatrix}\frac{\alpha-2}{2}\\2\\3\end{pmatrix}$ Since BD is the perpendicular bisector of AC, therefore B, X and D must be collinear. $\overrightarrow{BX}=\overrightarrow{OX}-\overrightarrow{OB}=\begin{pmatrix}\frac{\alpha-2}{2}\\2\\3\end{pmatrix}-\begin{pmatrix}1\\-2\\1\end{pmatrix}=\begin{pmatrix}\frac{\alpha-4}{2}\\4\\2\end{pmatrix}$ $\overrightarrow{BD}=\overrightarrow{OD}-\overrightarrow{OB}=\begin{pmatrix}1\\\beta\\7\end{pmatrix}-\begin{pmatrix}1\\-2\\1\end{pmatrix}=\begin{pmatrix}0\\\beta+2\\6\end{pmatrix}$ Since B, X and D are collinear, $\overrightarrow{BX}=\lambda\overrightarrow{BD}$ $\begin{pmatrix}\frac{\alpha-4}{2}\\4\\2\end{pmatrix}=\lambda\begin{pmatrix}0\\\beta+2\\6\end{pmatrix}$ $\Rightarrow\quad \alpha=4, \lambda=\dfrac{1}{3}, \beta=10$	

3 The hyperbola C passes through the point (2, 0) and has oblique asymptotes $y=-2x$ and $y=2x$.

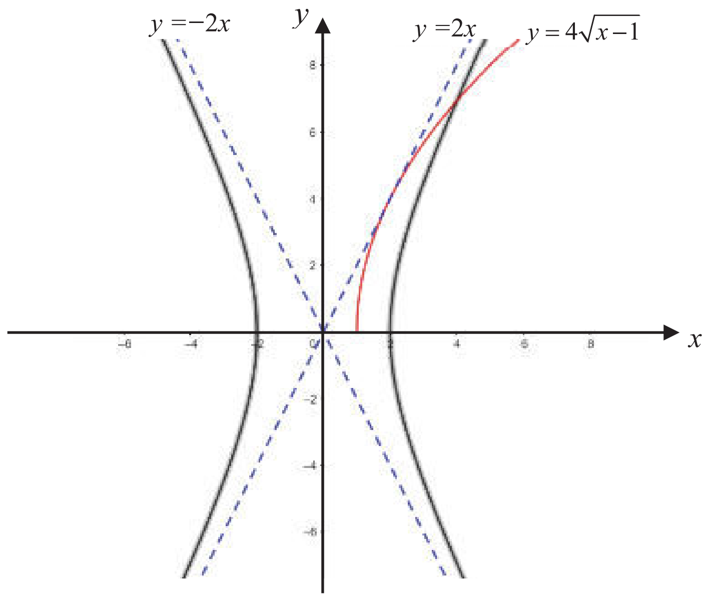
(i) Sketch C, showing the relevant features of the curve. [2]

(ii) Write down the equation of C. [1]

(iii) By adding a suitable curve to your sketch in part **(i)**, solve the inequality

$$\sqrt{\frac{x^2}{4}-1}<\sqrt{x-1}\,. \qquad [3]$$

Solution:

(i) 	Most students got this part correct. A small handful lost marks due to insufficient details, either missing out the equation of asymptotes or the x-intercept.
(ii) The hyperbola is of the form $\frac{x^2}{2^2}-\frac{y^2}{b^2}=1$ Oblique asymptotes is $y=\pm\frac{bx}{2}=\pm 2$ $\therefore \frac{b}{2}=2 \Rightarrow b=4$ Hence equation of hyperbola is $\frac{x^2}{4}-\frac{y^2}{16}=1$	A common error is to get b to be 1. Note that although b=1 will lead to the curve passing through (2,0), the asymptotes will not be $y=\pm 2x$.

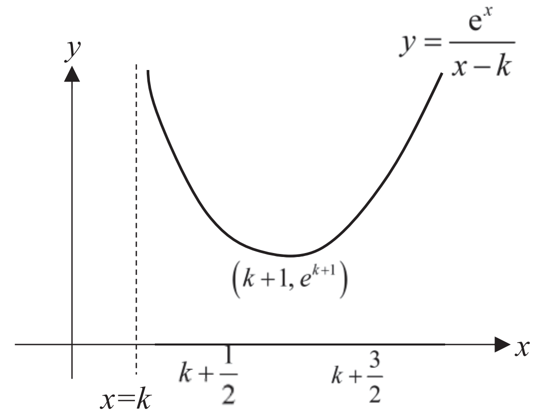
(iii) $\frac{y^2}{16}=\frac{x^2}{4}-1 \Rightarrow y^2=16\left(\frac{x^2}{4}-1\right) \Rightarrow y=\pm 4\sqrt{\frac{x^2}{4}-1}$ $4\sqrt{\frac{x^2}{4}-1}<4\sqrt{x-1}$ Sketch $y=4\sqrt{x-1}$ The 2 curves intersect at $x = 4$. Hence for $\sqrt{\frac{x^2}{4}-1}<\sqrt{x-1}$, $2\le x<4$	Students are lacking in details in their solution for this part. Equation of the suitable curve and details of the curve (such as intersection points) must be clearly indicated.

4 A curve C has equation $y=\dfrac{e^x}{x-k}$, $x \neq k$, where k is a positive real number.

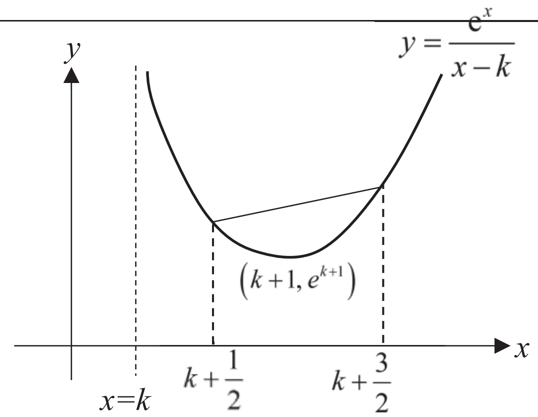
Show algebraically that C has exactly one stationary point, and show that the stationary point lies in the first quadrant. [3]

$y=\dfrac{e^x}{x-k}$ $\dfrac{dy}{dx}=\dfrac{(x-k)e^x-e^x}{(x-k)^2}=\dfrac{(x-k-1)e^x}{(x-k)^2}$ When $\dfrac{dy}{dx}=0$, $x=k+1$. Since there is only one real solution for $\dfrac{dy}{dx}=0$, there is exactly one stationary point. (Shown) Since $k>0$, $x=k+1>0$ and $y=\dfrac{e^{k+1}}{(k+1)-k}=e^{k+1}>0$ the turning point $\left(k+1, e^{k+1}\right)$ lies in the first quadrant. (Shown)	Most students are able to find $\dfrac{dy}{dx}$ and explain why there is only one solution. However, quite a handful of students failed to explain fully why it lies in the 1st quadrant. Many only indicated that the x-coordinate is +ve. Students need to know that for 1st quad, besides x-coordinates being positive, the y-coordinate must also be positive.

Sketch C for $x > k$, indicating clearly the equation of the asymptote and the coordinates of the stationary point. [2]

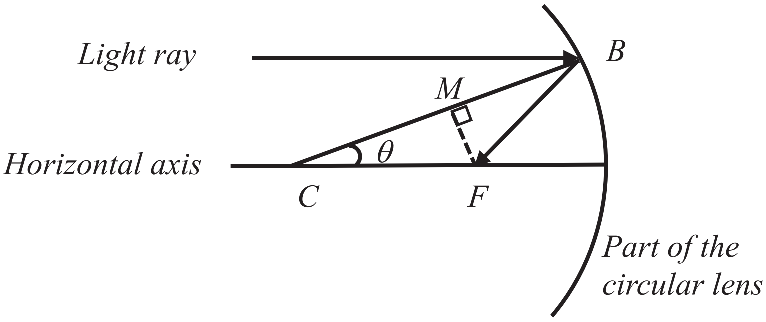
	Most students were able to sketch the curve correctly. Still, a handful lost marks due to missing details such as coordinates of stationary point and asymptote.

Deduce that $\displaystyle\int_{k+\frac{1}{2}}^{k+\frac{3}{2}} \frac{e^x}{x-k}\,dx < \frac{1}{3}\left(3e^{k+\frac{1}{2}} + e^{k+\frac{3}{2}}\right)$ for all positive real values of k. [2]

 Area under the curve from $k+\frac{1}{2}$ to $k+\frac{3}{2}$ = $\displaystyle\int_{k+\frac{1}{2}}^{k+\frac{3}{2}} \frac{e^x}{x-k}\,dx$ Area of trapezium = $\frac{1}{2}\left[\left(k+\frac{3}{2}\right)-\left(k+\frac{1}{2}\right)\right]\left(\dfrac{e^{k+\frac{1}{2}}}{\frac{1}{2}}+\dfrac{e^{k+\frac{3}{2}}}{\frac{3}{2}}\right)$ $= \frac{1}{2}(1)\left(2e^{k+\frac{1}{2}}+\dfrac{2e^{k+\frac{3}{2}}}{3}\right)$ Since area under graph < area of trapezium Therefore $\displaystyle\int_{k+\frac{1}{2}}^{k+\frac{3}{2}} \frac{e^x}{x-k}\,dx < \frac{1}{3}\left(3e^{k+\frac{1}{2}} + e^{k+\frac{3}{2}}\right)$	The last part is badly done. Students must get the hint from the earlier parts that the solution involves area under the curve. Instead of approaching this part using integration by parts, students should approach it by searching for an area of a known geometrical shape to make comparsion with the area under the curve.

5 In geometric optics, the paraxial approximation is a small-angle approximation used in Gaussian optics and ray tracing of light through an optical system such as a lens.

In the diagram below, a light ray parallel to the horizontal axis is reflected at point B on the circular lens centred at point C and has radius r cm. Let $\angle BCF = \theta$ radians. FM is the perpendicular bisector of CB.



(i) Show that $CF = \dfrac{r}{k\cos\theta}$, where k is a real constant to be determined. [1]

$\cos\theta = \dfrac{CM}{CF} \quad \Rightarrow \quad CF = \dfrac{CM}{\cos\theta} = \dfrac{r}{2\cos\theta}$	It is surprising that a small number of students could not get this part.

(ii) Hence find the series expansion for CF if θ is sufficiently small for θ^3 and terms in higher powers of θ to be neglected. [2]

(ii) Since θ is sufficiently small for powers of θ higher than θ^3 to be neglected, $CF = \dfrac{r}{k\cos\theta}$ $CF \approx \dfrac{r}{2\left(1-\dfrac{\theta^2}{2}\right)}$ $CF \approx \dfrac{r}{2}\left(1-\dfrac{\theta^2}{2}\right)^{-1}$ $CF \approx \dfrac{r}{2}\left[1+\dfrac{\theta^2}{2}+\ldots\right]$	A majority of students did not know that "series expansion" refers to a polynomial expression, and did not proceed beyond the first step.

Suppose that the source of the light ray is now repositioned such that $\angle BCF = \left(\theta + \frac{\pi}{6}\right)$ radians.

(iii) Find the corresponding series expansion for CF, up to and including the term in θ^2. [4]

(iii) $CF = \dfrac{r}{2\cos\left(\theta + \frac{\pi}{6}\right)}$ $CF = \dfrac{r}{2\left(\cos(\theta)\cos\left(\frac{\pi}{6}\right) - \sin(\theta)\sin\left(\frac{\pi}{6}\right)\right)}$ $CF = \dfrac{r}{2\left(\cos(\theta)\frac{\sqrt{3}}{2} - \sin(\theta)\frac{1}{2}\right)}$ $CF \approx \dfrac{r}{2\left(\frac{\sqrt{3}}{2}\left(1 - \frac{\theta^2}{2}\right) - \frac{1}{2}\theta\right)}$ $CF \approx \dfrac{r}{\sqrt{3}}\left(1 + \left(-\frac{1}{\sqrt{3}}\theta - \frac{1}{2}\theta^2\right)\right)^{-1}$ $CF \approx \dfrac{r}{\sqrt{3}}\left(1 - \left(-\frac{1}{\sqrt{3}}\theta - \frac{1}{2}\theta^2\right) + \left(-\frac{1}{\sqrt{3}}\theta - \frac{1}{2}\theta^2\right)^2 + \dots\right)$ $CF \approx \dfrac{r}{\sqrt{3}}\left(1 + \frac{1}{\sqrt{3}}\theta + \frac{1}{2}\theta^2 + \frac{1}{3}\theta^2 + \dots\right)$ $CF \approx \dfrac{r}{\sqrt{3}}\left(1 + \frac{1}{\sqrt{3}}\theta + \frac{5}{6}\theta^2 + \dots\right)$	Similarly, a large number of students did not remember that while θ is sufficiently small, $\left(\theta + \frac{\pi}{6}\right)$ is not, and the formula from MF26 cannot be applied directly.

Overall this question was poorly done, mainly because of **poor knowledge of a very early (and relatively easy) JC1 topic**. Students are apparently not revising early topics effectively.

6 An arithmetic sequence has first term a and common difference d, where a and d are non-zero. The ninth, tenth and thirteenth terms of the arithmetic sequence are the first three terms of a geometric sequence.

(i) Show that $a = -\frac{15}{2}d$. [3]

(ii) The sum of the first n terms of the arithmetic sequence is denoted by S_n. Find the value of S_{16}. [2]

(iii) Given that the k^{th} term of the arithmetic sequence is the fourth term of the geometric sequence, find the value of k. [3]

Solution:

(i) Let $u_n = a+(n-1)d$ denote the arithmetic sequence. $u_9 = a+8d$, $u_{10} = a+9d$, $u_{13} = a+12d$ Since these terms are the first three terms of a geometric sequence, $\dfrac{a+9d}{a+8d} = \dfrac{a+12d}{a+9d}$ $\Rightarrow (a+9d)^2 = (a+8d)(a+12d)$ $\Rightarrow a^2+18ad+81d^2 = a^2+20ad+96d^2$ $\Rightarrow 2ad = -15d^2$ $\therefore a = -\dfrac{15}{2}d$ since d is non-zero	Overall this question was well attempted. While most students were able to obtain the required expression, many failed to justify the rejection of $d = 0$.
(ii) $S_{16} = \dfrac{16}{2}[2a+(16-1)d]$ $= 8[-15d+15d]$ $= 0$	A surprising number of students thought that $8 \times 0 = 8\ldots$
(iii) Common ratio of GP: $r = \dfrac{a+9d}{a+8d} = \dfrac{-\frac{15}{2}d+9d}{-\frac{15}{2}d+8d} = 3$ Fourth term of GP = $(a+8d)r^3$ $= \left(-\dfrac{15}{2}d+8d\right)(3^3) = \dfrac{27}{2}d$ $u_k = -\dfrac{15}{2}d+(k-1)d = \dfrac{27}{2}d$ $\Rightarrow k-1 = 21$ $\Rightarrow k = 22$	While many students obtained the correct answer, most methods were extremely long, mainly because ***a* was not expressed in terms of *d*** immediately. Some students also confused the first terms of the AP and the GP, thinking they are the same term.

7 A curve C has parametric equations

$$x = t^2, \qquad y = t\,\mathrm{e}^{t^2}, \qquad \text{for } t \geq 0.$$

(i) Find the equation of the tangent to C at the point P with coordinates $\left(p^2, p\,\mathrm{e}^{p^2}\right)$, where $p \neq 0$. Hence, or otherwise, find the exact equation of the tangent L to C which passes through the origin. [5]

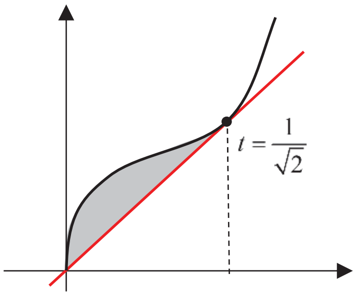
(i) $x = t^2 \Rightarrow \dfrac{\mathrm{d}x}{\mathrm{d}t} = 2t$ $y = t\mathrm{e}^{t^2} \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}t} = \mathrm{e}^{t^2} + 2t^2\mathrm{e}^{t^2} = \mathrm{e}^{t^2}(1+2t^2)$ $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{e}^{t^2}(1+2t^2)}{2t}$	A few students were not able to perform Product Rule correctly.
At $t = p$: $x = p^2$, $y = p\mathrm{e}^{p^2}$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{e}^{p^2}(1+2p^2)}{2p}$ Equation of tangent is $y - p\mathrm{e}^{p^2} = \dfrac{\mathrm{e}^{p^2}(1+2p^2)}{2p}\left(x - p^2\right)$ $y = \dfrac{\mathrm{e}^{p^2}(1+2p^2)}{2p}x - \dfrac{\mathrm{e}^{p^2}(1+2p^2)p}{2} + p\mathrm{e}^{p^2}$	Quite a number of students did not let $t = p$ for the gradient or for the coordinates. This reflects a lack of understanding of each value of t represents a point on the curve. Otherwise, finding the equation of tangent was well done in general.
Tangent passes through (0, 0): $0 = 0 - \dfrac{\mathrm{e}^{p^2}(1+2p^2)p}{2} + p\mathrm{e}^{p^2}$ $p\left(1 - \dfrac{1}{2} - p^2\right) = 0$ $p = \dfrac{1}{\sqrt{2}}$ or $p = -\dfrac{1}{\sqrt{2}}$ (reject as $p \geq 0$) or $p = 0$ (reject as tangent is not parallel to y-axis)	Many students substitute $x = 0$ and $y = 0$ in the parametric equation instead of the equation of tangent to get t. Many students were also not able to get all the values of p, mostly forgetting $p = \pm\dfrac{1}{\sqrt{2}}$.
Equation of tangent passing through origin: $y = \dfrac{\mathrm{e}^{\frac{1}{2}}\left(1+2\left(\frac{1}{2}\right)\right)}{2\left(\frac{1}{\sqrt{2}}\right)}x = \sqrt{2\mathrm{e}}\,x$	Since it is given that the tangent passes through the origin, students are expected to simplify to show that it is so.

$$x = t^2, \qquad y = t\mathrm{e}^{t^2}, \qquad \text{for } t \ge 0.$$

(ii) **(a)** Find the cartesian equation of C. [1]

$x = t^2 \Rightarrow t = \sqrt{x}$ (since $t \ge 0$) $y = t\mathrm{e}^{t^2} = \sqrt{x}\mathrm{e}^x$	This part was well done in general, although many inefficient methods were adopted.

(b) Find the exact volume of the solid formed when the region bounded by C and L is rotated through 2π radians about the x-axis. [5]

(ii)(b)  At $t = \frac{1}{\sqrt{2}}$: $x = \frac{1}{2}$, $y = \sqrt{\frac{1}{2}}\mathrm{e}^{\frac{1}{2}}$ Volume of solid formed $= \pi\int_0^{\frac{1}{2}} y^2\,\mathrm{d}x - \frac{1}{3}\pi\left(\sqrt{\frac{1}{2}}\mathrm{e}^{\frac{1}{2}}\right)^2\frac{1}{2}$ $= \pi\int_0^{\frac{1}{2}} x\mathrm{e}^{2x}\,\mathrm{d}x - \frac{\pi}{12}\mathrm{e}$ $= \pi\left\{\left[x\frac{\mathrm{e}^{2x}}{2}\right]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}}\frac{\mathrm{e}^{2x}}{2}\,\mathrm{d}x\right\} - \frac{\pi}{12}\mathrm{e}$ $= \pi\left[x\frac{\mathrm{e}^{2x}}{2} - \frac{\mathrm{e}^{2x}}{4}\right]_0^{\frac{1}{2}} - \frac{\pi}{12}\mathrm{e}$ $= \pi\left[\left(\frac{\mathrm{e}}{4} - \frac{\mathrm{e}}{4}\right) - \left(0 - \frac{1}{4}\right)\right] - \frac{\pi}{12}\mathrm{e}$ $= \frac{\pi}{4} - \frac{\pi}{12}\mathrm{e}$ units^3	More than half the students left this part blank. Many students did not realise that the intersection of the two curves is already found in (i). Students should use Zoombox to visualise the region bounded by the two curves. Many students also "subtracted" the two volumes wrongly. Almost all students who made this mistake did not sketch a graph. Many students who tried to use the volume of cone formula got the formula wrong. Very common algebraic error: $\left(\sqrt{x}\mathrm{e}^x\right)^2 = x\mathrm{e}^{2x}$, Not $\left(\sqrt{x}\mathrm{e}^x\right)^2 = x\mathrm{e}^{x^2}$! Many algebraic errors arose while performing by parts also.

General Comments

In (i) Some students approached by changing the parametric equations to Cartesian equation. Unless guided to do so (*just like in (ii)*), this is usually not required, especially since anot all parametric equation can be easily converted to Cartesian equation.

In (iii), students should refrain from leaving such technical questions totally blank.

An attempt to find the volumes generated by the region bounded by the two curves separately will get some marks. An attempt to take the volume of the bigger solid to subtract the volume of the smaller solid will get some marks also.

8 **Do not use a calculator in answering this question.**

The complex numbers z and w are given by $z=\dfrac{(1+\mathrm{i})^4}{(1-\mathrm{i})^2}$ and $w=\dfrac{8}{\left(\sqrt{3}+\mathrm{i}\right)^2}$.

(i) Express z and w in polar form $r(\cos\theta+\mathrm{i}\sin\theta)$, where $r>0$ and $-\pi<\theta\le\pi$. Give r and θ in exact form. [4]

(ii) Given that z^2, w and w^* are the roots of the equation $x^3+bx^2+cx+d=0$ where b, c and d are real values, find the equation. [3]

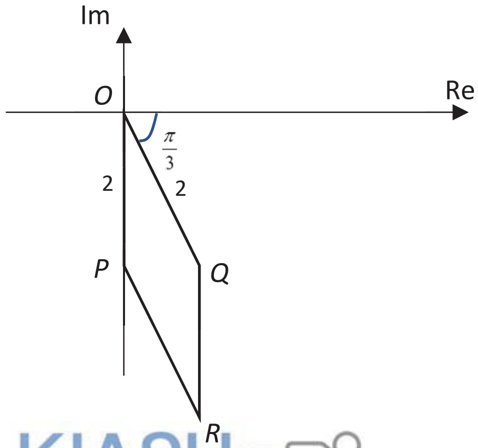
(iii) Sketch on an Argand diagram with origin O, the points P, Q and R representing the complex numbers z, w and $z+w$ respectively. [2]

(iv) By considering the quadrilateral $OPRQ$ and the argument of $z+w$, deduce that

$$\tan\frac{5\pi}{12}=2+\sqrt{3}\ .$$ [3]

Solution:

(i) $\lvert z\rvert=\dfrac{\lvert 1+i\rvert^4}{\lvert 1-i\rvert^2}=\dfrac{\left(\sqrt{2}\right)^4}{\left(\sqrt{2}\right)^2}=2$ $\arg z=4\arg(1+\mathrm{i})-2\arg(1-\mathrm{i})$ $=4\left(\dfrac{\pi}{4}\right)-2\left(-\dfrac{\pi}{4}\right)$ $=\dfrac{3\pi}{2}\notin(-\pi,\pi]$ $\therefore \arg z=-\dfrac{\pi}{2}$ $\therefore z=2\left(\cos\left(-\dfrac{\pi}{2}\right)+\mathrm{i}\sin\left(-\dfrac{\pi}{2}\right)\right)$ $\lvert w\rvert=\dfrac{8}{\left\lvert\sqrt{3}+\mathrm{i}\right\rvert^2}=\dfrac{8}{(2)^2}=2$ $\arg w=\arg 8-2\arg\left(\sqrt{3}+\mathrm{i}\right)=0-2\left(\dfrac{\pi}{6}\right)=-\dfrac{\pi}{3}$ $\therefore w=2\left(\cos\left(-\dfrac{\pi}{3}\right)+\mathrm{i}\sin\left(-\dfrac{\pi}{3}\right)\right)$	• Inefficient method: simplify z and w by expansion to find its modulus and argument. • Be reminded that GC is not allowed, hence all expansions like $(1-\mathrm{i})^2,(1+\mathrm{i})^2,(1+\mathrm{i})^4$, $\left(\sqrt{3}+\mathrm{i}\right)^2$ must be clearly shown with workings. • Answer must be in this form, $r(\cos\theta+\mathrm{i}\sin\theta)$. **Do not** simply to $z=2\left(\cos\left(\dfrac{\pi}{2}\right)-\mathrm{i}\sin\left(\dfrac{\pi}{2}\right)\right)$ OR $z=-2\mathrm{i}\sin\left(\dfrac{\pi}{2}\right)$ • There are still quite a handful in each class who failed to find the argument and modulus correctly.
(ii) $z^2=\left(2\cos\left(-\dfrac{\pi}{2}\right)+\mathrm{i}\sin\left(-\dfrac{\pi}{2}\right)\right)^2=(-2\mathrm{i})^2=-4$ Since z^2, w and w^* are the roots of the equation	• Students do not see the link between polar form and cartesian form of a complex number. Some

An equation is $\left(x-z^2\right)\left(x-w\right)\left(x-w^*\right)=0$ $\Rightarrow\left(x-(-4)\right)\left(x^2-\overbrace{\left(w+w^*\right)}^{\text{Sum of roots}}x+\overbrace{ww^*}^{\text{product of roots}}\right)=0$ $\Rightarrow\left(x+4\right)\left(x^2-2\left(2\cos\left(-\frac{\pi}{3}\right)\right)x+\lvert w\rvert^2\right)=0$ $\Rightarrow\left(x+4\right)\left(x^2-2x+4\right)=0$ $x^3-2x^2+4x+4x^2-8x+16=0$ $\therefore x^3+2x^2-4x+16=0$ **OR** $w=2\left(\cos\left(-\frac{\pi}{3}\right)+\mathrm{i}\sin\left(-\frac{\pi}{3}\right)\right)=2\left(\frac{1}{2}-\mathrm{i}\frac{\sqrt{3}}{2}\right)=1-\mathrm{i}\sqrt{3}$ Hence $w^*=1+\mathrm{i}\sqrt{3}$ An equation is $\left(x-z^2\right)\left(x-w\right)\left(x-w^*\right)=0$ $\left(x-(-4)\right)\left(x-\left(1-\mathrm{i}\sqrt{3}\right)\right)\left(x-\left(1+\mathrm{i}\sqrt{3}\right)\right)=0$ $\left(x+4\right)\left(\left(x-1\right)+\mathrm{i}\sqrt{3}\right)\left(\left(x-1\right)-\mathrm{i}\sqrt{3}\right)=0$ $\left(x+4\right)\left(\left(x-1\right)^2+3\right)=0$ $\left(x+4\right)\left(x^2-2x+4\right)=0$ $x^3-2x^2+4x+4x^2-8x+16=0$ $\therefore x^3+2x^2-4x+16=0$	went to simplify z^2, w and w^* from the original expressions instead of converting the polar form obtained in part (i) to get the cartesian form. • Many failed to realise that roots of the equation are given and thus one just need to get the 3 factors and expand to find the equation. A handful went to sub in the roots and attempt to form equations to solve for b, c and d. • Many did not answer the question. They simply leave their answer as $x^3+2x^2-4x+16$ which is an expression but not an equation. Some thought that the question is asking for the values of b, c and d.
(iii)  	• This is badly done. The points P, Q and R are often drawn with no relation to the complex numbers found in (i). • Points to take note: - $\lvert z\rvert=\lvert w\rvert=2 \Rightarrow OP$ and OQ must have the same length. - $\arg(w)=\frac{\pi}{3}$ and hence angle from real axis to OQ must look like 60°. Some angles drawn are less than 60°. - $\overrightarrow{OR}=\overrightarrow{OP}+\overrightarrow{OQ}$ and hence by vector addition, OR is the diagonal of rhombus $OPRQ$.

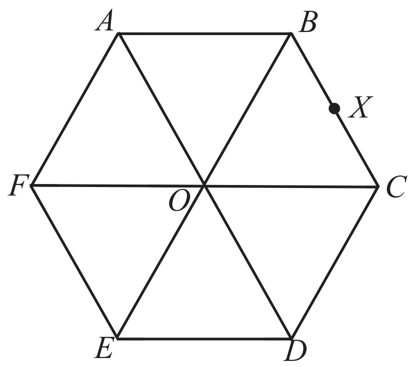
	• Students are reminded that they have to draw the diagram accurately.
(iv) *OPRQ* forms a rhombus where $\angle QOR=\frac{1}{2}\left(\pi-\frac{\pi}{3}\right)=\frac{\pi}{12}$ Hence $\arg(z+w)=-\frac{\pi}{3}-\frac{\pi}{12}=-\frac{5\pi}{12}$ And $z+w=2\cos\left(-\frac{\pi}{2}\right)+\mathrm{i}\sin\left(-\frac{\pi}{2}\right)+2\left(\cos\left(-\frac{\pi}{3}\right)+\mathrm{i}\sin\left(-\frac{\pi}{3}\right)\right)$ $=-2\mathrm{i}+2\left(\frac{1}{2}-\frac{\sqrt{3}}{2}\mathrm{i}\right)$ $=1-\left(2+\sqrt{3}\right)\mathrm{i}$ Hence $\tan\left(\frac{-5\pi}{12}\right)=\frac{-\left(2+\sqrt{3}\right)}{1}\ \therefore\ \tan\left(\frac{5\pi}{12}\right)=2+\sqrt{3}$	• This part was badly done. • NOTE that: $\arg(z+w)\neq\arg(z)+\arg(w)$ • Students failed to explain clearly why $\arg(z+w)=-\frac{5\pi}{12}$ • Many students state that $\arg(z+w)=\frac{5\pi}{12}$ but they should realise that from their diagram, $z+w$ lies in the 4th quadrant and hence its argument should be negative.
Generally, students are still quite weak in their basic foundation for complex numbers. They failed to find modulus and argument correctly at this stage and are not able to convert from one form to another. They struggle with the algebraic manipulations and are not able to interpret the geometrical representation of complex numbers well.	

9 **(a)** Vectors $\mathbf{u}$ and $\mathbf{v}$ are such that $\mathbf{u}.\mathbf{v} = -1$ and $(\mathbf{u}\times\mathbf{v})+\mathbf{u}$ is perpendicular to $(\mathbf{u}\times\mathbf{v})+\mathbf{v}$.

Show that $|\mathbf{u}\times\mathbf{v}| = 1$. [3]

Hence find the angle between $\mathbf{u}$ and $\mathbf{v}$. [3]

(b) The figure shows a regular hexagon *ABCDEF* with *O* at the centre of the hexagon. *X* is the midpoint of *BC*.



Given that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$, find $\overrightarrow{OF}$ and $\overrightarrow{OX}$ in terms of **a** and **b**. [2]

Line segments *AC* and *FX* intersect at the point *Y*. Determine the ratio *AY* : *YC*. [4]

Solution:

<table>
<tr><td colspan="2">This question is generally poorly attempted, with most students obtaining less than half the total score. The main challenge stems from the basic understanding of vector concepts and geometry.</td></tr>
<tr><td>

(a) Since $\mathbf{u}\times\mathbf{v}+\mathbf{u}$ is perpendicular to $\mathbf{u}\times\mathbf{v}+\mathbf{v}$, we have $((\mathbf{u}\times\mathbf{v})+\mathbf{u}).((\mathbf{u}\times\mathbf{v})+\mathbf{v}) = 0$

$\Rightarrow$ $(\mathbf{u}\times\mathbf{v}).(\mathbf{u}\times\mathbf{v})+(\mathbf{u}\times\mathbf{v}).\mathbf{v}+(\mathbf{u}\times\mathbf{v}).\mathbf{u}+\mathbf{u}.\mathbf{v} = 0$

$\Rightarrow$ $|\mathbf{u}\times\mathbf{v}|^2 + 0 + 0 - 1 = 0$

since $(\mathbf{u}\times\mathbf{v})\perp\mathbf{v}$ and $(\mathbf{u}\times\mathbf{v})\perp\mathbf{u}$

$\Rightarrow$ $|\mathbf{u}\times\mathbf{v}| = 1$ (shown)

Let θ be the angle between **u** and **v**.

$\mathbf{u}.\mathbf{v} = -1 \Rightarrow |\mathbf{u}||\mathbf{v}|\cos\theta = -1$ ---(1)

$|\mathbf{u}\times\mathbf{v}| = 1 \Rightarrow |\mathbf{u}||\mathbf{v}|\sin\theta = 1$ ---(2)

$\frac{(2)}{(1)}$: $\tan\theta = -1 \quad \Rightarrow \quad \theta = 135^\circ$

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Most were able to write out $((\mathbf{u}\times\mathbf{v})+\mathbf{u}).((\mathbf{u}\times\mathbf{v})+\mathbf{v}) = 0$ knowing that the vectors are perpendicular to each other. Subsequent expansion of the scalar product proves challenging for many, where only a small handful were able to write out sufficient details to arrive at the intended result.

Most seemed oblivious that $\mathbf{u}.\mathbf{v} = -1$ would imply that the angle between the vectors **u** and **v** is obtuse, thus only 135° is valid.

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(b) $\overrightarrow{OF}=\overrightarrow{OA}+\overrightarrow{AF}=\mathbf{a}-\mathbf{b}$ $\overrightarrow{OX}=\overrightarrow{OB}+\overrightarrow{BX}=\mathbf{b}-\frac{1}{2}\mathbf{a}$ **<u>Method 1</u>** Let $AY:YC=\lambda:1-\lambda$ and $FY:YX=\mu:1-\mu$ $\therefore\ \overrightarrow{OY}=\lambda\overrightarrow{OC}+(1-\lambda)\overrightarrow{OA}=\mu\overrightarrow{OX}+(1-\mu)\overrightarrow{OF}$ $\Rightarrow\ \lambda(\mathbf{b}-\mathbf{a})+(1-\lambda)\mathbf{a}=\mu\left(\mathbf{b}-\frac{1}{2}\mathbf{a}\right)+(1-\mu)(\mathbf{a}-\mathbf{b})$ $\Rightarrow\ (\lambda-\mu+1-\mu)\mathbf{b}=\left(\lambda-1+\lambda-\frac{1}{2}\mu+1-\mu\right)\mathbf{a}$ Since **a** and **b** are non-parallel, $\left.\begin{array}{l}\lambda-2\mu+1=0\\ 2\lambda-\frac{3}{2}\mu=0\end{array}\right\}$ solving gives $\lambda=\frac{3}{5},\quad \mu=\frac{4}{5}$ $\therefore AY:YC=\frac{3}{5}:1-\frac{3}{5}=3:2$ **<u>Method 2</u>** Line AC: $\mathbf{r}=\mathbf{a}+\lambda(\mathbf{b}-2\mathbf{a}),\ \lambda\in\mathbb{R}$ Line FX: $\mathbf{r}=\mathbf{a}-\mathbf{b}+\mu\left(2\mathbf{b}-\frac{3}{2}\mathbf{a}\right),\ \mu\in\mathbb{R}$ When the lines intersect at Y, $\mathbf{a}+\lambda(\mathbf{b}-2\mathbf{a})=\mathbf{a}-\mathbf{b}+\mu\left(2\mathbf{b}-\frac{3}{2}\mathbf{a}\right)$ $\left.\begin{array}{l}\lambda-2\mu+1=0\\ 2\lambda-\frac{3}{2}\mu=0\end{array}\right\}$ solving gives $\lambda=\frac{3}{5},\quad \mu=\frac{4}{5}$ $\overrightarrow{OY}=\mathbf{a}+\frac{3}{5}(\mathbf{b}-2\mathbf{a})=-\frac{1}{5}\mathbf{a}+\frac{3}{5}\mathbf{b}$ $\overrightarrow{AY}=-\frac{6}{5}\mathbf{a}+\frac{3}{5}\mathbf{b}$ and $\overrightarrow{YC}=-\frac{4}{5}\mathbf{a}+\frac{2}{5}\mathbf{b}$ $\therefore\ AY:YC=3:2$	Most are familiar with the triangle law for vectors to obtain $\overrightarrow{OF}$ and $\overrightarrow{OX}$, though some face difficulty with the arithmetic. Almost all presented either of the 2 methods while solving for the ratio of $AY:YC$, though it is worth noting that simple geometry based on regular polygons can also be applied to obtain the result. Students who attempted method 1 were generally more successful in obtaining the ratio than those who attempted method 2. Many who attempted method 2 had a misconception that $\overrightarrow{OY}$ can be found by equating the vectors $\overrightarrow{AC}$ and $\overrightarrow{FX}$, thus losing all the credits for this part of the question. It is also of interest that many attempted to find the ratio by dividing vectors on both sides of their equations, resulting in an immediate forfeit of the credits for this part of the question.

10 Mr Ng wants to hang a decoration on the vertical wall above his bookshelf. He needs a ladder to climb up.

The rectangle $ABCD$ is the side-view of the bookshelf and HK is the side-view of the ladder where $AB = 24$ cm and $BC = 192$ cm (see **Figure 1**). The ladder touches the wall at H, the edge of the top of the bookshelf at B and the floor at K.

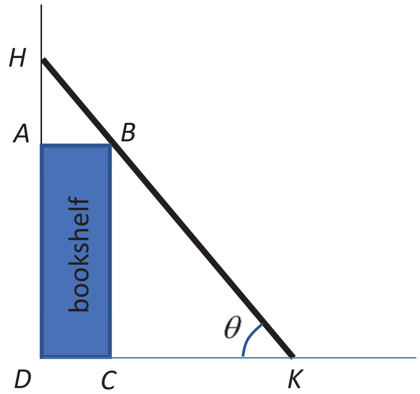

Figure 1

(i) Given that $\angle HKD = \theta$, show that the length , L cm of the ladder is given by

$$L = \frac{24}{\cos\theta} + \frac{192}{\sin\theta} .$$ [1]

(ii) Use differentiation to find the exact value of the shortest length of the ladder as θ varies. [4]
[You do not need to verify that this length of the ladder is the shortest.]

Take L to be 270 for the rest of this question.

The ladder starts to slide such that H moves away from the wall and K moves towards E (see **Figure 2**). The ladder maintains contact with the bookshelf at B.

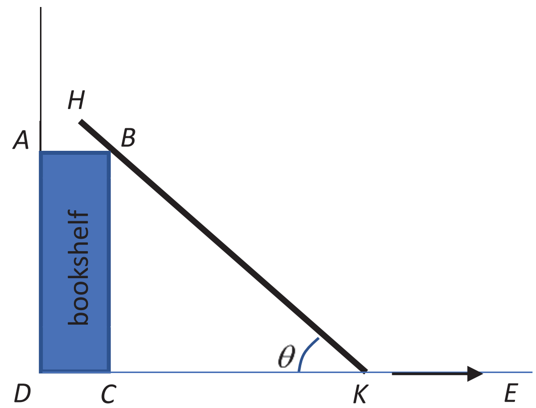

Figure 2

The horizontal distances from the wall to H and from the wall to K are x cm and y cm respectively.

(iii) By expressing $y - x$ in terms of θ, determine whether the rate of change of y is greater than the rate of change of x. [3]

(iv) Given that the rate of change of θ is -0.1 rad s^{-1} when $CK = 160$ cm, find the rate of change of x at this instant. [5]

[Solution]

The question was challenging for many students due to the context and the trigonometry involved.

(i) $L = HB + BK = \dfrac{24}{\cos\theta} + \dfrac{192}{\sin\theta}$	Most students were able to write out sufficient workings.
(ii) $\dfrac{dL}{d\theta} = -24\dfrac{(-\sin\theta)}{\cos^2\theta} - \dfrac{192(\cos\theta)}{\sin^2\theta}$ OR by rewriting $L = 24\sec\theta + 192\operatorname{cosec}\theta$, $\dfrac{dL}{d\theta} = 24\sec\theta\tan\theta - 192\operatorname{co}\sec\theta\cot\theta$ When L is shortest, $\dfrac{dL}{d\theta} = 0 \Rightarrow \dfrac{24\sin\theta}{\cos^2\theta} = \dfrac{192\cos\theta}{\sin^2\theta}$ $\Rightarrow \tan^3\theta = 8$ $\therefore \tan\theta = 2$ (since θ is acute) i.e. $\sin\theta = \dfrac{2}{\sqrt{5}}, \cos\theta = \dfrac{1}{\sqrt{5}}$ Shortest length of the ladder $= 24\left(\dfrac{\sqrt{5}}{1}\right) + 192\left(\dfrac{\sqrt{5}}{2}\right) = 120\sqrt{5}$ cm.	Poorly attempted due to poor differentiation techniques. Common errors: 1. $\dfrac{d}{d\theta}\left(\dfrac{1}{\cos\theta}\right) = \cos^{-2}\theta$ $\dfrac{d}{d\theta}\left(\dfrac{1}{\sin\theta}\right) = \sin^{-2}\theta$ Students did not apply chain rule. 2. $\dfrac{d}{d\theta}\left(\dfrac{1}{\sin\theta}\right) = \dfrac{1}{\sqrt{1-\theta^2}}$ Students thought that $\dfrac{1}{\sin\theta} = \sin^{-1}\theta$ A number of students also made algebraic slips while solving the equation $\dfrac{dL}{d\theta} = 0$. Part (ii) responses were varied, many students understood the that they need to find the first derivative and set to zero to solve for value of θ when L is shortest, but did not realise that they should leave their answer **exactly** in terms of $\tan\theta$ and work out the exact values of $\sin\theta$ and $\cos\theta$
(iii) $y - x = 270\cos\theta$ [B1] Formulate an expression for $y - x$ in terms of θ $\dfrac{dy}{dt} - \dfrac{dx}{dt} = -270\sin\theta\dfrac{d\theta}{dt}$ [B1]	Many students were able to follow instruction to express $y-x$ in terms of θ correctly and went on to differentiate the expression. However most of them were not able to

$\because \sin\theta > 0$ and $\dfrac{d\theta}{dt} < 0$ $\dfrac{dy}{dt} - \dfrac{dx}{dt} > 0$ i.e. $\dfrac{dy}{dt} > \dfrac{dx}{dt}$	arrive at the expression involving $\dfrac{dy}{dt}, \dfrac{dx}{dt}$. Some students attempted to multiply $\dfrac{d\theta}{dt}$ throughout but failed to realise that $\dfrac{d\theta}{dt} < 0$.
(iv) $y - x = 270\cos\theta \Rightarrow x = y - 270\cos\theta$ Since $y = 24 + \dfrac{192}{\tan\theta}$, $x = 24 + \dfrac{192}{\tan\theta} - 270\cos\theta$ $\dfrac{dx}{d\theta} = -\dfrac{192}{\tan^2\theta}\sec^2\theta + 270\sin\theta = 270\sin\theta - \dfrac{192}{\sin^2\theta}$ When $CK = 160$, $\tan\theta = \dfrac{192}{160} = \dfrac{6}{5}$ i.e. $\sin\theta = \dfrac{6}{\sqrt{61}}$ $\dfrac{dx}{dt} = \dfrac{dx}{d\theta} \times \dfrac{d\theta}{dt}$ $= \left[270\left(\dfrac{6}{\sqrt{61}}\right) - \dfrac{192}{\left(\dfrac{6}{\sqrt{61}}\right)^2} \right] \times (-0.1) \approx 11.791$ Rate of change of x is 11.8 cm s^{-1}.	Students need clarity in planning the solution for this part. Many of them did not realise that y is changing with respect to time and θ and wrote $y = 24 + 160 = 184$, which was a serious error arising from poor interpretation. They did not understand that rate of change of x, y varies as θ varies.

[Solution]

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(i) $L = HB + BK = \dfrac{24}{\cos\theta} + \dfrac{192}{\sin\theta}$

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Most students were able to write out sufficient workings.

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(ii) $\dfrac{dL}{d\theta} = -24\dfrac{(-\sin\theta)}{\cos^2\theta} - \dfrac{192(\cos\theta)}{\sin^2\theta}$

OR by rewriting $L = 24\sec\theta + 192\operatorname{cosec}\theta$,

$\dfrac{dL}{d\theta} = 24\sec\theta\tan\theta - 192\operatorname{cosec}\theta\cot\theta$

When L is shortest,

$\dfrac{dL}{d\theta} = 0 \Rightarrow \dfrac{24\sin\theta}{\cos^2\theta} = \dfrac{192\cos\theta}{\sin^2\theta}$

$\Rightarrow \tan^3\theta = 8$

$\therefore \tan\theta = 2$ (since θ is acute)

i.e. $\sin\theta = \dfrac{2}{\sqrt{5}}, \cos\theta = \dfrac{1}{\sqrt{5}}$

Shortest length of the ladder

$= 24\left(\dfrac{\sqrt{5}}{1}\right) + 192\left(\dfrac{\sqrt{5}}{2}\right) = 120\sqrt{5}$ cm.

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Poorly attempted due to poor differentiation techniques.

Common errors:

3. $\dfrac{d}{d\theta}\left(\dfrac{1}{\cos\theta}\right) = \cos^{-2}\theta$

$\dfrac{d}{d\theta}\left(\dfrac{1}{\sin\theta}\right) = \sin^{-2}\theta$

Students did not apply chain rule.

4. $\dfrac{d}{d\theta}\left(\dfrac{1}{\sin\theta}\right) = \dfrac{1}{\sqrt{1-\theta^2}}$

Students thought that $\dfrac{1}{\sin\theta} = \sin^{-1}\theta$

A number of students also made algebraic slips while solving the equation $\dfrac{dL}{d\theta} = 0$.

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(iii) $y - x = 270\cos\theta$

[B1] Formulate an expression for $y - x$ in terms of θ

$\dfrac{dy}{dt} - \dfrac{dx}{dt} = -270\sin\theta\dfrac{d\theta}{dt}$ [B1]

$\because \sin\theta > 0$ and $\dfrac{d\theta}{dt} < 0$

$\dfrac{dy}{dt} - \dfrac{dx}{dt} > 0$ i.e. $\dfrac{dy}{dt} > \dfrac{dx}{dt}$

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Many students were able to follow instruction to express $y - x$ in terms θ correctly and went on to differentiate the expression. However most of them were not able to arrive at the expression involving $\dfrac{dy}{dt}, \dfrac{dx}{dt}$. Some students attempted to multiply $\dfrac{d\theta}{dt}$ throughout but failed to realise that $\dfrac{d\theta}{dt} < 0$.

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(iv) $y - x = 270\cos\theta \Rightarrow x = y - 270\cos\theta$

Since $y = 24 + \dfrac{192}{\tan\theta}$, $x = 24 + \dfrac{192}{\tan\theta} - 270\cos\theta$

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Students need clarity in planning the solution for this part.

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$\dfrac{dx}{d\theta}=-\dfrac{192}{\tan^2\theta}\sec^2\theta+270\sin\theta=270\sin\theta-\dfrac{192}{\sin^2\theta}$ When $CK=160$, $\tan\theta=\dfrac{192}{160}=\dfrac{6}{5}$ i.e. $\sin\theta=\dfrac{6}{\sqrt{61}}$ $\dfrac{dx}{dt}=\dfrac{dx}{d\theta}\times\dfrac{d\theta}{dt}$ $=\left[270\left(\dfrac{6}{\sqrt{61}}\right)-\dfrac{192}{\left(\dfrac{6}{\sqrt{61}}\right)^2}\right]\times(-0.1)\approx 11.791$ Rate of change of x is $11.8\,\text{cm s}^{-1}$.	Many of them did not realise that y is changing with respect to time and θ and wrote $y=24+160=184$, which was a serious error arising from poor interpretation. They did not understand that rate of change of x, y varies as θ varies.

11 The daily food calories, L, taken in by a human body are partly used to fulfill the daily requirements of the body. The daily requirements is proportional to the body mass, M kg, with a constant of proportionality p. The rate of change in body mass is proportional to the remaining calories.

It is given that the body mass, M kg, at time t days satisfies the differential equation

$$\frac{dM}{dt}=k(L-pM),$$

where k and L are constants.

John's initial body mass is 100 kg. Find, in terms of p, the daily food calories needed to keep his body mass constant at 100 kg. [1]

To lose weight, John decides to start on a diet where his daily food calorie intake is 75% of the daily calories needed to keep his body mass constant at 100 kg.

(i) Show that $M=75+25e^{-pkt}$. [4]

(ii) John attained a body mass of 90 kg after 50 days on this diet. If it takes him n more days to lose at least another 10 kg, find the smallest integer value of n. [5]

(iii) John's goal with this diet plan is to achieve a body mass of 70 kg. With the aid of a graph, explain why he can never achieve his goal. [2]

(iv) By considering $\dfrac{d^2M}{dt^2}$, comment on his rate of body mass loss as time passes. [2]

General Comments

Question generally not well attempted. Many students were unable to

(i) interpret the question fully / correctly

(ii) solve the D.E. correctly

(iii) provide qualitative explanations as required in parts (iii) & (iv).

It is given that the body mass, M kg, at time t days satisfies the differential equation

$$\frac{\mathrm{d}M}{\mathrm{d}t} = k\left(L - pM\right),$$

where k and L are constants.

John's initial body mass is 100 kg (i.e. t = 0, M = 0). Find, in terms of p, the daily food calories (i.e. find L) needed to keep his body mass constant at 100 kg (i.e. $M = 100$, $\frac{\mathrm{d}M}{\mathrm{d}t} = 0$). [1]

Solution :

Mass constant at 100kg $\Rightarrow$ When $M = 100$, $\frac{\mathrm{d}M}{\mathrm{d}t} = 0$ $\Rightarrow$ $k\left(L - p(100)\right) = 0$ $\Rightarrow$ $L = 100p$ If calories L =100p, then M will remain constant at 100kg.	Generally well attempted. A few students could not deduce that $\frac{\mathrm{d}M}{\mathrm{d}t} = 0$.

To lose weight, John decides to start on a diet where his daily food calorie intake is 75% of the daily calories needed to keep his body mass constant at 100 kg.

(i) Show that $M = 75 + 25\mathrm{e}^{-pkt}$. [4]

Solution :

(i) L is reduced to 75% of above.

M is NOT constant at 100kg anymore as calories L has reduced. M is now a variable.

$$\frac{\mathrm{d}M}{\mathrm{d}t} = k\left(L - pM\right)$$

$$\Rightarrow \quad \int \frac{1}{L - pM}\,\mathrm{d}M = k\int \mathrm{d}t$$

$$\Rightarrow \quad -\frac{1}{p}\int \frac{-p}{L - pM}\,\mathrm{d}M = k\int \mathrm{d}t$$

$$\Rightarrow \quad \ln\left| L - pM \right| = -pkt + c$$

$$\Rightarrow \quad L - pM = \pm \mathrm{e}^{-pkt+c}$$

$$\Rightarrow \quad L - pM = A\mathrm{e}^{-pkt} \quad , \quad \text{where } A = \pm \mathrm{e}^{c}$$

Given: When $t = 0$, $M = 100$

$$L - p(100) = A$$

Thus, $L - pM = \left(L - 100p\right)\mathrm{e}^{-pkt}$

Sub $L = 0.75\left(100p\right) = 75p$:

$$75p - pM = \left(75p - 100p\right)\mathrm{e}^{-pkt}$$

$$\Rightarrow \quad 75 - M = -25\mathrm{e}^{-pkt}$$

$$\Rightarrow \quad M = 75 + 25\mathrm{e}^{-pkt} \quad \text{(shown)}$$

<u>Integration mistakes</u>

- Did not recognize

  $\int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)}\,\mathrm{d}x = \ln\left|\mathrm{f}(x)\right| + C$

  or applied formula incorrectly
- Forgot $\mathrm{f}'(x)$
- Forgot modulus $|\quad|$
- Forgot ‘ $+C$ ’ which is crucial
- Do not know how to manipulate modulus e.g. forgot $\pm$

- Could not interpret from question that $t = 0$, $M = 100$

- Could not interpret from qn that $L = 0.75(100p) = 75p$

(ii) John attained a body mass of 90 kg after 50 days on this diet. If it takes him n more days to lose at least another 10 kg, find the smallest integer value of n. [5]

Solution :

(ii) $M = 75 + 25\mathrm{e}^{-pkt}$ Given: When $t = 50$, $M = 90$ $90 = 75 + 25\mathrm{e}^{-50pk}$ $\Rightarrow \quad \mathrm{e}^{-50pk} = \frac{15}{25} = \frac{3}{5}$ $\Rightarrow \quad -50pk = \ln\left(\frac{3}{5}\right)$ $\Rightarrow \quad -pk = \frac{1}{50}\ln\left(\frac{3}{5}\right)$ Hence $M = 75 + 25\mathrm{e}^{t\frac{1}{50}\ln\left(\frac{3}{5}\right)}$.	Good - Students successfully attempted this part even if earlier part was unsuccessful. Students should pay attention to to avoid common algebraic errors • $\mathrm{e}^{-pk50} \neq \mathrm{e}^{-pk}\,\mathrm{e}^{50}$ • Did not solve for pk even though it was obvious that the value of pk was needed later on.
To lose at least another 10 kg: $75 + 25\mathrm{e}^{t\frac{1}{50}\ln\left(\frac{3}{5}\right)} \leq 80$ $\Rightarrow \quad \mathrm{e}^{t\frac{1}{50}\ln\left(\frac{3}{5}\right)} \leq \frac{5}{25}$ $\Rightarrow \quad t\frac{1}{50}\ln\left(\frac{3}{5}\right) \leq \ln\left(\frac{1}{5}\right)$ $\Rightarrow \quad t \geq \frac{\ln\left(\frac{1}{5}\right)}{\frac{1}{50}\ln\left(\frac{3}{5}\right)} = 157.53$ Thus, smallest integer n is $158 - 50 = 108$.	• Most students did not formulate an inequality which would have helped to identify the "smallest integer n". Instead they merely solved an equation and rounded off to 158 for t (108 for n). • Quite a number did not note that qn was "***n* <u>more</u>** days" and gave t (158) as the answer instead of n (i.e. $t - 50 = 108$).

(iii) John's goal with this diet plan is to achieve a body mass of 70 kg. With the aid of a graph, explain why he can never achieve his goal. [2]

Solution :

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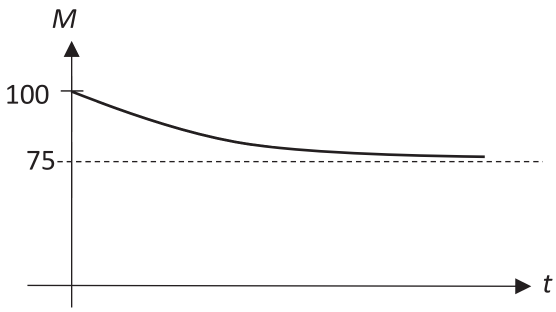
(iii) $M = 75 + 25\mathrm{e}^{t\frac{1}{50}\ln\left(\frac{3}{5}\right)}$



<u>From the graph</u>, it can be seen that John's body mass <u>decreases with time but will never go below 75 kg for all t</u>. Thus, he will never achieve his goal of 70kg.

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Students must use a graph as it is required by question.

Graphs must include key features

- M-intercept at M =100
- Graph must **approach** (not touch) the $M=75$ asymptote
- Axes are M-axis / t-axis and not y-axis / x-axis
- Authentic context: no negative t

- Poorly worded explanations. Note also we are not just interested in the long run ($t \to \infty$) but $M > 75$ for all t.

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(iv) By considering $\frac{d^2M}{dt^2}$, comment on his rate of body mass loss as time passes. [2]

Solution :

(iv) $M = 75 + 25e^{t\frac{1}{50}\ln\left(\frac{3}{5}\right)}$

$$\Rightarrow \frac{dM}{dt} = \frac{1}{2}\ln\left(\frac{3}{5}\right)e^{t\frac{1}{50}\ln\left(\frac{3}{5}\right)}$$

$$\Rightarrow \frac{d^2M}{dt^2} = \frac{1}{100}\left[\ln\left(\frac{3}{5}\right)\right]^2 e^{t\frac{1}{50}\ln\left(\frac{3}{5}\right)} > 0$$

$\Rightarrow \frac{dM}{dt}$ (negative) <u>increases</u> (become less negative) as t increases.

John's <u>rate</u> of body <u>mass loss</u> $\left|\frac{dM}{dt}\right|$ **decreases** as time passes.

<u>**Alternative**</u>

$$\frac{dM}{dt} = k\left(L - pM\right)$$

$$\frac{d^2M}{dt^2} = -kp\frac{dM}{dt} = \frac{1}{50}\ln\left(\frac{3}{5}\right)\frac{dM}{dt}$$

Since $\frac{1}{50}\ln\left(\frac{3}{5}\right) < 0$ and $\frac{dM}{dt} < 0$, we have $\frac{d^2M}{dt^2} > 0$

$\Rightarrow \frac{dM}{dt}$ (negative) <u>increases</u> (become less negative) as t increases.

John's <u>rate</u> of body <u>mass loss</u> $\left|\frac{dM}{dt}\right|$ **decreases** as time passes.

Very poorly attempted question.

- Few students found $\frac{d^2M}{dt^2}$ successfully.
  Some forgot that k & L are constants. Some differentiated with respect to M instead of t.
- Few students linked the sign of $\frac{d^2M}{dt^2}$ to rate of mass loss $\left(\left|\frac{dM}{dt}\right|\right)$.
- Note the difference between rate of mass change $\frac{dM}{dt}$ vs. rate of mass loss $\left|\frac{dM}{dt}\right|$.
- Many differentiated $\frac{dM}{dt}$ with respect to M instead of t:
  $\frac{dM}{dt} = k\left(L - pM\right)$
  $\frac{d^2M}{dt^2} \neq -kp$

Section A: Pure Mathematics [40 marks]

1 The function f is defined by

$$\mathrm{f}: x \mapsto \frac{x^2}{2-x},\ x \in \mathbb{R},\ 0 \le x < 2\ .$$

(i) Find $\mathrm{f}^{-1}(x)$ and write down the domain of f^{-1}. [4]

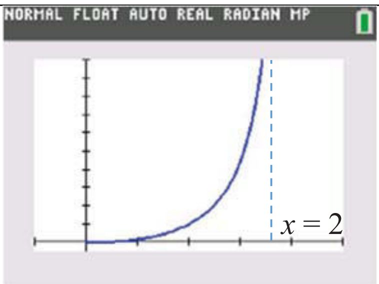
It is given that

$$\mathrm{g}: x \mapsto \frac{1}{1+\mathrm{e}^{-x}},\ x \in \mathbb{R},\ x \ge 0\ .$$

(ii) Show that fg exists. [2]

(iii) Find the range of fg. [2]

1(i)	Let $y = \dfrac{x^2}{2-x}$. $2y - xy = x^2 \quad \Rightarrow \quad x^2 + xy - 2y = 0$ Then $x = \dfrac{-y \pm \sqrt{y^2+8y}}{2}$ Alternatively, $2y - xy = x^2 \quad \Rightarrow \quad x^2 + xy - 2y = 0$ $\Rightarrow \left(x + \dfrac{y}{2}\right)^2 - \left(\dfrac{y}{2}\right)^2 - 2y = 0$ $\Rightarrow \left(x + \dfrac{y}{2}\right)^2 = \dfrac{y^2}{4} + 2y$ $\Rightarrow x + \dfrac{y}{2} = \pm\sqrt{\dfrac{y^2}{4} + 2y}$ $\Rightarrow x + \dfrac{y}{2} = \pm\sqrt{\dfrac{y^2}{4} + 2y}$ $\Rightarrow x = \dfrac{-y \pm \sqrt{y^2+8y}}{2}$ Since $0 \le x < 2$, $x = \dfrac{-y + \sqrt{y^2+8y}}{2}$. $\mathrm{f}^{-1}(x) = \dfrac{-x + \sqrt{x^2+8x}}{2}$ $\mathrm{D}_{\mathrm{f}^{-1}} = \mathrm{R}_\mathrm{f} = [0, \infty)$	Many students are uncertain how to make x the subject. Note that this can be done by either using the quadratic formula or completing square. Students must note that it is essential to explain why $x = \dfrac{-y - \sqrt{y^2+8y}}{2}$ is rejected. Some students did not provide the final answer. While they able to obtain $x = \dfrac{-y + \sqrt{y^2+8y}}{2}$, they fail to write down the expression for $\mathrm{f}^{-1}(x)$ and $\mathrm{D}_{\mathrm{f}^{-1}}$ to answer the question completely.
1(ii)	$\mathrm{R}_\mathrm{g} = [\tfrac{1}{2}, 1)$ $\mathrm{D}_\mathrm{f} = [0, 2)$	It is necessary to write down the $\mathrm{R_g}$ and $\mathrm{D_f}$ to answer this

	Since $\mathrm{R_g} \subset \mathrm{D_f}$, the function fg exists.	part of the question. Simply stating $\mathrm{R_g} \subset \mathrm{D_f}$ is insufficient.
1(iii)	 Observe that $y = \mathrm{f}(x)$ is a strictly increasing function; so we only need to find $\mathrm{f}(\frac{1}{2}) = \frac{1}{6}$ and $\mathrm{f}(1) = 1$. $\mathrm{R_{fg}} = [\frac{1}{6}, 1)$	Many students did not indicate that f is an increasing function. Simply writing $\mathrm{R_g} = [\frac{1}{2}, 1)$ leading to $\mathrm{R_{fg}} = [\frac{1}{6}, 1)$ is insufficient to obtain the full credit.

2 Express $\dfrac{6r+7}{r(r+1)}$ as partial fractions. [1]

(i) Hence use method of differences to find $\displaystyle\sum_{r=1}^{N}\left(\left(\frac{1}{7}\right)^{r+1}\frac{6r+7}{r(r+1)}\right)$ in terms of N. (There is no need to express your answer as a single algebraic fraction.) [3]

(ii) Give a reason why the series $\displaystyle\sum_{r=1}^{\infty}\left(\left(\frac{1}{7}\right)^{r+1}\frac{6r+7}{r(r+1)}\right)$ converges, and write down its value. [2]

(iii) Use your answer in part **(i)** to find $\displaystyle\sum_{r=1}^{N}\left(\left(\frac{1}{7}\right)^{r+1}\frac{6r+13}{(r+1)(r+2)}\right)$. [3]

2	$\dfrac{6r+7}{r(r+1)} = \dfrac{7}{r} - \dfrac{1}{r+1}$	Most students got this right.

(i)	$\sum_{r=1}^{N}\left(\left(\frac{1}{7}\right)^{r+1}\frac{6r+7}{r(r+1)}\right)=\sum_{r=1}^{N}\left(\left(\frac{1}{7}\right)^{r+1}\left(\frac{7}{r}-\frac{1}{r+1}\right)\right)$ $=\sum_{r=1}^{N}\left(\left(\frac{7^{-r}}{r}-\frac{7^{-r-1}}{r+1}\right)\right)$ $=\ (\frac{7^{-1}}{1}\ -\ \frac{7^{-2}}{2}$ $+\ \frac{7^{-2}}{2}\ -\ \frac{7^{-3}}{3}$ $+\ \frac{7^{-3}}{3}\ -\ \frac{7^{-4}}{4}$ $+\cdots$ $+\ \frac{7^{-N+1}}{N-1}\ -\ \frac{7^{-N}}{N}$ $+\ \frac{7^{-N}}{N}\ -\ \frac{7^{-N-1}}{N+1})$ $=\frac{7^{-1}}{1}-\frac{7^{-N-1}}{N+1}=\frac{1}{7}-\frac{1}{7^{N+1}(N+1)}$	A common error here is to express $\sum_{r=1}^{N}\left(\left(\frac{1}{7}\right)^{r+1}\left(\frac{7}{r}-\frac{1}{r+1}\right)\right)$ as $\sum_{r=1}^{N}\left(\frac{1}{7}\right)^{r+1}\cdot\sum_{r=1}^{N}\left(\frac{7}{r}-\frac{1}{r+1}\right)$. Students must take note that $\sum_{r=1}^{N}\left(f(r)\cdot g(r)\right)$ is not equal to $\sum_{r=1}^{N}f(r)\cdot\sum_{r=1}^{N}g(r)$.
(ii)	As $N\to\infty,\left(\frac{1}{7}\right)^{N+1}\to 0,\frac{1}{N+1}\to 0$ Hence as $N\to\infty$, $\frac{7^{-N-1}}{N+1}=\left(\frac{1}{7}\right)^{N+1}\left(\frac{1}{N+1}\right)\to 0$ Then $\sum_{r=1}^{\infty}\left(\left(\frac{1}{7}\right)^{r+1}\frac{6r+7}{r(r+1)}\right)=\frac{1}{7}$, $\therefore\sum_{r=1}^{\infty}\left(\left(\frac{1}{7}\right)^{r+1}\frac{6r+7}{r(r+1)}\right)$ converges and it converges to $\frac{1}{7}$.	Students must know that the highlighted statement is the reason why the series converges, and therefore must be clearly written down in your answer.
(iii)	Method 1: $\sum_{r=1}^{N}\left(\left(\frac{1}{7}\right)^{r+1}\frac{6r+13}{(r+1)(r+2)}\right)$ replace r by r-1 $=\sum_{r-1=1}^{r-1=N}\left(\left(\frac{1}{7}\right)^{r-1+1}\frac{6(r-1)+13}{(r-1+1)(r-1+2)}\right)$ $=\sum_{r=2}^{N+1}\left(\left(\frac{1}{7}\right)^{r}\frac{6r+7}{(r)(r+1)}\right)=7\sum_{r=2}^{N+1}\left(\left(\frac{1}{7}\right)^{r+1}\frac{6r+7}{(r)(r+1)}\right)$ $=7\left[\sum_{r=1}^{N+1}\left(\left(\frac{1}{7}\right)^{r+1}\frac{6r+7}{(r)(r+1)}\right)-\left(\frac{1}{7}\right)^{2}\frac{13}{(1)(2)}\right]$	Many students are aware that they must replace r by r–1 in Method 1 or replace r by r+1 in Method 2. However, the plan is poorly executed with many making errors in its replacement. Some who have correctly replaced the r, are unsure how to proceed from there.

	$= 7\left[\sum_{r=1}^{N+1}\left(\left(\frac{1}{7}\right)^{r+1}\frac{6r+7}{(r)(r+1)}\right)-\left(\frac{1}{7}\right)^{2}\frac{13}{(1)(2)}\right]$ $= 7\left[\frac{1}{7}-\frac{7^{-(N+1)-1}}{(N+1)+1}-\frac{13}{49(2)}\right]=\frac{1}{14}-\frac{7^{-N-1}}{N+2}=\frac{1}{14}-\frac{1}{7^{N+1}(N+2)}$ Method 2: From (i), $\sum_{r=1}^{N}\left(\left(\frac{1}{7}\right)^{r+1}\frac{6r+7}{r(r+1)}\right)=\frac{1}{7}-\frac{1}{7^{N+1}(N+1)}$ Replace r by r+1, we have $\sum_{r+1=1}^{r+1=N}\left(\left(\frac{1}{7}\right)^{(r+1)+1}\frac{6(r+1)+7}{(r+1)((r+1)+1)}\right)=\frac{1}{7}-\frac{1}{7^{N+1}(N+1)}$ $\sum_{r=0}^{N-1}\left(\left(\frac{1}{7}\right)^{r+2}\frac{6r+13}{(r+1)(r+2)}\right)=\frac{1}{7}-\frac{1}{7^{N+1}(N+1)}$ Replace N by N+1, we have $\sum_{r=0}^{N}\left(\left(\frac{1}{7}\right)^{r+2}\frac{6r+13}{(r+1)(r+2)}\right)=\frac{1}{7}-\frac{1}{7^{N+2}(N+2)}$ $\left[\sum_{r=1}^{N}\left(\left(\frac{1}{7}\right)^{r+2}\frac{6r+13}{(r+1)(r+2)}\right)\right]+\underbrace{\left(\frac{1}{7}\right)^{2}\frac{13}{2}}_{\uparrow \text{ Term when } r=0}=\frac{1}{7}-\frac{1}{7^{N+2}(N+2)}$ $\frac{1}{7}\sum_{r=1}^{N}\left(\left(\frac{1}{7}\right)^{r+1}\frac{6r+13}{(r+1)(r+2)}\right)=\frac{1}{7}-\frac{1}{7^{N+2}(N+2)}-\frac{13}{98}$ $\sum_{r=1}^{N}\left(\left(\frac{1}{7}\right)^{r+1}\frac{6r+13}{(r+1)(r+2)}\right)=7\left(\frac{1}{98}-\frac{1}{7^{N+2}(N+2)}\right)$ $=\frac{1}{14}-\frac{1}{7^{N+1}(N+2)}$	

3 A curve C with equation $y = \mathrm{f}(x)$ satisfies the equation

$$(x^2 + 2x + 2)\frac{\mathrm{d}y}{\mathrm{d}x} = 2$$

and passes through the point $(0, \pi)$.

(i) By further differentiation, find the Maclaurin expansion of $\mathrm{f}(x)$ in ascending powers of x up to and including the term x^3. [5]

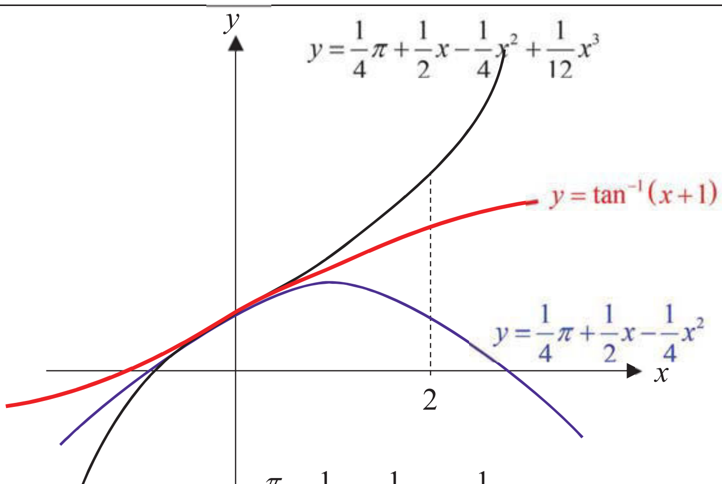
(ii) Solve the differential equation $(x^2 + 2x + 2)\frac{\mathrm{d}y}{\mathrm{d}x} = 2$, given that $y = \pi$ when $x = 0$, leaving y in terms of x.
Hence show that

$$\tan^{-1}(x+1) \approx \frac{\pi}{4} + \frac{1}{2}x - \frac{1}{4}x^2 + \frac{1}{12}x^3$$

for small values of x. [4]

(iii) With the aid of a sketch, explain why $\int_0^2 \frac{\pi}{4} + \frac{1}{2}x - \frac{1}{4}x^2 + \frac{1}{12}x^3 \, \mathrm{d}x$ gives a more accurate approximation of $\int_0^2 \tan^{-1}(x+1) \, \mathrm{d}x$ than $\int_0^2 \frac{\pi}{4} + \frac{1}{2}x - \frac{1}{4}x^2 \, \mathrm{d}x$. [2]

3(i)	(i) $(x^2+2x+2)\frac{\mathrm{d}y}{\mathrm{d}x} = 2$ Differentiating wrt x, $(x^2+2x+2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + (2x+2)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ Differentiating wrt x, $(x^2+2x+2)\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + (2x+2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + (2x+2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ $\Rightarrow (x^2+2x+2)\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + (4x+4)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ When $x=0$, $y=\pi$ since curve passes through $(0,\pi)$. $2\frac{\mathrm{d}y}{\mathrm{d}x} = 2 \quad \Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = 1$ $2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2(1) = 0 \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -1$ $2\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + 4(-1) + 2(1) = 0 \Rightarrow \frac{\mathrm{d}^3y}{\mathrm{d}x^3} = 1$ Thus the Maclaurin expansion is $y = \mathrm{f}(x) = \pi + x - \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots$ i.e., $y = \mathrm{f}(x) = \pi + x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + \ldots$	Question was very well done. Only a handful of students expressed $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{x^2+2x+2}$ and performed direct differentiation. Many students integrate the expression to find y, which is the question for (ii). This is an indicator that they do not know what the question is asking for.

(ii)	(ii) $(x^2+2x+2)\dfrac{\mathrm{d}y}{\mathrm{d}x}=2$ $\Rightarrow \int \mathrm{d}y = 2\int \dfrac{1}{x^2+2x+2}\,\mathrm{d}x$ $\Rightarrow y = 2\int \dfrac{1}{(x+1)^2+1^2}\,\mathrm{d}x$ $\Rightarrow y=2\tan^{-1}(x+1)+c$ Sub $x=0$, $y=\pi$: $\pi = 2\tan^{-1}(1)+c$ $\Rightarrow c=\pi-2\left(\dfrac{\pi}{4}\right)=\dfrac{\pi}{2}$ Thus, $y=2\tan^{-1}(x+1)+\dfrac{\pi}{2}$. $\Rightarrow 2\tan^{-1}(x+1)+\dfrac{\pi}{2}=\pi+x-\dfrac{1}{2}x^2+\dfrac{1}{6}x^3+\ldots$ $\Rightarrow \tan^{-1}(x+1)=\dfrac{1}{2}\left(\pi-\dfrac{\pi}{2}+x-\dfrac{1}{2}x^2+\dfrac{1}{6}x^3+\ldots\right)$ $\Rightarrow \tan^{-1}(x+1)\approx\dfrac{1}{4}\pi+\dfrac{1}{2}x-\dfrac{1}{4}x^2+\dfrac{1}{12}x^3$ (Shown) for small values of x	Apart from algebraic errors, students were able to find the particular solution. Quite a number of students forgot to "$+C$", which affected the rest of their answers. Most students were able to see the relationship between their answers in (i) and (ii).
(iii)	 Since the curve $y=\dfrac{\pi}{4}+\dfrac{1}{2}x-\dfrac{1}{4}x^2+\dfrac{1}{12}x^3$ is **closer to** $y=\tan^{-1}(x+1)$ than the curve $y=\dfrac{\pi}{4}+\dfrac{1}{2}x-\dfrac{1}{4}x^2$ **for** $0\le x\le 2$, the **area under the cubic curve**, $\int_0^2 \dfrac{\pi}{4}+\dfrac{1}{2}x-\dfrac{1}{4}x^2+\dfrac{1}{12}x^3\,\mathrm{d}x$, will give a <u>better</u> approximation to $\int_0^2 \tan^{-1}(x+1)\,\mathrm{d}x$, the area under the curve $y=\tan^{-1}(x+1)$, <u>than</u> the area under the quadratic curve, $\int_0^2 \dfrac{\pi}{4}+\dfrac{1}{2}x-\dfrac{1}{4}x^2\,\mathrm{d}x$. 	This part was poorly performed. Students should realise that they must sketch $y=\tan^{-1}(x+1)$ in order to make any comparison. For the sketches, many students did not pay attention to the details which led to loss of marks. The more common ones are: 1. not realizing that they have the same y-intercept, 2. Not showing that the quadratic curve has a turning point before $x=2$, and cuts the x-axis after $x=2$. 3. Not showing that the cubic curve is concave up.

	Students plotted / labelled the graphs as (for e.g.) $\int_0^2 \frac{\pi}{4}+\frac{1}{2}x-\frac{1}{4}x^2+\frac{1}{12}x^3\ \mathrm{d}x$ which is a definite integral (i.e. just a value). This has no meaning for comparison. A number of students drew rectangles under the graph, which clearly shows regurgitation without understanding. For the qualitative explanation, students should make it clear that the comparison of how close the curves are is limited from $x=0$ to $x=2$. Also, the link between the definite integral to the area must be clear.

4 The points A, B, C and D have coordinates $(1,0,3)$, $(-1,0,1)$, $(1,1,3)$ and $(1,k,0)$ respectively, where k is a positive real number. The plane p_1 contains A, B and C while the plane p_2 contains A, B and D.

Given that p_1 makes an angle of $\frac{\pi}{3}$ with p_2, show that $k=\frac{\sqrt{6}}{2}$. [5]

The point X lies on p_2 such that the vector $\overrightarrow{XC}$ is perpendicular to p_1. Find $\overrightarrow{XC}$. [5]

Hence find the exact area of the triangle AXC. [2]

Solution:

A standard question which can provide a good source of marks, but students seemed to be unprepared for the third question on Vectors in Paper 2.

4	$\overrightarrow{AB}=\begin{pmatrix}-1\\0\\1\end{pmatrix}-\begin{pmatrix}1\\0\\3\end{pmatrix}=-2\begin{pmatrix}1\\0\\1\end{pmatrix}$, $\overrightarrow{AC}=\begin{pmatrix}1\\1\\3\end{pmatrix}-\begin{pmatrix}1\\0\\3\end{pmatrix}=\begin{pmatrix}0\\1\\0\end{pmatrix}$, $\overrightarrow{AD}=\begin{pmatrix}1\\k\\0\end{pmatrix}-\begin{pmatrix}1\\0\\3\end{pmatrix}=\begin{pmatrix}0\\k\\-3\end{pmatrix}$ Normal vector $\mathbf{n}_1$ of $p_1=\begin{pmatrix}1\\0\\1\end{pmatrix}\times\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}-1\\0\\1\end{pmatrix}$ Normal vector $\mathbf{n}_2$ of $p_2=\begin{pmatrix}1\\0\\1\end{pmatrix}\times\begin{pmatrix}0\\k\\-3\end{pmatrix}=\begin{pmatrix}-k\\3\\k\end{pmatrix}$ Since p_1 makes an angle of $\frac{\pi}{3}$ with p_2, $\frac{\lvert \mathbf{n}_1.\mathbf{n}_2\rvert}{\lvert\mathbf{n}_1\rvert\lvert\mathbf{n}_2\rvert}=\cos\frac{\pi}{3}$	Many students used the correct method but made numerous errors seen in vector product operation to obtain the normal to planes p_1 and p_2. This had a knock-on effect on the other parts of the question. This part required the direct application of formula to obtain the angle between planes in order to show that $k=\frac{\sqrt{6}}{2}$. However students made errors in obtaining normals to planes p_1 and p_2. Also some students wrongly

	$\dfrac{\left\lvert \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} . \begin{pmatrix} -k \\ 3 \\ k \end{pmatrix} \right\rvert}{\sqrt{2}\sqrt{2k^2+9}} = \dfrac{1}{2}$ $\dfrac{\lvert 2k \rvert}{\sqrt{2}\sqrt{2k^2+9}} = \dfrac{1}{2}$ Squaring both sides, we have $\dfrac{4k^2}{2\left(2k^2+9\right)} = \dfrac{1}{4}$ $8k^2 = 2k^2 + 9$ $k^2 = \dfrac{3}{2}$ Since k is a positive real number, we have $k = \sqrt{\dfrac{3}{2}} = \dfrac{\sqrt{6}}{2}$ (shown)	stated that the formula gave the sine of the angle. A handful students were not able to find the correct pair of vectors to "cross-product" Example, to find normal to p_1 which contains *A, B and C*: $\overrightarrow{OA} \times \overrightarrow{OB}$ [WRONG] A mark was deducted for not including $k = -\dfrac{\sqrt{6}}{2}$ and rejecting the answer accordingly.
	$\overrightarrow{XC}$ is perpendicular to p_1 $\Rightarrow$ $\overrightarrow{XC}$ is parallel to $\mathbf{n}_1$ $\Rightarrow \overrightarrow{XC} = m\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ $\Rightarrow \overrightarrow{OX} = \overrightarrow{OC} - m\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} - m\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ 	This part was badly attempted. There was a lack of clarity in presentation for this part. Many students were able to write out the equation of line *XC*: **r** = $\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, but instead of $\overrightarrow{OX} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ [RIGHT], many wrote $\overrightarrow{XC} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ [WRONG] This was an indication of poor understanding of the equation of line!

	Equation of p_2 is $\mathbf{r}.\begin{pmatrix}-\frac{\sqrt{6}}{2}\\3\\\frac{\sqrt{6}}{2}\end{pmatrix}=\begin{pmatrix}1\\0\\3\end{pmatrix}.\begin{pmatrix}-\frac{\sqrt{6}}{2}\\3\\\frac{\sqrt{6}}{2}\end{pmatrix}=\sqrt{6}$ **i.e.** $\mathbf{r}.\begin{pmatrix}-\frac{\sqrt{6}}{2}\\3\\\frac{\sqrt{6}}{2}\end{pmatrix}=\sqrt{6}$ Since X lies on p_2, we have $\left[\begin{pmatrix}1\\1\\3\end{pmatrix}-m\begin{pmatrix}-1\\0\\1\end{pmatrix}\right].\begin{pmatrix}-\frac{\sqrt{6}}{2}\\3\\\frac{\sqrt{6}}{2}\end{pmatrix}=\sqrt{6}$ $\Rightarrow\left(-\frac{\sqrt{6}}{2}+3+\frac{3\sqrt{6}}{2}\right)-m\left(\frac{\sqrt{6}}{2}+0+\frac{\sqrt{6}}{2}\right)=\sqrt{6}$ $\Rightarrow\sqrt{6}+3-m\left(\sqrt{6}\right)=\sqrt{6}$ $\Rightarrow m\left(\sqrt{6}\right)=3$ $\therefore m=\frac{3}{\sqrt{6}}$ $\Rightarrow\ \overrightarrow{XC}=\frac{3}{\sqrt{6}}\begin{pmatrix}-1\\0\\1\end{pmatrix}$	
	$\left\|\overrightarrow{XC}\right\|=\left\|\sqrt{\frac{3}{2}}\begin{pmatrix}-1\\0\\1\end{pmatrix}\right\|=\sqrt{\frac{3}{2}}\sqrt{2}=\sqrt{3}$ $\left\|\overrightarrow{AC}\right\|=\left\|\begin{pmatrix}0\\1\\0\end{pmatrix}\right\|=1$ Since XC is perpendicular to p_1 and AC lies on p_1, we have XC perpendicular to AC. Therefore, area of triangle AXC is $\frac{1}{2}\left\|\overrightarrow{XC}\right\|\left\|\overrightarrow{AC}\right\|=\frac{1}{2}\left(\sqrt{3}\right)(1)=\frac{\sqrt{3}}{2}$	Many students failed to see that triangle *AXC* is a right-angled triangle and the area can be quite easily obtained by $\frac{1}{2}(base)(\perp height)$ i.e. $\frac{1}{2}(AC)(XC)$. Instead many students found the area by finding $\frac{1}{2}\left\|\overrightarrow{AC}\times\overrightarrow{XC}\right\|$.

Section B: Probability and Statistics [60 marks]

5 Anand, Beng, Charlie, Dayanah and 6 other people attend a banquet dinner, and are to sit at a round table.

(i) Dayanah will only sit next to Anand, Beng or Charlie (and no one else), and Anand, Beng and Charlie do not want to sit next to each other. Find the number of ways the 10 people can seat themselves around the table. [4]

(ii) As part of dinner entertainment, 4 people from the table are chosen to participate in a game.

Among Anand, Beng and Charlie, if any one of them is chosen, the other two will refuse to participate in the game. Furthermore, Dayanah refuses to participate unless at least one of Anand, Beng or Charlie is also chosen.

Find the number of ways the 4 people can be chosen for the game. [3]

5(i)	No. of ways to arrange 2 of A/B/C next to D, to form a unit = ${}^3C_2 \times 2!$ No. of ways to arrange the unit and 6 other people in a circle = $(7 - 1)!$ No. of ways to insert the remaining person from A/B/C = ${}^5C_1$ $\therefore$ Total no. of ways = ${}^3C_2 \times 2! \times (7-1)! \times {}^5C_1 = 21600$	**In general, students lost marks for this whole question because they failed to understand the question.** For this part, many students interpreted "sit next to" as applying to one side (should be both sides) and therefore were incorrect. **Students should also take the effort to describe/explain their steps** to possibly gain partial credit for their methodology.
(ii)	Case 1: A, B or C not chosen Number of ways = ${}^6C_4 = 15$ Case 2: Exactly one of A, B or C chosen Number of ways = ${}^3C_1 \times {}^7C_3 = 105$ Total number of ways = 15 + 105 = 120 [Note: Case 2 may be split into: Case 2a: Exactly one of A, B or C chosen but not Dayanah, (Number of ways = ${}^3C_1 \times {}^6C_3 = 60$) And Case 2b: Dayanah and exactly one of A, B or C chosen (Number of ways = ${}^3C_1 \times {}^6C_2 = 45$).]	A large number of students assumed that when A, B or C were chosen, D must be chosen when this is not the case. Otherwise this part was the better done of the two.

6 A bag contains four identical counters labelled with the digits 0, 1, 2, and 3. In a game, Amira chooses one counter randomly from the bag and then tosses a fair coin. If the coin shows a Head, her score in the game is the digit labelled on the counter chosen. If the coin shows a Tail, her score in the game is the negative of the digit labelled on the counter chosen. T denotes the score in a game.

(i) Find the probability distribution of T. [2]

(ii) Amira tosses the coin and it shows a Tail. Find the probability that $T < -1$. [3]

(iii) Amira plays the game twice. Find the probability that the sum of her two scores is positive. [3]

<table>
<tr>
<td>6(i)</td>
<td>

t	-3	-2	-1	0	1	2	3
$\mathrm{P}(T=t)$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$

</td>
<td>Most students could do this part correctly.</td>
</tr>
<tr>
<td>(ii)</td>
<td>

$\mathrm{P}(T<-1 \mid \text{coin shows a tail})$

$= \dfrac{\mathrm{P}(\text{counter shows 2 or 3 and coin shows a Tail})}{\mathrm{P}(\text{coin shows a Tail})}$

$\dfrac{\left(\frac{2}{4}\right)\times\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)} = \dfrac{1}{2}$

</td>
<td>Many students could not identify that this was a conditional probability question. Details were also insufficient among the rest of the answers.</td>
</tr>
<tr>
<td>(iii)</td>
<td>

$$\text{Required probability} = \frac{1}{2}\left[1-\mathrm{P}(\text{sum of scores}=0)\right]$$

$$= \frac{1}{2}\left[1-\left(\frac{1}{4}\times\frac{1}{4}+\frac{1}{8}\times\frac{1}{8}\times 6\right)\right]$$

$$= \frac{27}{64}$$

Alternative:
Required probability

$$= 2\left[\mathrm{P}(0,1)+\mathrm{P}(0,2)+\mathrm{P}(0,3)\right]+2\left[\mathrm{P}(-1,2)+\mathrm{P}(-1,3)\right]+2\mathrm{P}(-2,3)$$

$$+2\left[\mathrm{P}(1,2)+\mathrm{P}(1,3)+\mathrm{P}(2,3)\right]+\left[\mathrm{P}(1,1)+\mathrm{P}(2,2)+\mathrm{P}(3,3)\right]$$

$$= 2\left[\left(\frac{1}{4}\times\frac{1}{8}\times 3\right)\right]+2\left[\left(\frac{1}{8}\times\frac{1}{8}\times 2\right)\right]+2\left(\frac{1}{8}\times\frac{1}{8}\right)+2\left(\frac{1}{8}\times\frac{1}{8}\times 3\right)+3\left(\frac{1}{8}\times\frac{1}{8}\right)$$

$$= \frac{27}{64}$$

</td>
<td>Most students attempted the second method, but there were many common errors:
(1) not considering permutations for each case,
(2) missing out cases (such as 1 + 1)
(3) forgetting that $\mathrm{P}(T{=}0) = \frac{1}{4}$.

A smaller number tried to use a table of outcomes, but were not very successful as they did not notice that not all outcomes had equal probabilities.</td>
</tr>
</table>

7 It is generally accepted that a person's diet and cardiorespiratory fitness affects his cholesterol levels. The results of a study on the relationship between the cholesterol levels, C mmol/L, and cardiorespiratory fitness, F, measured in suitable units, on 8 individuals with similar diets are given in the following table.

Cardiorespiratory Fitness (F units)	55.0	50.7	45.3	40.2	34.7	31.9	27.9	26.0
Cholesterol (C mmol/L)	4.70	4.98	5.30	5.64	6.04	6.30	6.99	6.79

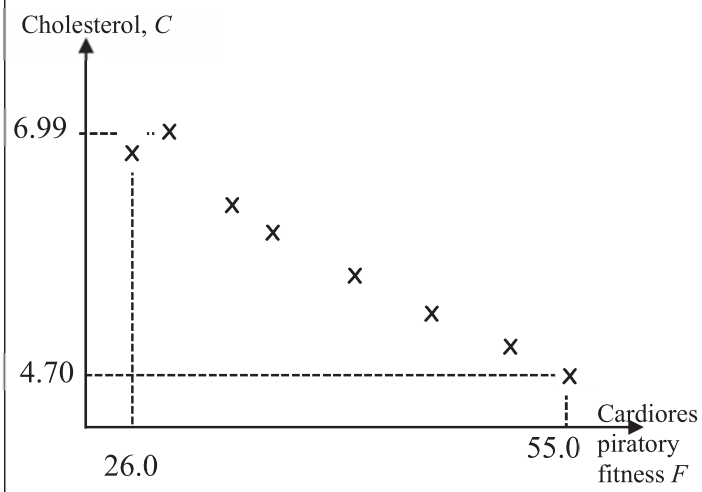
(i) Draw a scatter diagram of these data. Suppose that the relationship between F and C is modelled by an equation of the form $\ln C = aF + b$, where a and b are constants. Use your diagram to explain whether a is positive or negative. [4]

(ii) Find the product moment correlation coefficient between ln C and F, and the constants a and b for the model in part **(i)**. [3]

(iii) Bronz is a fitness instructor. His cardiorespiratory fitness is 52.0 units. Estimate Bronz's cholesterol level using the model in **(i)** and the values of a and b in part **(ii)**. Comment on the reliability of the estimate. [2]

(iv) Bronz then had a medical checkup and found his actual cholesterol level to be 6.2 mmol/L. Assuming his cholesterol level is measured accurately, explain why there is a great difference between Bronz's cholesterol level and the estimated value in **(iii)**. [1]

(i)



$\ln C = aF + b \quad \Rightarrow \quad C = e^{aF+b}$

From the scatter diagram, as F increases, <u>C decreases at a decreasing rate</u>, hence $a < 0$.

- Many students gave a scatter diagram of ln$C$ vs F. However, the question asked for a scatter diagram of *these data* & the given data is C and F.
- Please ensure that the **relative positions** of the 8 data points (and not 7 or 9 points) are correct.
- Many students did not label the max/min values of C and F. Amongst those who did, a large majority did not label the max C (6.99) and/or min F (26) correctly.

This part was poorly attempted.

- Many students say that the gradient of the points in the scatter diagram (C against F) are negatively correlated so it has a negative gradient and thus a is negative. BUT "a" is the gradient of the regression line of ln$C$ on F and <u>not</u> C on F.

		• <u>Alternative solution given by some students</u> ln$C$ decreases as C decreases. Thus as F increases, ln$C$ decreases. This means that the gradient of the regression line of ln$C$ on F is negative. However, they failed to **explain <u>clearly</u>** that "a" is the gradient of the regression line <u>before concluding</u> that "a" is negative.
(ii)	Product moment correlation $r = -0.992$ (3 sf) Model is $\ln C = -0.013371F + 2.2772$ (5 sf) Thus, $a = -0.0134$ (3 sf) and $b = 2.28$ (3 sf)	Some students identified the values of a and b wrongly, i.e., wrote $a = 2.28$ in spite of the fact they had claimed that $a < 0$ in (i).
(iii)	$\ln C = -0.013371(52.0) + 2.2772$ $C = \mathrm{e}^{1.581908} = 4.86$ Estimate of Bronz's cholesterol level is 4.86 mmol/L Since $r = -0.992$ is close to -1 which suggests that the linear model is a good one <u>and</u> Bronz's cardiorespiratory fitness level, $F = 52.0$ lies within the data range of $[26.0, 55.0]$, the estimate is reliable.	Some students used the inaccurate values of a and b, i.e., used $\ln C = -0.0134F + 2.28$ to compute C and got an inaccurate value. Many students mentioned only one of the 2 conditions needed for a reliable estimate with the regression line.
(iv)	Bronz's **diet** could be very high in cholesterol **compared to the 8 individuals** which resulted in his actual cholesterol level of 6.20 to be higher than the value of 4.86 mmol/L estimated using the values in (iv) which are based on the 8 individuals.	This part was very poorly attempted. Students need to check if there is a reason provided by the context before giving any other reasonable explanation. In this question, a reason was given (diet) and so this is the only answer accepted. The regression line that was used to find the estimate was based on the data from the 8 people with a similar diet. So the estimate of 4.86 mmol/L would be Bronz's cholesterol level if he was on the same diet.

8 A research laboratory uses a data probe to collect data for its experiments. There is a probability of 0.04 that the probe will give an incorrect reading. In a particular experiment, the probe is used to take 80 readings, and X denotes the number of times the probe gives an incorrect reading.

(i) State, in context, two assumptions (not conditions) necessary for X to be well modelled by a binomial distribution. [2]

(ii) Find the probability that between 5 and 10 (inclusive) incorrect readings are obtained in the experiment. [3]

8(i)	1. The **probability** of the probe giving an incorrect reading is **constant** at 0.04 for each reading. 2. Trial of whether a reading by the probe is incorrect is **independent** of other readings.	Not well answered. Despite the many reminders, common errors are: 1. Probability of getting an incorrect reading is independent of … 2. Two outcomes: correct or incorrect reading (given in question, not assumption) 3. Fixed number (80) of readings (given in question, not assumption) 4. Trial is on "reading correct or incorrect", not the probe (only one probe) nor the experiment (only one experiment).
(ii)	$X \sim B(80, 0.05)$ $P(5 \leq X \leq 10) = P(X \leq 10) - P(X \leq 4)$ $= 0.216$ (3sf)	Generally well done for this part. Common error is interpreting "inclusive" to apply only to 10 and not 5, i.e. wrongly gave $P(5 < X \leq 10)$. Some students wrongly gave $P(5 \leq X \leq 10) = P(X \leq 10) \times P(X \geq 5)$ which is not true as the events $X \leq 10$ and $X \geq 5$ are not independent.

When the probe gives an incorrect reading, it will give a reading that is 5% greater than the actual value.

(iii) Suppose the 80 readings are multiplied together to obtain a Calculated Value. Find the probability that the Calculated Value is **at least** 50% more than the product of the 80 actual values. [5]

Solution

(iii)	Let the actual i^{th} value be V_i. Product of 80 actual values = $V_1 \times V_2 \times \ldots \times V_{80}$ For an incorrect reading, V_i is read as $(1.05)V_i$ (5% greater) If there are x incorrect readings, Calculated Value = $V_1 \times V_2 \times \ldots \times V_{80}\,(1.05)^x$. $V_1 \times V_2 \times \ldots \times V_{80}(1.05)^x \geq 1.5\,(V_1 \times V_2 \times \ldots \times V_{80})$ $\Rightarrow (1.05)^x \geq 1.5$ $\Rightarrow x\ln(1.05) \geq \ln 1.5$ $\Rightarrow x \geq \dfrac{\ln 1.5}{\ln 1.05} = 8.31$ i.e. at least 9 incorrect readings Required probability = $P(X \geq 9)$ $= 1 - P(X \leq 8)$ $= 0.00468$ (3sf)	Most students did not know how to interpret the question. Many students tried to apply Central Limit Theorem for sample sum – note that CLT does not apply to product of sample. Of those who could interpret the question, many did not explain how they obtained $(1.05)^x \geq 1.5$ Some students presented 'product of the 80 actual values' wrongly as V^{80} instead of $V_1 \times V_2 \times \ldots \times V_{80}$ (80 different readings)

9 A Wheel Set refers to a set of wheel rim and tyre. The three types of wheel sets are the Clincher Bike Wheel Set, Tubular Bike Wheel Set and Mountain Bike Wheel Set.
The weight of a rim of a Clincher Bike Wheel Set follows a normal distribution with mean 1.5 kg and standard deviation 0.01 kg. The weight of its tyre follows a normal distribution with mean 110 g and standard deviation 5 g.

(i) Let C be the total weight in grams of a randomly chosen Clincher Bike Wheel Set in grams. Find $\mathrm{P}(C > 1620)$. [3]

(ii) State, in the context of the question, an assumption required in your calculation in **(i)**. [1]

Let T be the total weight in grams of a Tubular Bike Wheel Set, where $T \sim \mathrm{N}(\mu,\ 15^2)$.

(iii) The probability that the weight of a randomly chosen Clincher Bike Wheel Set exceeds a randomly chosen Tubular Bike Wheel Set by more than 150 g is smaller than 0.70351 correct to 5 decimal places. Find the range of values that μ can take. [5]

Let M be the total weight in grams of a randomly chosen Mountain Bike Wheel Set with mean 1800 g and standard deviation 20 g.

(iv) Find the probability that the mean weight of 50 randomly chosen Mountain Bike Wheel Sets is more than 1795 g. [3]

9(i)	$C \sim \mathrm{N}(1500 + 110,\ 10^2 + 5^2)$ $\mathrm{P}(A > 1620) = 0.185546 \approx 0.186$ (3sf)	Well attempted. Only a few students did not convert kilograms to grams. Note that $10^2 + 5^2 \neq (10+5)^2$
(ii)	Assume that the weight of a randomly chosen Clincher Bike rim and tyre are independent of each other. (This is required for the calculation of Var(C)) (Recall Var(X+Y) = Var(X) + Var(Y) if X and Y are independent)	Poorly done. Many gave assumptions such as: - The weight of a randomly chosen Clincher Bike Wheelset is independent of another set. (There's only one set in (i)) - Assume that a set has only one rim and one wheel (This was already stated in the question)

(iii)	$C-T \sim N\left(1610-\mu,\ 125+15^2\right) \Rightarrow C-T \sim N\left(1610-\mu,\ 350\right)$ $\mathrm{P}\left(C-T>150\right)<0.70351$ $\mathrm{P}\left(Z \geq \frac{150-1610+\mu}{\sqrt{350}}\right)<0.70351$ $\mathrm{P}\left(Z \geq \frac{-1460+\mu}{\sqrt{350}}\right)<0.70351$ $\frac{-1460+\mu}{\sqrt{350}} \geq -0.534523$ $\mu \geq 1449.9999$ $\mu \geq 1450$	Note that GC table is not allowed as μ is not an integer value. Many students failed to standardise correctly. Many wrote 0.534523 instead of – 0.534523, indicating that they are still not able to identify the correct area for the invnorm command in GC.
(iv)	Since $n=50$ is large, by Central Limit Theorem, $\overline{M} \sim N\left(1800,\ \frac{400}{50}\right)$ approximately $\overline{M} \sim N\left(1800,\ 8\right)$ approximately $P\left(\overline{M}>1795\right)=0.96146 \approx 0.961\,(3\text{sf})$	Many students wrote $M \sim N\left(1800,\ 20^2\right)$ which is incorrect as it is not given in the question that M is normally distributed. Not many students knew that CLT has to be used for this part. Some applied CLT to *M*.
Students are able to find the variance and expectation of the random variables but would need to work on problems which require them to standardise (especially those that involve inequality)		

10 (a) Two random samples of different sample sizes of households in the town of Aimek were taken to find out the mean number of computers per household there. The first sample of 50 households gave the following results.

Number of computers	0	1	2	3	4
Number of households	5	12	18	10	5

The results of the second sample of 60 households were summarised as follows.

$$\sum y = 118 \qquad \sum y^2 = 314,$$

where y is the number of computers in a household.

(i) By combining the two samples, find unbiased estimates of the population mean and variance of the number of computers per household in the town. [4]

(ii) Describe what you understand by 'population' in the context of this question. [1]

(b) Past data has shown that the working hours of teachers in a city are normally distributed with mean 48 hours per week. In a recent study, a large random sample of n teachers in the city was surveyed and the number of working hours per week was recorded. The sample mean was 46 hours and the sample variance was 131.1 hours^2. A hypothesis test is carried out to determine whether the mean working hours per week of teachers has been reduced.

(i) State appropriate hypotheses for the test. [1]

The calculated value of the test statistic is $z = -1.78133$ correct to 5 decimal places.

(ii) Deduce the conclusion of the test at the 2.5 % level of significance. [2]

(iii) Find the value of n. [3]

(iv) In another test, using the same sample, there is significant evidence at the $\alpha\%$ level that there is a change in the mean working hours per week of teachers in that city. Find the smallest possible integral value of α . [2]

This question is poorly done in general. Students display a lack of understanding of hypothesis testing concepts and procedures, coupled with a perpetual incompetency in obtaining the unbiased estimates from given data. Many of the solutions presented also reveal a lack in comprehension by students on what the questions are asking for.		
10(a)(i)	Let X be the number of computers in each household in the first sample. Using GC, $\sum fx = 98$, $\sum fx^2 = 254$ Unbiased estimate of the population mean $= \dfrac{98+118}{50+60} = \dfrac{108}{55} (\approx 1.96)$ Unbiased estimate of the population variance $= \dfrac{1}{110-1}\left[(254+314) - \dfrac{(98+118)^2}{110}\right] = \dfrac{7912}{5995} (\approx 1.32)$	The most common error for the unbiased estimate for the population mean from the combined sample is taking the averages of the unbiased estimates for the respective sample sizes, i.e. $\dfrac{1}{2}\left(\dfrac{98}{50}+\dfrac{118}{60}\right)$. The most common error for the unbiased estimate for the population variance is using the formula for pooled variance given in the MF26.
(a)(ii)	The population in this question refers to <u>all the households in the town</u>.	Majority of students are able to correctly identify what the population is. Common errors include failing to state that the households are from the town (and not anywhere else) and for referring the population to computers instead. A small handful gave the definition of a random sample here, which indicates a lack in comprehension what the question is asking for.
(b)(i)	Let μ be the population mean working hours of teachers in the city. $H_0 : \mu = 48$ $H_1 : \mu < 48$	This part is generally well done.
(ii)	Critical region $= \{z : z \le -1.960\}$ Since $z_{cal} = -1.781 > -1.960$, we do not reject H_0 **OR** p-value $= P(Z \le -1.781) = 0.0375 > 0.025$, we do not reject H_0 There is <u>insufficient</u> evidence at <u>2.5% level of significance</u> that the <u>mean working hours per week of teachers in the city has been reduced</u>.	Most students know that the finding the critical region is key to stating the correct conclusion, and most are able to obtain the correct critical value (not the region though). This leads to an overall confusion with whether to reject/accept H_0,

		or whether to reject/accept H_1. Subsequently, the confusion with whether there is sufficient/insufficient evidence persists as well. There is a sizable number of students who are of the impression that the decision for rejection of H_0 is the conclusion.
(iii)	$s^2 = \frac{n}{n-1}(131.1)$ Given z= $-1.781 \Rightarrow \frac{46-48}{\left(\sqrt{\frac{131.1}{n-1}}\right)} = -1.781$ $-2 = -1.781\sqrt{\frac{131.1}{n-1}}$ $\sqrt{n-1} = \frac{-1.781\sqrt{131.1}}{-2}$ $n = 105$	Most students use 131.1 as the population variance, therefore losing marks for accuracy.
(iv)	p-value for the two tailed test = $2 \times \mathrm{P}(Z \le -1.781) = 0.0749$ (accept 0.075) For H_0 to be rejected, $p\text{-value} \le \frac{\alpha}{100}$ $\alpha \ge 7.49$ Smallest value of α = 8	Students who rightly approach this part using a 2-tailed test are able to obtain the correct result. There is a small handful who mistakenly compared the test statistic –1.78133 to the level of significance.