Y2024 AM 4049 PRELIM PAPER 1 SOLUTIONS

1 It is given that $\mathrm{f}(x) = x^3 - 3x^2 - 25x - 21$.

(a) Solve the equation $\mathrm{f}(x) = 0$. [5]

[Turn over

(b) Hence, solve the equation $21y^3 + 25y^2 + 3y - 1 = 0$ where $y \neq 0$. [3]

[Turn over

2 **(a)** Given that $\sin^2\theta = \cos\theta(2-\cos\theta)$, show that $\cos\theta = \frac{1}{2}$. [2]

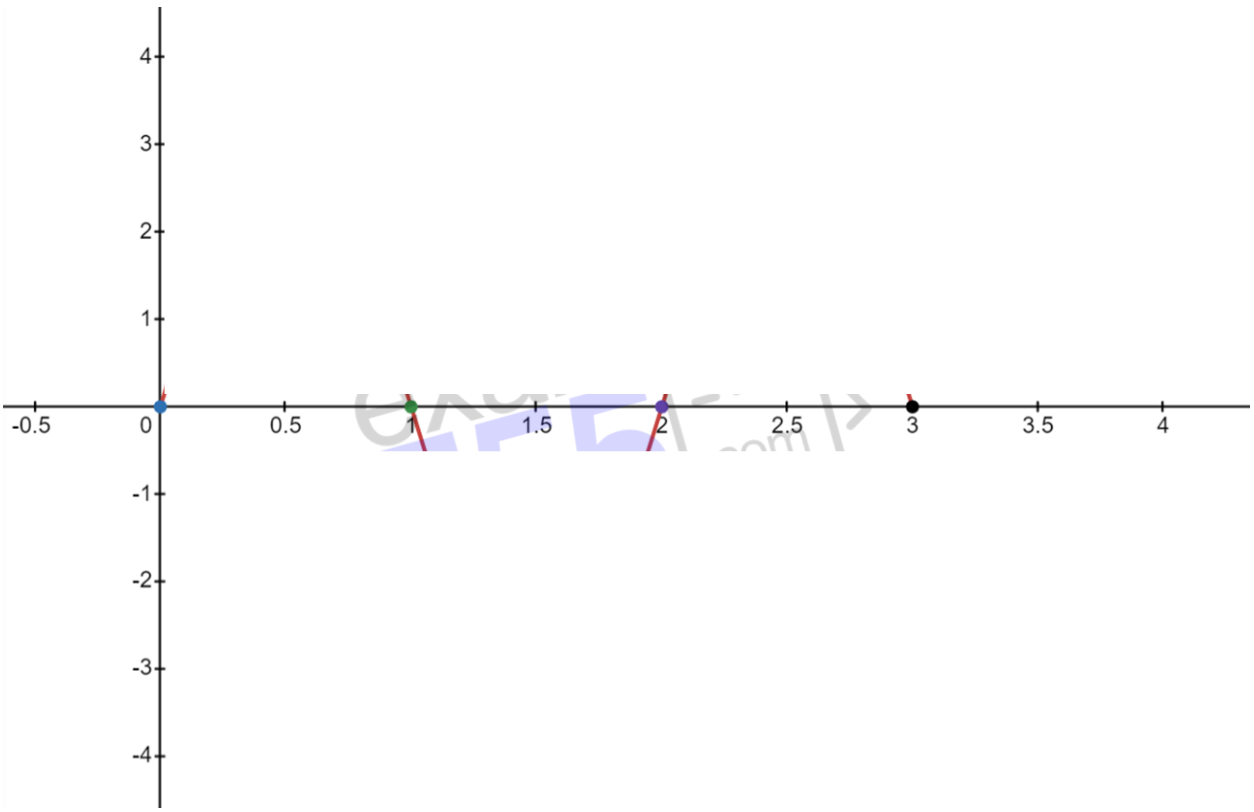
(b) Hence solve the equation $\sin^2 2x = \cos 2x(2-\cos 2x)$

in the interval $0 \le x \le \pi$, giving your answers in exact form. [3]

[Turn over

3 **(a)** Find the amplitude and period of $y = 3\sin(\pi x)$. [2]

(b) On the grid below, sketch the graph of $y = 3\sin(\pi x)$ for $0 \le x \le 3$. [4]



[Turn over

4 **(a)** Given that $\log_9 x^2 = \log_{27} u$, show that $u = x^3$. [3]

(b) Hence, solve the equation $\log_{27}\left(6x^2 + 8x\right) - \log_9 x^2 = \dfrac{1}{\log_2 27}$. [5]

5 A circle C has equation $x^2 + y^2 - 2y - 4 = 0$.

Find the equations of the two circles centred at point $(8, 5)$ and touching C. [8]

6 **(a)** Express $\dfrac{5x^2+11x}{(4+x)(1+x)^2}$ in partial fractions. [5]

[Turn over

(b) Hence, find $\int \frac{10x^2 + 22x}{(4+x)(1+x)^2}\,\mathrm{d}x$. [4]

[Turn over

7 The following diagram shows the graph of the function

$f(x) = 10x^3 - 30x^2 - 240x + 150$, for $-5 \leq x \leq 7$

(a) State whether the function is increasing or decreasing at $x = -3$.

Give a reason for your answer. [2]

[Turn over

(b) Find the coordinates of the turning points. [5]

(c) Find the equation of the tangent to the graph at the point $(-1, 350)$.

Give your answer in the form $ax + by + d = 0$. [3]

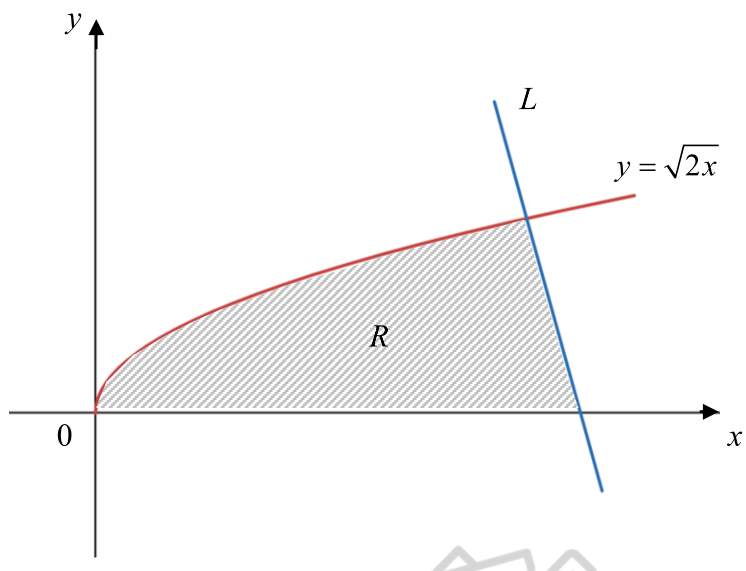
[Turn over

8 **(a)** Prove the identity $\dfrac{\tan 3x + \tan x}{\tan 3x - \tan x} = 2\cos 2x$. [4]

(b) Hence, find the exact value of $\tan 15°$ in the form $a + b\sqrt{c}$, where a, b and c are constants. [4]

9 Line L is the normal to the graph $y = \sqrt{2x}$ at $x = 2$.

Find the area of the shaded region R. [8]



10 The velocity v ms^{-1} of a particle t seconds after passing through O is given by

[illegible]

[2]

When $t = 4$,

$a = 0.6e^{2(4)} = 1790$ ms^{-2} (3 sf)

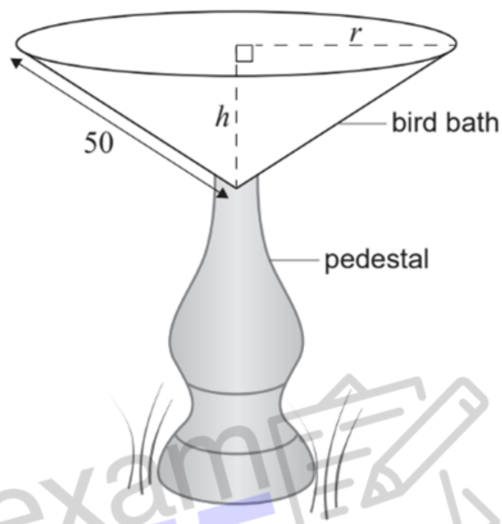
(b) Find the value of t when the particle is at instantaneous rest. [2]

(c) Find the distance travelled by the particle in the first 4 seconds. [6]

11 [The volume of cone of height h and base radius r is $\frac{1}{3}\pi r^2 h$]

Joe designs a concrete bird bath. The bird bath is supported by a pedestal.

This is shown in the diagram.



The interior of the bird bath is in the shape of a cone with radius r, height h and a constant slant height of 50 cm.
Let V be the volume of the bird bath.

(a) Show that $V = \frac{2500\pi h}{3} - \frac{\pi h^3}{3}$. [3]

Joe wants the bird bath to have maximum volume.

(b) Find the value of h for which V is a maximum. [4]

(c) Find the maximum volume of the bird bath, justifying that this value is a maximum. [3]

Therefore volume is maximum.

[Turn over

ALTERNATIVE METHOD

Therefore volume is maximum.

Y2024 AM 4049 PRELIM PAPER 1 SOLUTIONS

1 It is given that $f(x) = x^3 - 3x^2 - 25x - 21$.

(a) Solve the equation $f(x) = 0$. [5]

$f(x) = x^3 - 3x^2 - 25x - 21$

$f(-1) = (-1)^3 - 3(-1)^2 - 25(-1) - 21 = 0$

Therefore $(x+1)$ is a factor.

$x^3 - 3x^2 - 25x - 21 = (x+1)(ax^2 + bx + c)$

By comparing coefficients,

$a = 1$

$c = -21$

$b + a = -3$

$b + 1 = -3$

$b = -4$

$x^3 - 3x^2 - 25x - 21 = 0$

$(x+1)(x^2 - 4x - 21) = 0$

$(x+1)(x-7)(x+3) = 0$

$x = -1$ $x = 7$ $x = -3$

[Turn over

(b) Hence, solve the equation $21y^3+25y^2+3y-1=0$ where $y \neq 0$. [3]

$$21y^3+25y^2+3y-1=0$$

$$1-3y-25y^2-21y^3=0$$

$$y^3\left(1\left(\frac{1}{y}\right)^3-3\left(\frac{1}{y}\right)^2-25\left(\frac{1}{y}\right)-21\right)=0$$

Let $x=\frac{1}{y}$,

$\frac{1}{y}=-1$ $\quad\quad$ $\frac{1}{y}=7$ $\quad\quad$ $\frac{1}{y}=-3$

$y=-1$ $\quad\quad$ $y=\frac{1}{7}$ $\quad\quad$ $y=-\frac{1}{3}$

[Turn over

2 **(a)** Given that $\sin^2\theta = \cos\theta(2-\cos\theta)$, show that $\cos\theta = \frac{1}{2}$. [2]

$\sin^2\theta = \cos\theta(2-\cos\theta)$

$1-\cos^2\theta = 2\cos\theta - \cos^2\theta$

$2\cos\theta = 1$

$\cos\theta = \frac{1}{2}$

(b) Hence solve the equation $\sin^2 2x = \cos 2x(2-\cos 2x)$

in the interval $0 \le x \le \pi$, giving your answers in exact form. [3]

$\sin^2 2x = \cos 2x(2-\cos 2x)$

$\cos 2x = \frac{1}{2}$

Basic angle $= \frac{\pi}{3}$ $\quad 0 \le 2x \le 2\pi$

$2x = \frac{\pi}{3}, \frac{5\pi}{3}$

$x = \frac{\pi}{6}, \frac{5\pi}{6}$

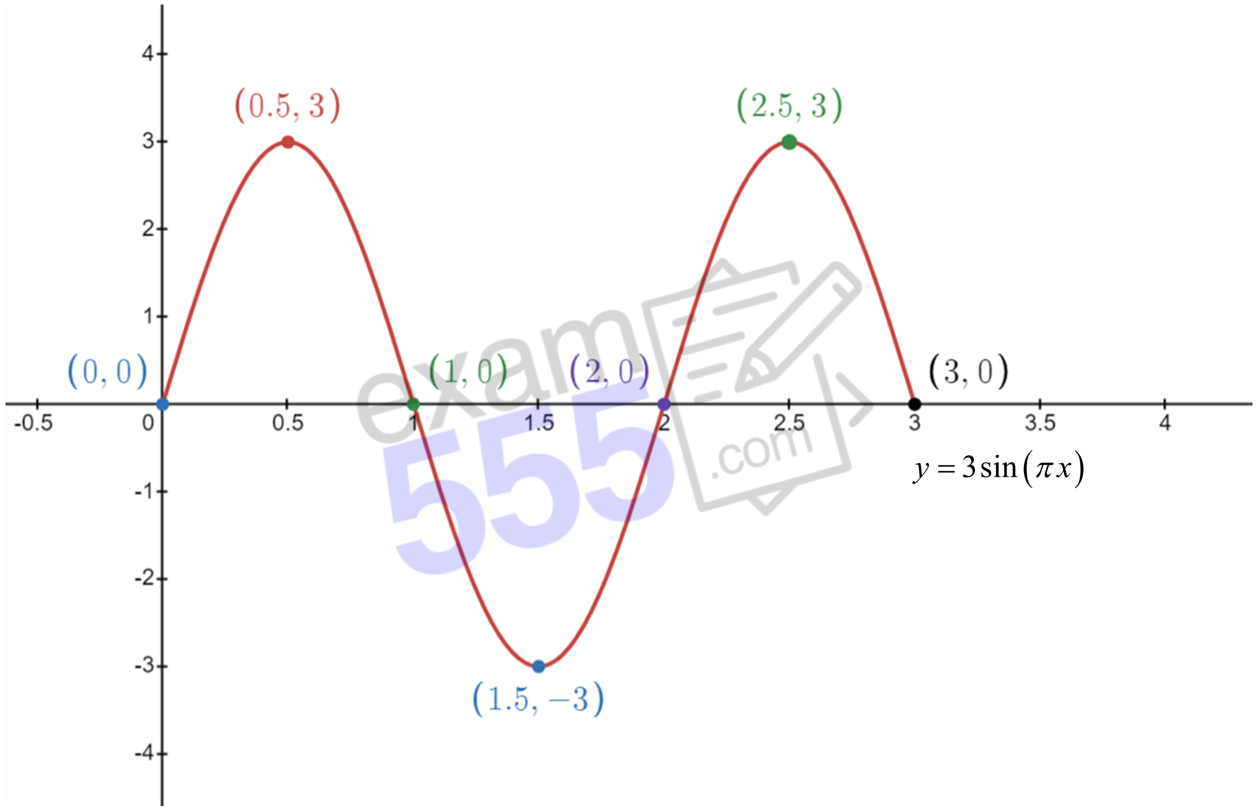
[Turn over

3 **(a)** Find the amplitude and period of $y = 3\sin(\pi x)$. [2]

amplitude = 3

period $= \dfrac{2\pi}{\pi} = 2$

(b) On the grid below, sketch the graph of $y = 3\sin(\pi x)$ for $0 \le x \le 3$. [4]



[Turn over

4 **(a)** Given that $\log_9 x^2 = \log_{27} u$, show that $u = x^3$. [3]

$\log_9 x^2 = \log_{27} u$

$\frac{\log_3 x^2}{\log_3 9} = \frac{\log_3 u}{\log_3 27}$

$\frac{\log_3 x^2}{2} = \frac{\log_3 u}{3}$

$\frac{3}{2}\log_3 x^2 = \log_3 u$

$\log_3 x^3 = \log_3 u$

Therefore, $u = x^3$

(b) Hence, solve the equation $\log_{27}\left(6x^2+8x\right) - \log_9 x^2 = \frac{1}{\log_2 27}$. [5]

$\log_{27}\left(6x^2+8x\right) - \log_9 x^2 = \frac{1}{\log_2 27}$

$\log_{27}\left(6x^2+8x\right) - \log_{27} u = \log_{27} 2$

$\log_{27}\left(6x^2+8x\right) - \log_{27} x^3 = \log_{27} 2$

$\log_{27}\frac{6x^2+8x}{x^3} = \log_{27} 2$

$\frac{6x^2+8x}{x^3} = 2$

$6x^2+8x = 2x^3$

$2x^3 - 6x^2 - 8x = 0$

$x\left(x^2-3x-4\right) = 0$

$x(x-4)(x+1) = 0$

$x = 0$ (NA), $x = 4$, $x = -1$ (NA)

[Turn over

5 A circle C has equation $x^2 + y^2 - 2y - 4 = 0$.

Find the equations of the two circles centred at point $(8,5)$ and touching C. [8]

$x^2 + y^2 - 2y - 4 = 0$

$(x-0)^2 + (y-1)^2 - 1 - 4 = 0$

$(x-0)^2 + (y-1)^2 = 5$

Centre = $(0,1)$

Radius = $\sqrt{5}$

Distance between two centres

$= \sqrt{(8-0)^2 + (5-1)^2}$

$= \sqrt{80}$

$= 4\sqrt{5}$

Radius of Circle 1 $= 4\sqrt{5} - \sqrt{5} = 3\sqrt{5}$

Equation: $(x-8)^2 + (y-5)^2 = (3\sqrt{5})^2$

$(x-8)^2 + (y-5)^2 = 45$

Radius of Circle 2 $= 4\sqrt{5} + \sqrt{5} = 5\sqrt{5}$

Equation: $(x-8)^2 + (y-5)^2 = (5\sqrt{5})^2$

$(x-8)^2 + (y-5)^2 = 125$

[Turn over

6 **(a)** Express $\dfrac{5x^2+11x}{(4+x)(1+x)^2}$ in partial fractions. [5]

$$\frac{5x^2+11x}{(4+x)(1+x)^2}=\frac{A}{4+x}+\frac{B}{(1+x)}+\frac{C}{(1+x)^2}$$

$$5x^2+11x=A(1+x)^2+B(4+x)(1+x)+C(4+x)$$

When $x=-1$,

$$5(-1)^2+11(-1)=C(3)$$
$$3C=-6$$
$$C=-2$$

When $x=-4$,

$$5(-4)^2+11(-4)=A(-3)^2$$
$$9A=36$$
$$A=4$$

When $x=0$,

$$0=4(1)^2+B(4)(1)-2(4)$$
$$4B=4$$
$$B=1$$

$$\frac{5x^2+11x}{(4+x)(1+x)^2}=\frac{4}{4+x}+\frac{1}{(1+x)}-\frac{2}{(1+x)^2}$$

[Turn over

(b) Hence, find $\int \frac{10x^2+22x}{(4+x)(1+x)^2}\,\mathrm{d}x$. [4]

$$\int \frac{10x^2+22x}{(4+x)(1+x)^2}\,\mathrm{d}x$$

$$=2\int \frac{5x^2+11x}{(4+x)(1+x)^2}\,\mathrm{d}x$$

$$=2\int \frac{4}{4+x}+\frac{1}{(1+x)}-\frac{2}{(1+x)^2}\,\mathrm{d}x$$

$$=2\int \frac{4}{4+x}+\frac{1}{(1+x)}-2(1+x)^{-2}\,\mathrm{d}x$$

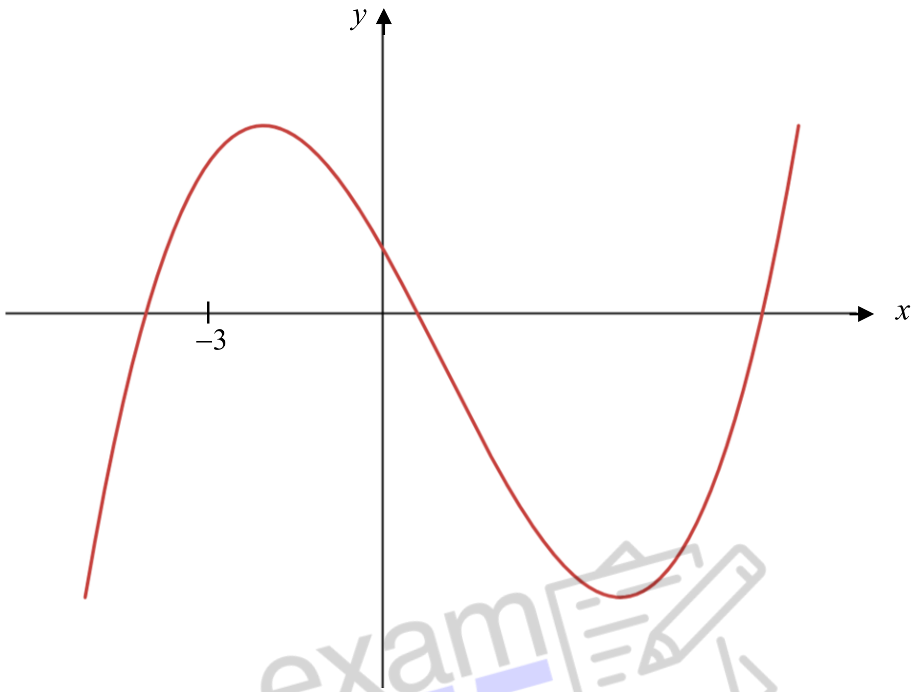
$$=2\left[4\ln(4+x)+\ln(1+x)-\frac{2(1+x)^{-1}}{-1}\right]+c$$

$$=8\ln(4+x)+2\ln(1+x)+\frac{4}{(1+x)}+c$$

[Turn over

7 The following diagram shows the graph of the function

$f(x) = 10x^3 - 30x^2 - 240x + 150$, for $-5 \le x \le 7$



(a) State whether the function is increasing or decreasing at $x = -3$.

Give a reason for your answer. [2]

Increasing.

Gradient of the tangent line is positive.

(b) Find the coordinates of the turning points. [5]

$$f(x) = 10x^3 - 30x^2 - 240x + 150$$

$$f'(x) = 30x^2 - 60x - 240$$

$$f'(x) = 0$$

$$30x^2 - 60x - 240 = 0$$

$$x^2 - 2x - 8 = 0$$

$$(x+2)(x-4) = 0$$

$$x = -2 \qquad x = 4$$

When $x = -2$

$$f(x) = 10(-2)^3 - 30(-2)^2 - 240(-2) + 150$$
$$= 430$$

Max point $(-2, 430)$

When $x = 4$

$$f(x) = 10(4)^3 - 30(4)^2 - 240(4) + 150$$
$$= -650$$

Min point $(4, -650)$

(c) Find the equation of the tangent to the graph at the point $(-1, 350)$.

Give your answer in the form $ax + by + d = 0$. [3]

When $x = -1$

$f'(-1) = 30(-1)^2 - 60(-1) - 240 = -150$

$\frac{y - 350}{x + 1} = -150$

$y - 350 = -150x - 150$

$150x + y - 200 = 0$

8 **(a)** Prove the identity $\dfrac{\tan 3x + \tan x}{\tan 3x - \tan x} = 2\cos 2x$. [4]

LHS

$$= \frac{\tan 3x + \tan x}{\tan 3x - \tan x}$$

$$= \frac{\frac{\sin 3x}{\cos 3x} + \frac{\sin x}{\cos x}}{\frac{\sin 3x}{\cos 3x} - \frac{\sin x}{\cos x}}$$

$$= \frac{\frac{\sin 3x \cos x + \cos 3x \sin x}{\cos 3x \cos x}}{\frac{\sin 3x \cos x - \cos 3x \sin x}{\cos 3x \cos x}}$$

$$= \frac{\sin 3x \cos x + \cos 3x \sin x}{\sin 3x \cos x - \cos 3x \sin x}$$

$$= \frac{\sin(3x + x)}{\sin(3x - x)}$$

$$= \frac{\sin 4x}{\sin 2x}$$

$$= \frac{2\sin 2x \cos 2x}{\sin 2x}$$

$$= 2\cos 2x$$

= RHS

[Turn over

(b) Hence, find the exact value of $\tan 15°$ in the form $a+b\sqrt{c}$, where a, b and c are constants. [4]

$$\frac{\tan 3(15°)+\tan 15°}{\tan 3(15°)-\tan 15°}=2\cos 2(15°)$$

$$\frac{\tan(45°)+\tan 15°}{\tan(45°)-\tan 15°}=2\cos(30°)$$

$$\frac{1+\tan 15°}{1-\tan 15°}=2\left(\frac{\sqrt{3}}{2}\right)$$

$$1+\tan 15°=\sqrt{3}(1-\tan 15°)$$

$$1+\tan 15°=\sqrt{3}-\sqrt{3}\tan 15°$$

$$\sqrt{3}\tan 15°+\tan 15°=\sqrt{3}-1$$

$$\tan 15°\left(\sqrt{3}+1\right)=\sqrt{3}-1$$

$$\tan 15°=\frac{\sqrt{3}-1}{\sqrt{3}+1}$$

$$=\frac{\sqrt{3}-1\times\left(\sqrt{3}-1\right)}{\sqrt{3}+1\times\left(\sqrt{3}-1\right)}$$

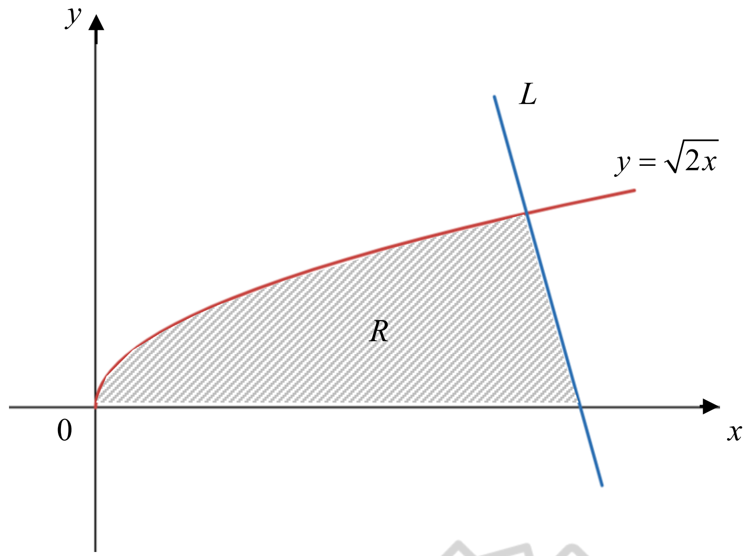
$$=\frac{3-\sqrt{3}-\sqrt{3}+1}{2}$$

$$=2-\sqrt{3}$$

[Turn over

9 Line L is the normal to the graph $y=\sqrt{2x}$ at $x=2$.

Find the area of the shaded region R. [8]



$y=\sqrt{2x}$

$\frac{dy}{dx}=\frac{\sqrt{2}}{2}x^{-\frac{1}{2}}$

When $x=2$,

$\frac{dy}{dx}=\frac{\sqrt{2}}{2}\frac{1}{2^{\frac{1}{2}}}=\frac{1}{2}$

Gradient of normal $=-2$

$y=\sqrt{2(2)}=2$

When $y=0$,

$\frac{0-2}{x-2}=-2$

$-2=-2x+4$

$x=3$

[Turn over

Area of shaded region

$$= \int_0^2 \sqrt{2x}\ \mathrm{d}x + \frac{1}{2}(1)(2)$$

$$= \left[\frac{\sqrt{2}x^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^2 + 1$$

$$= \left[\frac{2\sqrt{2}\sqrt{x^3}}{3} \right]_0^2 + 1$$

$$= \left[\frac{2\sqrt{2}\sqrt{2^3}}{3} \right] + 1$$

$$= 3\frac{2}{3} \text{ sq units}$$

[Turn over

10 The velocity v ms^{-1} of a particle t seconds after passing through O is given by $v = 0.3e^{2t} - 4$, for $0 \le t \le 4$.

(a) Find the acceleration of the particle when $t = 4$. [2]

$v = 0.3e^{2t} - 4$

$a = 0.3e^{2t}(2) = 0.6e^{2t}$

When $t = 4$,

$a = 0.6e^{2(4)} = 1790$ ms^{-2} (3 sf)

(b) Find the value of t when the particle is at instantaneous rest. [2]

When $v = 0$,

$0 = 0.3e^{2t} - 4$

$0.3e^{2t} = 4$

$e^{2t} = \frac{4}{0.3}$

$2t = \ln\frac{4}{0.3}$

$t = 1.2951$

$= 1.30$ (3 sf)

(c) Find the distance travelled by the particle in the first 4 seconds. [6]

Distance travelled

$$=\left|\int_0^{1.2951} 0.3e^{2t}-4\ \mathrm{d}t\right|+\int_{1.2951}^{4} 0.3e^{2t}-4\ \mathrm{d}t$$

$$=\left|\left[\frac{0.3e^{2t}}{2}-4t\right]_0^{1.2951}\right|+\left[\frac{0.3e^{2t}}{2}-4t\right]_{1.2951}^{4}$$

$$=\left|\left(\frac{0.3e^{2(1.2951)}}{2}-4(1.2951)\right)-\frac{0.3e^{2(0)}}{2}\right|$$

$$+\left(\left(\frac{0.3e^{2(4)}}{2}-4(4)\right)-\left(\frac{0.3e^{2(1.2951)}}{2}-4(1.2951)\right)\right)$$

$$=\left|-3.3305\right|+434.32$$

$$=438\text{ m}\quad(3\text{ sf})$$

ALTERNATIVE METHOD

$$s=\int 0.3e^{2t}-4\ \mathrm{d}t$$
$$=0.15e^{2t}-4t+c$$

When $t=0,\ s=0,$

$0=0.15+c$

$c=-0.15$

$s=0.15e^{2t}-4t-0.15$

When $t=0,\ s=0$

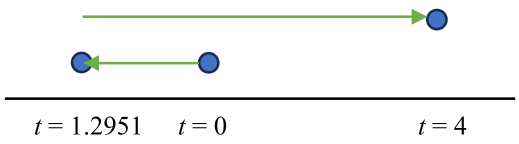
When $t=1.2951,$

$s=0.15e^{2(1.2951)}-4(1.2951)-0.15=-3.3305$

When $t=4,$

$s=0.15e^{2(4)}-4(4)-0.15=430.99$

[Turn over



Distance travelled

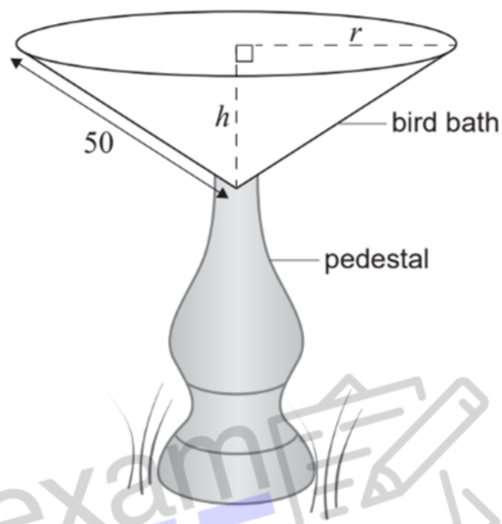
$= 2(3.3305) + 430.99$

$= 438$ m (3 sf)

11 [The volume of cone of height h and base radius r is $\frac{1}{3}\pi r^2 h$]

Joe designs a concrete bird bath. The bird bath is supported by a pedestal.

This is shown in the diagram.



The interior of the bird bath is in the shape of a cone with radius r, height h and a constant slant height of 50 cm.

Let V be the volume of the bird bath.

(a) Show that $V = \dfrac{2500\pi h}{3} - \dfrac{\pi h^3}{3}$. [3]

$$h^2 + r^2 = 50^2$$
$$r^2 = 2500 - h^2$$

$$V = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3}\pi\left(2500 - h^2\right)h$$
$$= \frac{2500\pi h}{3} - \frac{\pi h^3}{3}$$

[Turn over

Joe wants the bird bath to have maximum volume.

(b) Find the value of h for which V is a maximum. [4]

$$V = \frac{2500\pi h}{3} - \frac{\pi h^3}{3}$$

$$\frac{\mathrm{d}V}{\mathrm{d}h} = \frac{2500\pi}{3} - \pi h^2$$

When $\frac{dV}{dh} = 0$,

$$\frac{2500\pi}{3} - \pi h^2 = 0$$

$$h^2 = \frac{2500}{3}$$

$$h = \sqrt{\frac{2500}{3}}$$

$$h = 28.868 = 28.9 \quad (\text{3 sf})$$

(c) Find the maximum volume of the bird bath, justifying that this value is a maximum. [3]

$$V = \frac{2500\pi(28.868)}{3} - \frac{\pi(28.868)^3}{3}$$
$$= 50383.3$$
$$= 50400 \quad (\text{3 sf})$$

$$\frac{\mathrm{d}^2V}{\mathrm{d}h^2} = -2\pi h$$
$$= -2\pi(28.868)$$
$$= -181 < 0$$

Therefore volume is maximum.

[Turn over

ALTERNATIVE METHOD

$$V = \frac{2500\pi(28.868)}{3} - \frac{\pi(28.868)^3}{3}$$
$$= 50383.3$$
$$= 50400 \quad (\text{3 sf})$$

$$\frac{dV}{dh} = \frac{2500\pi}{3} - \pi h^2$$

x	28.868^-	28.868	28.868^+
$\frac{dV}{dh}$	+ve	0	-ve
	/	—	\

Therefore volume is maximum.