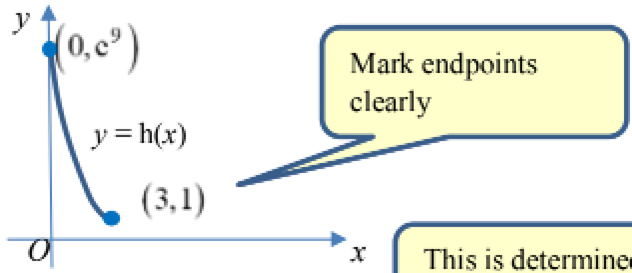
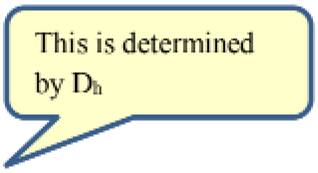
2022 C1 Block Test Revision Package Solutions
Chapter 4 Functions

1(i) **AJC12/C1BT/Q9(a)**

$\text{h} : x \mapsto \text{e}^{(3-x)^2}, \quad 0 \le x \le 3.$



Let $y = \text{h}(x) = \text{e}^{(3-x)^2}$



$x = 3 \pm \sqrt{\ln y}$

Since $0 \le x \le 3$, $x = 3 - \sqrt{\ln y}$

Domain of h^{-1} = Range of h = $[1, \text{e}^9]$

$\text{h}^{-1}(x) = 3 - \sqrt{\ln x}, \qquad 1 \le x \le \text{e}^9$

1(ii) $\text{h}^{-1}\text{h}(x) = \text{h}\,\text{h}^{-1}(x) = x$

Thus for $\text{h}^{-1}\text{h}(x) = \text{h}\,\text{h}^{-1}(x)$, $\text{D}_\text{h} = \text{D}_{\text{h}^{-1}} = \text{R}_\text{h}$

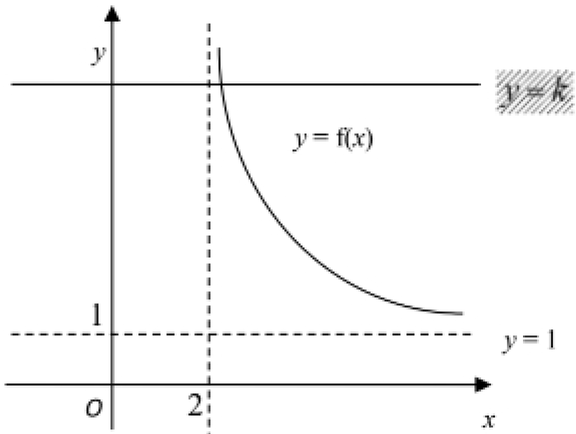
Since $\text{D}_\text{h} = [0,3]$ and $\text{R}_\text{h} = [1, e^9]$ i.e. $x \in [0,3] \cap [1, \text{e}^9] = [1,3]$

$\therefore 1 \le x \le 3$

2(i) **DHS10/ C1BT/Q6**

Any horizontal line $y = k$, $k \in \mathbb{R}$ cuts the graph of $y = \text{f}(x)$ at most once.

$\therefore$ f is one-one, hence f^{-1} exists.



Let $y = \dfrac{1-x}{2-x} \Rightarrow 2y - xy = 1 - x$ i.e. $x = \dfrac{2y-1}{y-1}$

$\therefore \text{f}^{-1} : x \mapsto \dfrac{2x-1}{x-1}, \quad x > 1 \qquad (\text{D}_{\text{f}^{-1}} = \text{R}_\text{f} = (1, \infty])$

2(ii)

Use $\dfrac{1-x}{2-x}=x$ or $\dfrac{2x-1}{x-1}=x$ or $\dfrac{1-x}{2-x}=\dfrac{2x-1}{x-1}$

$x^2-3x+1=0$

$x=\dfrac{3\pm\sqrt{5}}{2}$

$=\dfrac{3+\sqrt{5}}{2}$ (rej $\dfrac{3-\sqrt{5}}{2}$ as $x>1$)

2(iii)

Since $R_f=(1,\infty)\subseteq[1,\infty)=D_g \Rightarrow$ gf exists.

$(\mathrm{gf})^{-1}(x)=3$

$x=\mathrm{gf}(3)$

$=\mathrm{g}(2)$

$=2\ln 2$

The alternative method of finding $(\mathrm{gf})^{-1}(x)$ is too complex

3(a)

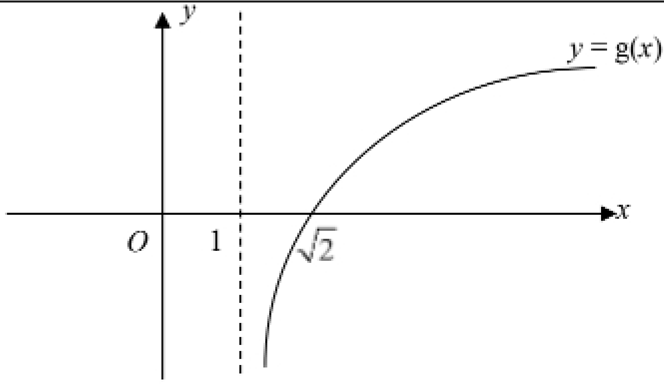
NJC11/C1BT/Q11(b)&(c)

Let $\mathrm{h}^{-1}\left(-\dfrac{1}{2}\right)=m$

Then $\mathrm{h}(m)=-\dfrac{1}{2}$

$\mathrm{h}(m)=m\left(m^2-m-1\right)=-\dfrac{1}{2}$

Using G.C. $m^3-m^2-m+\dfrac{1}{2}=0$,



$m=0.403$ or 1.45 (rejected $\because -\dfrac{1}{3}<x<1$) or -0.855(rejected $\because -\dfrac{1}{3}<x<1$)

$m=0.403$ (to 3 s.f.)

3(b)

$R_h=\left(-1,\tfrac{5}{27}\right)$

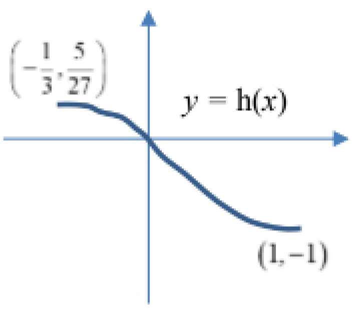
$D_g=\left(-2,\infty\right)$

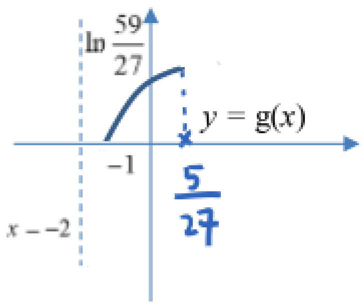
Since $R_h\subseteq D_g$, composite function gh exists.

$\mathrm{gh}(x)=\mathrm{g}\left(x\left(x^2-x-1\right)\right)$

$=\ln\left(x^3-x^2-x+2\right)$

$D_{gh}=D_h=\left(-\tfrac{1}{3},1\right)$





Range of gh $=\left(0,\ln\tfrac{59}{27}\right)$

4(i)	**VJC11/C1BT/Q7** Let $y = \mathrm{h}(x) = ax + b$ $x = \dfrac{y-b}{a}$ $\therefore \mathrm{h}^{-1}(x) = \dfrac{x-b}{a},\ x \in \mathbb{R}$ Since h^{-1} and g meet on the y axis
4(ii)	$\mathrm{h}^{-1}(2) = \mathrm{g}(2) \quad \Rightarrow \quad \dfrac{2-b}{a} = 3^2$ $\Rightarrow \quad 2 - b = 9a \quad \cdots\cdots\cdots(1)$ $\mathrm{h}^{-1}(0) = \mathrm{g}(0)$ $\Rightarrow \quad -\dfrac{b}{a} = 3^0$ $\Rightarrow -b = a \quad \cdots\cdots\cdots\cdots\cdots(2)$ Solving (1) & (2), $\quad a = \dfrac{1}{4} \quad$ & $\quad b = -\dfrac{1}{4}$
4(iii)	Let $(\mathrm{gh})^{-1}(3) = k$ $\Rightarrow \mathrm{gh}(k) = 3 \quad \Rightarrow 3^{ak+b} = 3$ $\Rightarrow ak + b = 1 \quad \Rightarrow k = \dfrac{1-b}{a} = \dfrac{1+\frac{1}{4}}{\frac{1}{4}} = 5$
5(i)	**DHS11/C1BT/Q8** Let $y = -x^3 + 1$. $x^3 = 1 - y$ $x = (1-y)^{\frac{1}{3}}$ $\mathrm{D}_{\mathrm{f}^{-1}} = \mathrm{R}_{\mathrm{f}} = \mathbb{R}$ $\therefore \mathrm{f}^{-1} : x \mapsto (1-x)^{\frac{1}{3}},\ x \in \mathbb{R}$
5(ii)	$\mathrm{fg}^{-1}(x) + 7 = 0$ $\mathrm{fg}^{-1}(x) = -7$ $\mathrm{f}\left(\mathrm{g}^{-1}(x)\right) = -7$ $\mathrm{f}^{-1}\left[\mathrm{f}\left(\mathrm{g}^{-1}(x)\right)\right] = \mathrm{f}^{-1}(-7)$ $\mathrm{g}^{-1}(x) = \left(1-(-7)\right)^{\frac{1}{3}} = 2$ $2 = \mathrm{g}^{-1}(x)$ $x = \mathrm{g}(2) = \mathrm{e}^4 - 2$ Without finding g^{-1} Student will need to take f^{-1} on both sides Note: $\mathrm{f}^{-1}\left[\mathrm{f}(x)\right] = x$

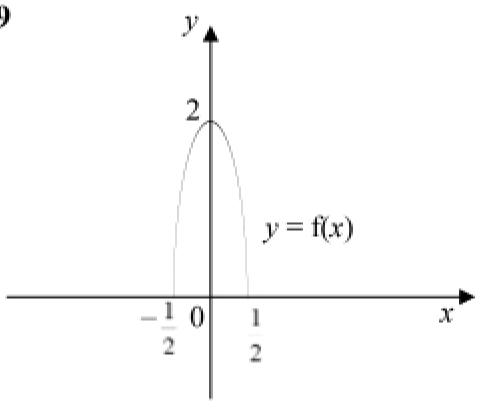
6(ia) **TJC10/C1BT/Q9**

For $-\frac{1}{2} \le x \le \frac{1}{2}$,

$y = 2\sqrt{1-4x^2}$

$y^2 = 4\left(1-4x^2\right)$

$\frac{y^2}{4} + \frac{x^2}{\frac{1}{4}} = 1$



Note that $y = 2\sqrt{1-4x^2}$ is the upper half of the ellipse

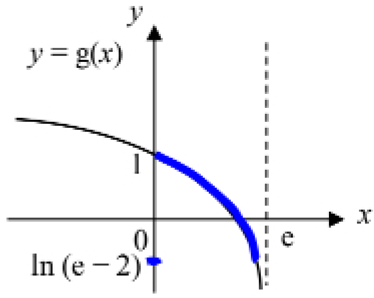
6(ib)

$R_f = [0, 2]$

$D_g = (-\infty, e)$

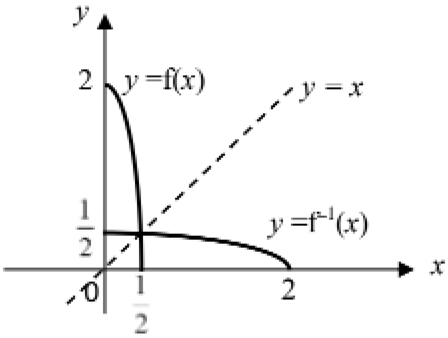
Since $R_f \subseteq D_g$ $\therefore$ gf exists

Range of gf = [ln(e – 2), 1]



6(ii)

For f^{-1} exists, f has to be one-one. Minimum value of $c = 0$.



- Use equal scales on both axes
- Label the graphs (f must be 1−1 for f^{-1} to exist)
- Show end points
- Show symmetry about line $y=x$
- Show intersection on the line $y=x$ (if any)

7(a) **AJC11/C1BT/Q11**

$$h^2(x) = a + \frac{1}{a + \frac{1}{x-a} - a} = a + x - a = x$$

Must simplify!

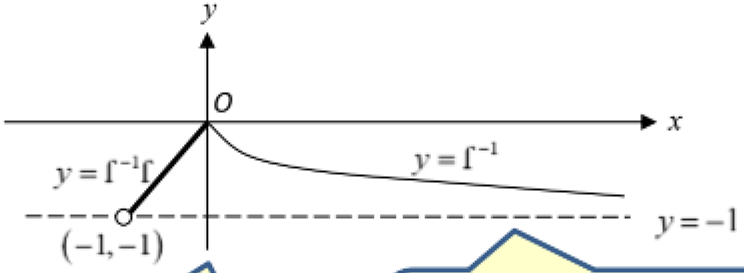
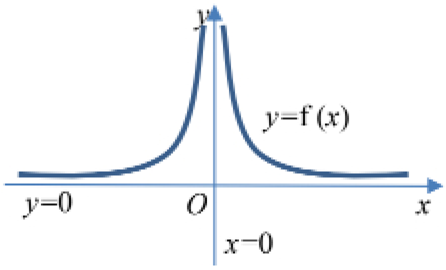
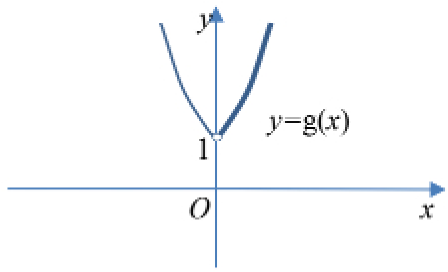
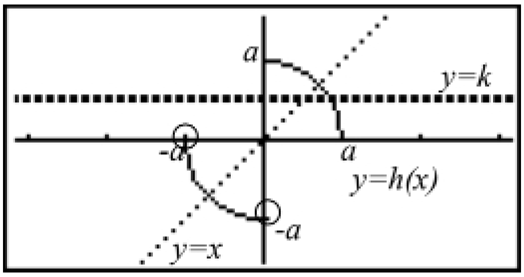
$$h^3(x) = h(x) = a + \frac{1}{x-a}$$

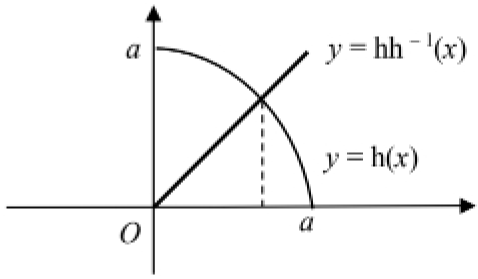
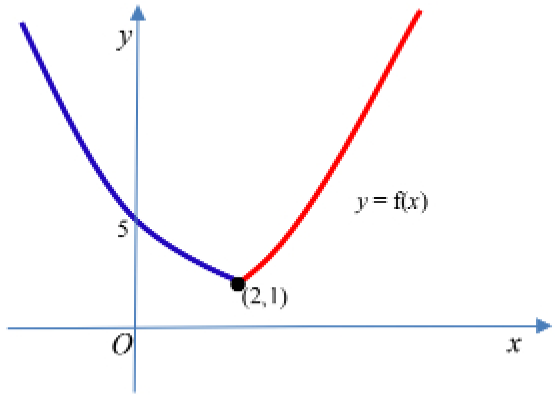
$$h^4(x) = x$$

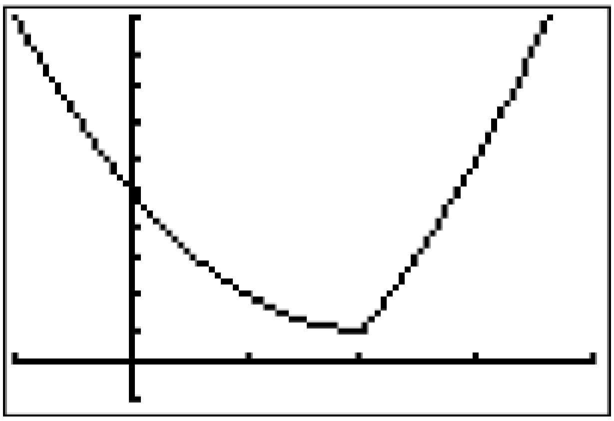
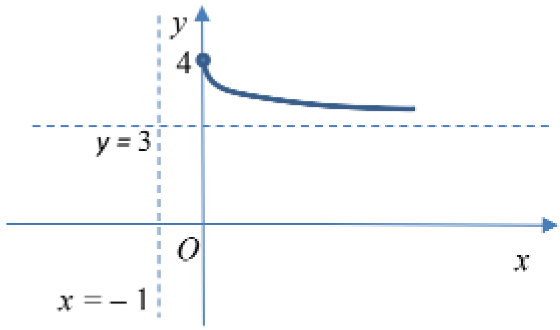
$$h^7(x) = h(x) = a + \frac{1}{x-a}$$

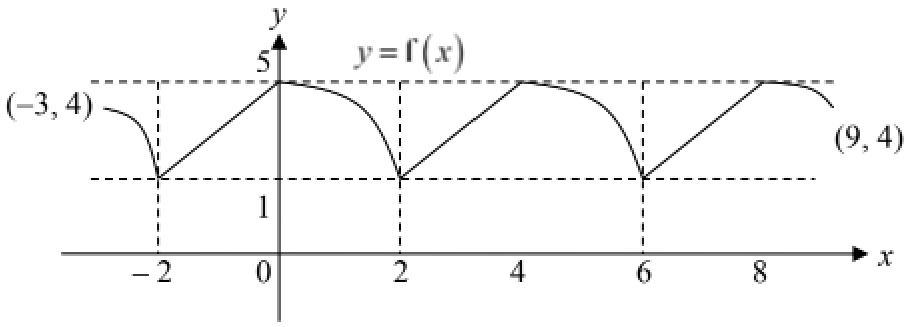
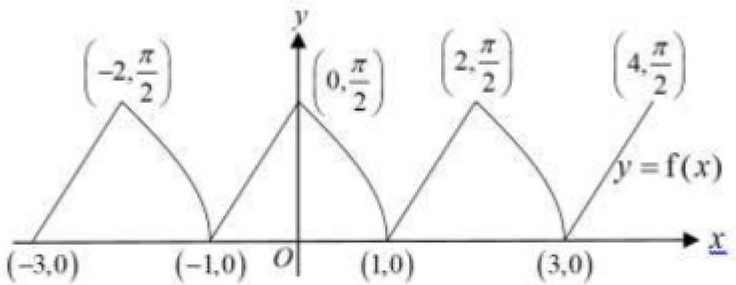
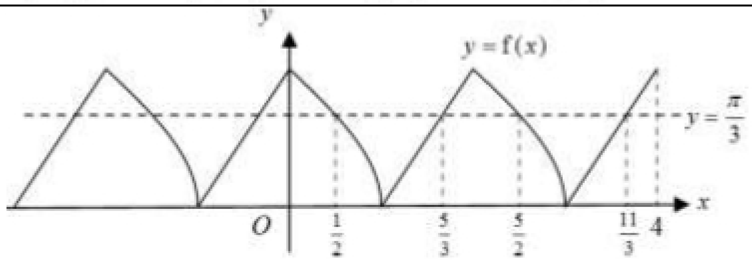
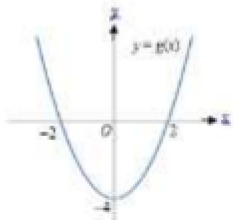
For this domain, range remains unchanged at $[0, \infty)$

7(bi) domain of f = $(-1, 0]$

7(bii)	 Note that $f^{-1}f(x) = x$ Note that the graph of f^{-1} is a reflection of the graph of f in the line $y = x$
8(a)	**AJC10/C1BT/Q12** $f(x) = \frac{1}{x^2}, x \in \mathbb{R}, x \neq 0$ and $g(x) = e^{\lvert x \rvert}, x \in \mathbb{R}$  $R_f = (0, \infty)$ $D_g = \mathbb{R}$, Since $R_f \subseteq D_g$ $\therefore$ gf exists.  $gf(x) = g\left[\frac{1}{x^2}\right] = e^{\left\lvert \frac{1}{x^2} \right\rvert} = e^{\frac{1}{x^2}}$ $\therefore gf(x) = e^{\frac{1}{x^2}}, x \in \mathbb{R}, x \neq 0$ From the graph, $R_{gf} = (1, \infty)$
8(bi)	 Since every horizontal line $y = k$ cuts the graph of $y = h(x)$ at most once, then h is one-one and thus, h^{-1} exists. Graph of $y = h(x)$ is symmetric about the line $y = x$. $\Rightarrow$ Reflection of h (i.e. h^{-1}) in the line $y = x$ is the same graph (as h) $\Rightarrow h^{-1} = h$
8(bii)	Since $h^{-1} = h$, $h^2(x) = h^{-1}h(x) = x$. $h^5\left(-\frac{a}{2}\right) = h^4\left[h\left(-\frac{a}{2}\right)\right]$ $= h\left(-\frac{a}{2}\right)$

	$= -\sqrt{a^2 - \left(-\frac{a}{2}\right)^2} = -\sqrt{\frac{3}{4}a^2}$ $= -\frac{\sqrt{3}}{2}a$
8(biii)	At point of intersection: $x = \sqrt{a^2 - x^2}$ $x^2 = a^2 - x^2$ $2x^2 = a^2$ $x = \frac{a}{\sqrt{2}}$ 
9(i)	**VJC10/C1BT/Q11** $\mathrm{f}(3x) = 9x^2 + 1$
9(ii)	$\mathrm{fg}(x) = \mathrm{f}(x-3) = (x-3)^2 + 1$ $\mathrm{gf}(x) = g(x^2+1) = x^2 - 2$ $\lvert \mathrm{fg}(x) - \mathrm{gf}(x) \rvert \le \sqrt{3}x + 4$ $\lvert x^2 - 6x + 9 + 1 - x^2 + 2 \rvert \le \sqrt{3}x + 4$ $\lvert -6x + 12 \rvert \le \sqrt{3}x + 4$ $-\sqrt{3}x - 4 \le -6x + 12 \le \sqrt{3}x + 4$ $(6-\sqrt{3})x \le 16$ and $(6+\sqrt{3})x \ge 8$ $x \le \frac{16}{6-\sqrt{3}}$ and $x \ge \frac{8}{6+\sqrt{3}}$ $\therefore \frac{8}{6+\sqrt{3}} \le x \le \frac{16}{6-\sqrt{3}}$ 
9(iii)	$\frac{5x-8}{x-1} = x$ $5x - 8 = x^2 - x$ $x^2 - 6x + 8 = 0$ $(x-2)(x-4) = 0$ Since $x > 3$, $x = 4$
9(iv)	$y = \frac{5x-8}{x-1}$ $xy - y = 5x - 8$ $x = \frac{y-8}{y-5}$ $\therefore \mathrm{h}^{-1}(x) = \frac{x-8}{x-5}, \ \frac{7}{2} < x < 5$ Remember to include domain

10(i)	**RI11/C1BTP1/Q9** $\mathrm{f}(x)=\begin{cases}(x-2)^2+1 & \text{for } x\in\mathbb{R},\ x\le 2,\\ x^2-3 & \text{for } x\in\mathbb{R},\ x>2.\end{cases}$ 
10(ii)	$\mathrm{g}(x)=\dfrac{3x+4}{x+1}$ for $x\in\mathbb{R}, x\ge 0$. $\mathrm{R_g}=(3,4]$ $\mathrm{fg}(x)=\left(\dfrac{3x+4}{x+1}\right)^2-3$ $\mathrm{fg}(x)=\dfrac{6x^2+18x+13}{x^2+2x+1}$ $\mathrm{fg}(x)=\dfrac{6x^2+18x+13}{(x+1)^2}$ 
10(iii)	Horizontal asymptote is $y=6$ Note: Since $x\ge 0$, there is no vertical asymptote.
10(iv)	Largest value of k is 2.
10(v)	$y=(x-2)^2+1$ $x=2\pm\sqrt{y-1}$ Since $x\le 2 \Rightarrow x=2-\sqrt{y-1}$ i.e. $\mathrm{h}^{-1}(x)=2-\sqrt{x-1}$ $\mathrm{h}^{-1}: x\to 2-\sqrt{x-1},\quad x\in\mathbb{R}, x\ge 1.$
11(i)	**TPJC15/C1BT/Q1** $\mathrm{f}(x)=\begin{cases}5-x^2 & \text{for } 0<x\le 2\ ,\\ 2x-3 & \text{for } 2<x\le 4\ .\end{cases}$ $\mathrm{f}(5)+\mathrm{f}(2015)=\mathrm{f}(1)+\mathrm{f}(3)$ $=(5-1^2)+(2(3)-3)$ $=7$

11(ii)	
12(i)	**VJC15/C1BT/Q2** 
12(ii)	 For $0 \le x < 1,\ \cos^{-1} x = \frac{\pi}{3} \Rightarrow x = \frac{1}{2}$ For $1 \le x < 2,,\ \frac{\pi}{2}x - \frac{\pi}{2} = \frac{\pi}{3} \Rightarrow x = \frac{5}{3}$ From the graph, $0 \le x < \frac{1}{2}$ or $\frac{5}{3} < x < \frac{5}{2}$ or $\frac{11}{3} < x \le 4$.
13(i)	**CJC16/C1BT/Q11** $\mathrm{R_g} = [-4, \infty)$ $\mathrm{D_f} = (-5, \infty)$ Since $\mathrm{R_g} \subseteq \mathrm{D_f}$, fg exists.  $\mathrm{fg}(x) = \mathrm{f}(x^2-4)$ $= \ln\left[(x^2-4)^2 + 2(x^2-4) + 5\right]$ $= \ln(x^4 - 8x^2 + 16 + 2x^2 - 8 + 5)$ $= \ln(x^4 - 6x^2 + 13)$

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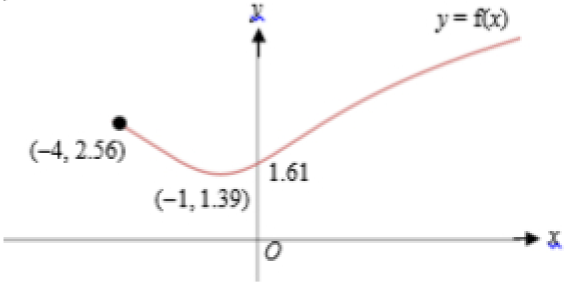
$D_{fg} = D_g = (-\infty, \infty)$
To find range of fg:
Mapping $R_g = [-4, \infty)$ as domain for f


Consider $x \geq -4$ in the graph of $y = f(x)$, $R_{fg} = [1.39, \infty)$

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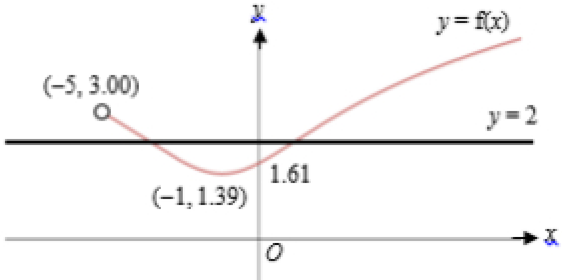
<tr><td>13(ii)</td><td>



Since the horizontal line $y = 2$ cuts the graph twice, f is not a one-one function, thus the inverse does not exist.

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<tr><td>13(iii)</td><td>

Least value of $k = -1$.

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<tr><td>13(iv)</td><td>

Let $y = f(x) \Rightarrow f^{-1}(y) = x$
$y = \ln(x^2 + 2x + 5)$
$e^y = x^2 + 2x + 5$
$e^y = (x+1)^2 + 4$
$(x+1)^2 = e^y - 4$
$x + 1 = \pm\sqrt{e^y - 4}$
$x = -1 + \sqrt{e^y - 4}$ or $-1 - \sqrt{e^y - 4}$ (rej. $\because x > -1$)
$f^{-1}(x) = -1 + \sqrt{e^x - 4}$
$D_{f^{-1}} = R_f = (1.39, \infty)$

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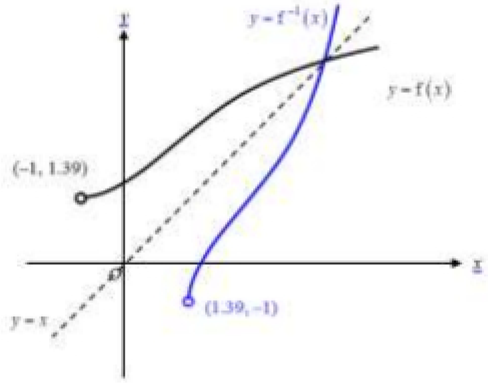
13(v)



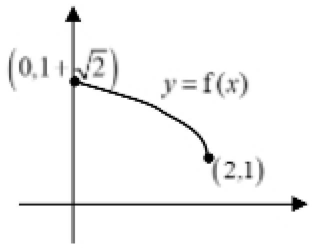
14(i) **TJC16/C1BT/Q11**

Let $y=\mathrm{f}(x)=1+\sqrt{2-x}$, $\quad 0\le x\le 2$

$\Rightarrow y-1=\sqrt{2-x}$

$\Rightarrow 2-x=(y-1)^2$

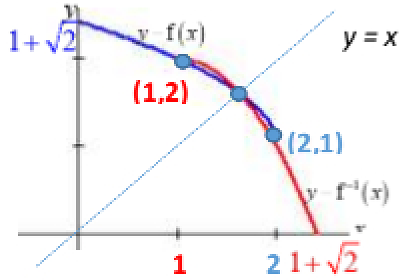
$\Rightarrow x=2-(y-1)^2$



Therefore, $\mathrm{f}^{-1}(x)=2-(x-1)^2$

$D_{f^{-1}}=R_f=\left[1,1+\sqrt{2}\right]$

14(ii)



For $\mathrm{f}(x)-\mathrm{f}^{-1}(x)=0\Rightarrow \mathrm{f}(x)=\mathrm{f}^{-1}(x)$

Consider the graph of $y=\mathrm{f}(x)$ and the graph of $y=\mathrm{f}^{-1}(x)$ on the same diagram.

From diagram, there are <u>three intersections</u>, hence $\mathrm{f}(x)-\mathrm{f}^{-1}(x)=0$ has 3 real roots.

Consider $\mathrm{f}^{-1}(x)=x$ for one of the intersection:

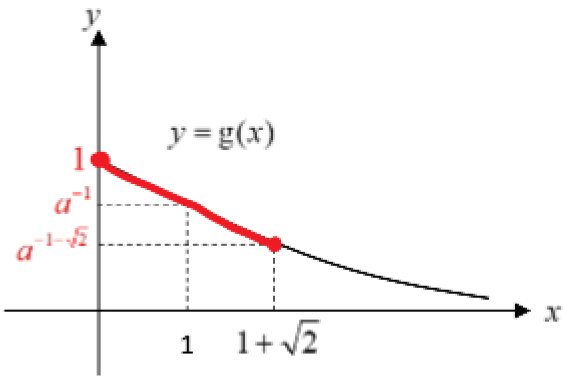
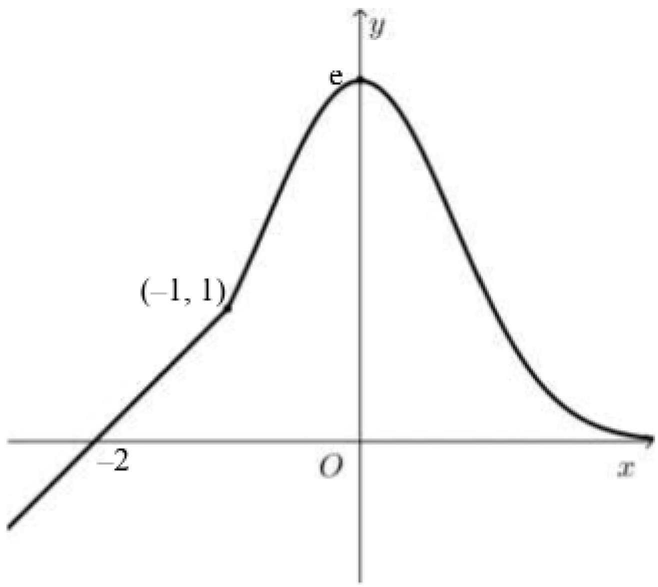
$\mathrm{f}^{-1}(x)=x\Rightarrow 2-(x-1)^2=x$

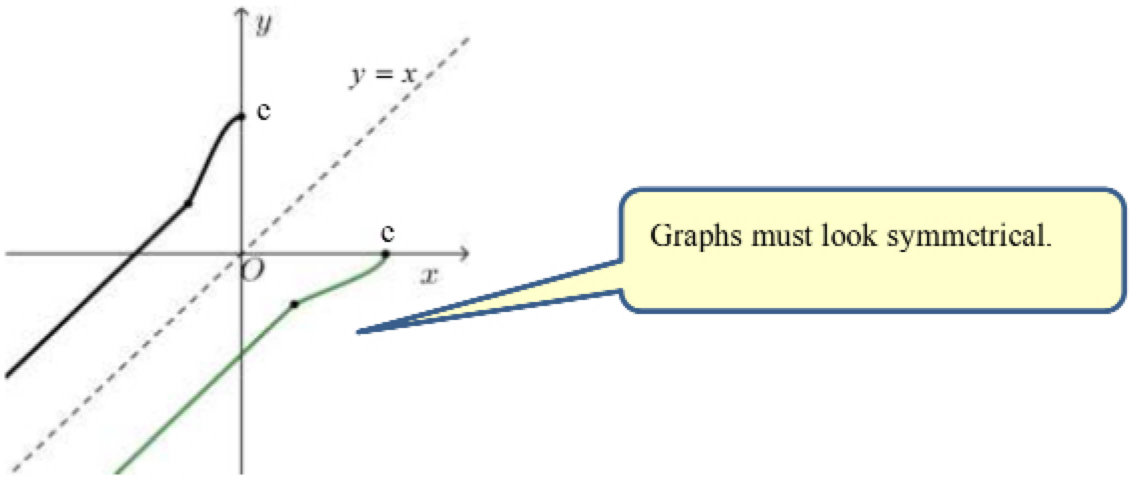
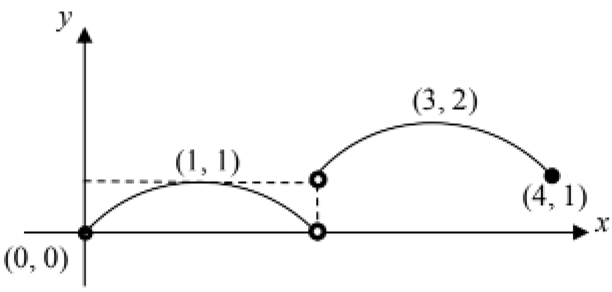
$\Rightarrow x^2-x-1=0$

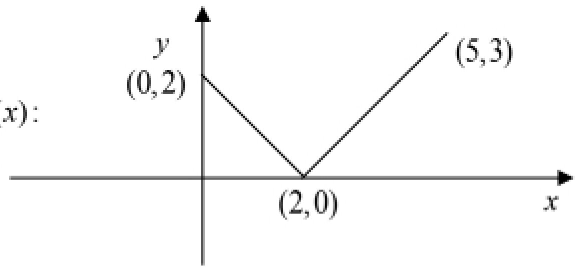
$\Rightarrow x=\dfrac{1+\sqrt{5}}{2}$ or $x=\dfrac{1-\sqrt{5}}{2}$

Since $x\in\left[1,1+\sqrt{2}\right]$, $\therefore x=\dfrac{1+\sqrt{5}}{2}$ (reject $x=\dfrac{1-\sqrt{5}}{2}$)

The roots of the equation are $x=1$, $x=\dfrac{1+\sqrt{5}}{2}$ or $x=2$

14(iii)	$R_{\mathrm{f}}=\left[1,1+\sqrt{2}\right]$ $D_{\mathrm{g}}=[0,\infty)$ Since $R_{\mathrm{f}}\subset D_{\mathrm{g}}$, the composite function gf exists. (Shown) $\mathrm{gf}(x)=\mathrm{g}\left(1+\sqrt{2-x}\right)=a^{-1-\sqrt{2-x}}$ $D_{\mathrm{gf}}=D_{\mathrm{f}}=[0,2]$ Thus, $\mathrm{gf}:x\rightarrow a^{-1-\sqrt{2-x}},x\in\mathbb{R},\ 0\le x\le 2$
14(iv)	$[0,2]\xrightarrow{\mathrm{f}}\left[1,1+\sqrt{2}\right]\xrightarrow{\mathrm{g}}\left[a^{-1-\sqrt{2}},a^{-1}\right]$ Thus, $\mathrm{R}_{\mathrm{gf}}=\left[a^{-1-\sqrt{2}},a^{-1}\right]$ 
15(i)	**NJC18/C1BT/Q5**  From the graph, $\mathrm{R}_{\mathrm{f}}=(-\infty,\mathrm{e}]$
15(ii)	The line $y=1$ cuts the graph of f twice so f is not a one-one function and its inverse does not exist.
15(iii)	$k=0$
15(iv)	Let $y=x+2$ for $x<-1$, the range for this piece is $(-\infty,1)$. $x=y-2$

	$\mathrm{g}^{-1}(x)=x-2$ for $x<1$. Let $y=\mathrm{e}^{1-x^2}$ for $-1\le x\le 0$, the range for this piece is $[1,\mathrm{e}]$. $1-x^2=\ln y$ $x^2=1-\ln y$ $x=\pm\sqrt{1-\ln y}$ (reject positive as $-1\le x\le 0$) $\mathrm{g}^{-1}(x)=-\sqrt{1-\ln x}$ for $1\le x\le \mathrm{e}$. $\mathrm{g}:x\mapsto\begin{cases} x-2 & x<1, \\ -\sqrt{1-\ln x} & 1\le x\le \mathrm{e}. \end{cases}$
15(v)	
16(i)	**VJC20/C1BT/Q5** $\mathrm{f}(2)=0,\ \mathrm{f}(4)=\mathrm{f}(2)+1=1$
16(ii)	

16(iii)	$y = \mathrm{g}(x)$:  $\mathrm{R_g} = [0,3)$ $\mathrm{D_f} = (0,4]$ Since $0 \in \mathrm{R}_g$ but $0 \notin \mathrm{D}_f$, hence $\mathrm{R}_g \not\subseteq \mathrm{D}_f$, fg does not exist.