ANDERSON SERANGOON JUNIOR COLLEGE

H2 MATHEMATICS **9758**

JC2 Prelim Paper 1 (100 marks) **9 Sept 2024**

3 hours

Additional Material(s): List of Formulae (MF 26)

CANDIDATE NAME

CLASS /

READ THESE INSTRUCTIONS FIRST

Write your name and class in the boxes above.
Please write clearly and use capital letters.
Write in dark blue or black pen. HB pencil may be used for graphs and diagrams only.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions and write your answers in this booklet.
Do not tear out any part of this booklet.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

All work must be handed in at the end of the examination. If you have used any additional paper, please insert them inside this booklet.
The number of marks is given in brackets [ ] at the end of each question or part question.

Question number	Marks
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
Total	

This document consists of 23 printed pages and 5 blank pages.

[Turn Over

1 **(a)** Sketch the graphs of $y = 3e^x$ and $y = x + 3$ on the same diagram. Indicate clearly the coordinates of the points of intersection between the 2 graphs.

Solve the inequality $3e^x > x + 3$. [3]

(b) Hence find $\int_{-2}^{2} \left|3e^x - x - 3\right| \mathrm{d}x$, giving your answer in an exact form. [2]

2 **(i)** Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(e^{\sin^{-1}x}\sqrt{1-x^2}\right)$. [1]

(ii) Hence using integration by parts, find $\int x \dfrac{e^{\sin^{-1}x}}{\sqrt{1-x^2}} \mathrm{d}x$. [3]

3 The curve C has parametric equations

$$x = 2t - \frac{1}{t^2}, \quad y = 2t + \frac{1}{t}, \quad t \in \mathbb{R}, t \neq 0.$$

The point P on the curve has parameter $t = 1$.

(i) Find the equation of tangent and normal to C at the point P. [4]

(ii) The tangent at P meets the y-axis at B. The normal at P meets the x-axis at A. If O is the origin, find the area of the quadrilateral $OAPB$. [2]

4 A sequence is such that $u_0 = 2$ and $u_n = u_{n-1} + n^3 + \left(\frac{1}{2}\right)^n$ for $n \geq 1$.

(a) It is given that $\sum_{r=1}^{n} r^3 = \dfrac{n^2(n+1)^2}{4}$. By considering $\sum_{r=1}^{n}(u_r - u_{r-1})$, find a formula for u_n in terms of n. [4]

(b) Hence, using the formula of u_n found in **(a)**, find $\sum_{r=9}^{n}\left((r+2)^3 + \left(\frac{1}{2}\right)^{r+2}\right)$ exactly. [3]

5 **(a)** It is given that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are non-zero vectors.

If $|\mathbf{a}+\mathbf{b}| = |\mathbf{a}-\mathbf{b}|$, show that the two vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular to each other. [4]

(b) (i) Explain why the result of

$$\mathbf{a}\times(\mathbf{b}+\mathbf{c})+\mathbf{b}\times(\mathbf{c}+\mathbf{a})+\mathbf{c}\times(\mathbf{a}+\mathbf{b})$$

is a vector. [1]

(ii) Simplify $\mathbf{a}\times(\mathbf{b}+\mathbf{c})+\mathbf{b}\times(\mathbf{c}+\mathbf{a})+\mathbf{c}\times(\mathbf{a}+\mathbf{b})$. Show your workings clearly. [3]

6 **(i)** The variables *x* and *y* are related by

$$(x+y)\frac{\mathrm{d}y}{\mathrm{d}x}+ky=2 \text{ and } y=1 \text{ at } x=0,$$

where *k* is a constant. Show that $(x+y)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+(1+k)\frac{\mathrm{d}y}{\mathrm{d}x}+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=0$. [1]

(ii) Given that *x* is small, find the series expansion of $\mathrm{g}(x)=\dfrac{1}{\sin^2\left(2x+\frac{\pi}{2}\right)}$ in ascending powers of *x*, up to and including the term in x^2.

If the coefficient of x^2 in the expansion of $\mathrm{g}(x)$ is equal to twice the coefficient of x^2 in the Maclaurin series for *y* in **(i)**, find the value of *k*. [5]

(iii) By further differentiation of the result found in **(i)**, and taking $k=1$, find the Maclaurin series for *y*, up to and including the term in x^3. [3]

7 **(a)** State a sequence of transformations that will transform the curve with equation $y^2-x^2=1$ on to the curve with equation

$9y^2-54y-x^2-2x+79=0$. [4]

(b) A curve *C* has equation

$$9y^2-54y-x^2-2x+79=0.$$

(i) For real values *x*, use a non-graphical method to determine that *y* cannot lie between *a* and *b*, where *a* and *b* are exact real constants to be determined. [3]

(ii) Sketch the curve *C*, indicating clearly the equations of all asymptotes and the coordinates of the turning points. [3]

(iii) By adding a suitable curve, determine the number of real roots of the equation,

$$9\left[(x+1)^2+3\right]^2-54\left[(x+1)^2+3\right]-x^2-2x+79=0.$$ [2]

[Turn Over

8 The functions f and g are defined by

$$\mathrm{f}: x \mapsto \left|4+2x-x^2\right|,\ x \in \mathbb{R},\ x \geq 3.5,$$

$$\mathrm{g}: x \mapsto 4+\mathrm{e}^{ax},\ x \in \mathbb{R},\ x \geq -1,$$

where $a > 0$.

(a) Find $\mathrm{f}^{-1}(x)$ and state its domain. [3]

(b) Find the value of x for which $\mathrm{f}^{-1}(x) = \mathrm{f}(x)$. [2]

(c) Show that the composite function fg exists and express the exact range of fg in the form of $A + B\mathrm{e}^{-a} + C\mathrm{e}^{-2a}$, where A, B and C are real constants. [4]

(d) Without the use of a graphing calculator, solve the inequality $\dfrac{\mathrm{g}(x)}{x^2-2x-2} \geq 0$.

Leave your answer in exact form. [3]

9 **(a)** The complex numbers z_1 and z_2 are given by

$$z_1 = \frac{1+\mathrm{i}}{1-\mathrm{i}} \text{ and } z_2 = \cos\left(\frac{\pi}{4}\right) + \mathrm{i}\sin\left(\frac{\pi}{4}\right).$$

(i) Find $z_1 + z_2$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where r is an exact real constant in trigonometric form such that $r > 0$, and θ is in the form $k\pi$ where k is an exact real constant such that $-1 < k \leq 1$. [3]

(ii) Find also $z_1 + z_2$ in the form $x + \mathrm{i}y$, where x and y are exact real constant.

Hence show that $\tan\left(\dfrac{3\pi}{8}\right) = 1 + \sqrt{2}$. [2]

(b) The complex number w is given by $w = \cos\theta + \mathrm{i}\sin\theta$, where $0 < \theta < \dfrac{\pi}{2}$.

(i) Show that $1 - w^2 = -2\mathrm{i}w\sin\theta$. [2]

(ii) Hence find the modulus and argument of $1 - w^2$ in terms of θ. [2]

(iii) Given that $\left(\dfrac{1-w^2}{\mathrm{i}w^*}\right)^n$ is real and negative and that $\theta = \dfrac{\pi}{5}$, find the three smallest positive integer values of n. [3]

10 A rice retailer pledges to donate a bowl of rice for every kilometre run by participants in a service-learning project. Donations will be made in complete bowls, based on the cumulative distance each individual ran by the end of the 28-day period. Distances ran by multiple individuals will not be combined. For example, if person A runs 18.8 km and person B runs 11.2 km, the retailer will donate a total of 29 bowls. Two such participants, athlete A and B, will each accumulate the distance they run for a total of 28 days via a plan each devised.

- Athlete A plans to run 5 km on the first day and then increase the distance by a fixed 0.65 km more than the previous day.
- Athlete B plans to run 7 km on the first day and then increase the distance by 4% more than the previous day.

(a) Determine the least number of days required for the cumulative distance of athlete A to exceed that of athlete B. [3]

(b) How many bowls of rice will both athletes contribute, in total, at the end of the 28-day period? [3]

(c) Suppose athlete A plans to cover at least 400 km by the end of the 28-day period, what is the minimum distance he should run in day 1 if the plan to increase by 0.65 km more than the previous day remains the same. Give your answer to the nearest metres. [3]

(d) On days where the distance athlete B is supposed to run exceeds 10 km based on his own plan, he will limit it to exactly 10 km instead. Given this change, how many bowls of rice will he contribute at the end of the 28-day period? [3]

11 In a large town, the number of people infected by a particular virus t days after the virus was first discovered is x. It is assumed that the rate of infection is proportional to x. Initially there are 5 people who are infected by the virus, and there are 5120 people who are infected by the virus 30 days after the virus was first discovered.

(i) Show that $x = 5(2)^{\frac{t}{3}}$. [5]

A cure and vaccine for the virus were discovered and administered to the population 30 days after the virus was discovered. Individuals who were cured are not at risk of reinfection. The number of people infected by the virus p days after the cure and vaccine were administered is represented by y. It is believed that the new rate of infection from then on is proportional to $6400y - y^2$.

It is given that 30 days after the cure or vaccine was administered, 3200 people remain infected with the virus.

(ii) Show that $y = \dfrac{6400}{1+2^{\left(\frac{p}{15}+H\right)}}$, where H is a constant to be determined. [6]

(iii) By finding the number of people in the town that will be infected by the virus in the long term, comment on the effectiveness of the cure and vaccine administered. [2]

End of Paper

ANDERSON SERANGOON JUNIOR COLLEGE

H2 MATHEMATICS **9758/2**

JC2 Prelim Paper 2 (100 marks)

16 Sept 2024

3 hours

Additional Material(s): List of Formulae (MF26)

CANDIDATE NAME

CLASS /

READ THESE INSTRUCTIONS FIRST

Write your name and class in the boxes above.
Please write clearly and use capital letters.
Write in dark blue or black pen. HB pencil may be used for graphs and diagrams only.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions and write your answers in this booklet.
Do not tear out any part of this booklet.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

All work must be handed in at the end of the examination. If you have used any additional paper, please insert them inside this booklet.
The number of marks is given in brackets [ ] at the end of each question or part question.

Question number	Marks
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
Total	

This document consists of 22 printed pages and 6 blank pages.

[Turn Over

Section A: Pure Mathematics [40 marks]

1 By using the substitution $x = \cot\theta$, for $0 < \theta < \dfrac{\pi}{4}$, find $\displaystyle\int \frac{1}{x^2\sqrt{1+x^2}}\,\mathrm{d}x.$ [4]

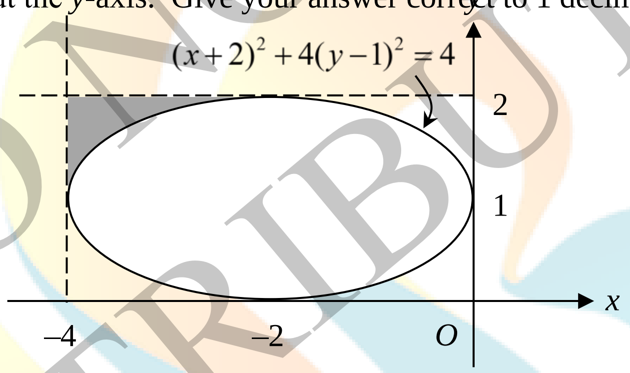
2 **(a)(i)** Find $\displaystyle\int \frac{9u-8}{4+9u^2}\,\mathrm{d}u$. [3]

(ii) The curve C is given by the parametric equations

$$x = u^2 + u + 1,\ y = \frac{9u}{4+9u^2},\ \text{where } u \geq 0.$$

Find the exact area bounded by C, the x-axis and the line $x = 3$. [4]

(b) Find the volume of the solid formed when the shaded region bounded by the lines $x = -4$, $y = 2$ and the ellipse $(x+2)^2 + 4(y-1)^2 = 4$ is rotated through 2π radians about the y-axis. Give your answer correct to 1 decimal place.



[3]

3 With respect to the origin O, the points A, B, C, D and E have coordinates $A(2,3,4)$, $B(6,5,7)$, $C(8,9,6)$, $D(4,7,3)$ and $E(5,6,10)$.

(a) Show that the cartesian equation of the surface containing the points A, B and E is $x - 5y + 2z = -5$. [2]

A line passes through the point D and the midpoint M of the line segment EC.

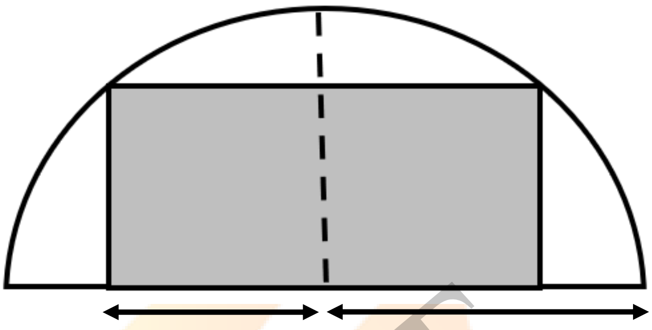
(b) Find the vector equation of the line DM. [3]

(c) Find the exact coordinates of the foot of the perpendicular from the point M to the surface found in part **(a)**. [3]

(d) Hence find the exact shortest distance from the point M to the surface found in part **(a).** [2]

(e) Verify the line DM intersects the surface found in part **(a)** at the point P with coordinates $(9,8,13)$. Hence find the vector equation of the reflection of the line DM about this slant surface. [4]

4 A popular toy company is designing a new water play feature for children. The toy consists of a cylindrical water container that will hold water for various playful activities. This cylindrical container is designed to be inscribed within a fixed, rigid hemispherical shell made of durable plastic of negligible thickness.



The shaded region in the diagram above shows the cross-sectional view of the upright cylindrical container that is inscribed in a hemisphere with fixed radius $2a$ cm.

(a) If the radius of the cylindrical water container is r cm, show that the volume V cm^3 of the water container is given by $V = \pi r^2\sqrt{4a^2 - r^2}$. [1]

The unique feature of this toy is that the height of the cylindrical container is adjustable, allowing it to expand or contract while always touching the inner surface of the hemisphere.

(b) Water is pumped into the container at a rate of 100π cm^{3}s^{-1} while the adjustment is taking place. If a = 50, find the exact rate of change of the radius of the container at the instant when the height of the water container is 80 cm. [5]

(c) Using differentiation, find in terms of a, the value of r which gives a maximum value of V. Justify that this value indeed gives a maximum V. Hence write down the exact maximum volume of the cylinder in terms of a. [4]

(d) Sketch the graph showing the volume of the cylinder as the radius of the cylinder varies. [2]

Section B: Probability and Statistics [60 marks]

5 An amateur music composer is arranging a sequence of four musical notes followed by three beats. There are 7 possible notes (labeled A to G) and 5 possible beats (labeled 1 to 5). The order of the notes and beats is important in the composition. Find the probability that a randomly chosen sequence has

(i) the third beat being a higher number than the second beat, [1]

(ii) exactly two notes the same or exactly two beats the same, but not both. [3]

6 Tetrahedral dice have four faces. Two fair tetrahedral dice, one red and one blue, have faces numbered 0, 1, 2, and 3 respectively. The dice are rolled, and the numbers faced down on the two dice are recorded. The random variable T is defined as the score on the red die multiplied by the score on the blue die.

(i) Find the probability distribution of T. [3]

(ii) Find $\mathrm{E}(T)$ and show that $\mathrm{Var}(T) = \frac{115}{16}$. Show your workings clearly. [2]

(iii) Evaluate $\mathrm{P}\left(|T-2\mu| > \sigma\right)$, where $\mu = \mathrm{E}(T)$ and $\sigma^2 = \mathrm{Var}(T)$. [2]

7 The masses, in grams, of the packets of semolina flour follow the distribution $\mathrm{N}(225, 25^2)$ and the masses, in grams, of the packets of millet flour follow the distribution $\mathrm{N}(\mu, \sigma^2)$.

(a) Find the probability that 4 times the mass of a packet of semolina flour is between 0.85 kilograms and 1.05 kilograms. [2]

(b) Let M be the mean mass of 3 packets of semolina flour and 2 packets of millet flour. Given that $\mathrm{P}(M < 125) = \mathrm{P}(M > 265) = 0.02$, show that the value of μ is 150. Hence, by finding an equation involving σ, find the value of σ. [5]

8 An office team of 10 people includes 7 men and 3 women named Anne, Beth, and Cathie. For an upcoming fire drill exercise, 5 individuals will be chosen, each assigned a unique role, to carry out the drill. Determine the number of possible ways to select 5 people from this group of 10

(i) to conduct the fire drill, [1]

(ii) such that at most 1 woman is selected to conduct the fire drill. [2]

(iii) After the fire drill exercise, the 10 people are to hold a discussion at a round table with 10 identical seats. Determine the number of ways in which Beth is seated between Anne and Cathie. [1]

(iv) A group photo of the 10 people, arranged in two rows of five, was taken after the discussion. Determine the number of ways in which Beth is not standing beside Anne or Cathie. [4]

9 A bakery produces batches of cookies. On average, the proportion of flawed cookies produced is p, where $0 < p < 1$. The cookies are packed in boxes of 20. The number of flawed cookies in a box of cookies is denoted by C.

(a) State, in context, one assumption needed for the number of flawed cookies in a box to be well-modelled by a binomial distribution. [1]

(b) Given that $\mathrm{P}(C = 0 \text{ or } 1) = 0.15$, write down an equation for the value of p, and find this value numerically. [2]

For **(c)** and **(d)**, take $p = 0.08$.

(c) Ten boxes of cookies are randomly chosen. As part of the bakery's quality control process, a box of cookies will be accepted if it contains fewer than 4 flawed cookies, otherwise it will be rejected. Find the probability there are at least 2 but no more than 5 rejected boxes. [4]

(d) A random sample of 15 boxes of cookies is taken and 3 of the boxes are found to be rejected. Find the probability that the third rejected box occurs on the fifteenth box. [3]

10 **(a)** Observations of 8 pairs of values (u, g), representing the hours of internet usage per week (u) and academic performance (g) in terms of Grade Point Average (GPA), are shown in the table below.

Internet usage (u)	4.0	6.0	8.0	a	12.0	16.0	18.0	20.0
GPA (g)	3.7	3.5	3.4	3.2	3.0	2.7	2.6	2.5

It is known that the equation of the linear regression line of g on u is $g = -0.0765u + 3.99$, find the value of a correct to 1 decimal place. [2]

(b) A researcher is studying the relationship between the battery life (y, in hours) of a new smartphone model and the screen brightness setting (x, in %). The following data was collected from the tests conducted at different brightness levels.

Screen Brightness (x)	10	20	30	40	50	60	70
Battery life (y)	48.2	47.4	45.5	37.3	35.6	31.1	24.3

(i) Draw a scatter diagram for these values. [2]

(ii) One of the values of y appears to be incorrect. Circle this point on your diagram and label it P. [1]

(iii) Explain why a linear model $y = a + bx$ is not a suitable model. [1]

(iv) It is thought that the battery life (y) can be modelled by one of the formulae after removing the point P.

(A) $y = a + bx^2$,

(B) $y = a + b\ln x$,

where a and b are non-zero constants.

Find, correct to 4 decimal places, the product moment correlation coefficient between y and x^2 as well as y and $\ln x$.

Explain clearly which model is a better model for this set of data.

For the case identified, find the equation of a suitable regression line. [3]

(v) Using the regression line found in **(iv)**, estimate the battery life when the screen brightness is set to 80%. [1]

(vi) Comment on the reliability of your answer in part **(v)**. [1]

11 **(a)** The leaves of a particular plant species have an average length of 12 cm with a standard deviation of 3.5 cm. If a random sample of 100 leaves is selected, estimate the probability that their total length is at least 1138 cm. [2]

(b) An operator of a public workspace at location A claims that users of its one-seater pods spend an average of 131 minutes using the facilities. To test this claim, a random sample of 64 users was observed, revealing a mean usage time of 127 minutes with a standard deviation of 16.4 minutes.

(i) Test at 3% level of significance whether the workspace operator's claim is overstated. You should state the hypotheses and define any symbols you use. [5]

(ii) Explain the meaning of '3% level of significance' in the context of the question. [1]

(iii) The workspace operator at location B claims that the mean time spent by users of its one-seater pods is 140 minutes, with a known population standard deviation of 20.1 minutes. A new sample of 15 pod users is taken, and the sample mean usage time, $\bar{w}$, is reported. A hypothesis test is conducted at a 5% significance level, and the operator's claim is not rejected.

State two necessary assumptions for the test and determine the range of values that $\bar{w}$ can take. Give your answer correct to one decimal place. [5]

BLANK PAGE

ANDERSON SERANGOON JUNIOR COLLEGE

H2 MATHEMATICS **9758**

JC2 Prelim Paper 1 (100 marks) **9 Sept 2024**

3 hours

Additional Material(s): List of Formulae (MF 26)

CANDIDATE NAME

CLASS

READ THESE INSTRUCTIONS FIRST

Write your name and class in the boxes above.
Please write clearly and use capital letters.
Write in dark blue or black pen. HB pencil may be used for graphs and diagrams only.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions and write your answers in this booklet.
Do not tear out any part of this booklet.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

All work must be handed in at the end of the examination. If you have used any additional paper, please insert them inside this booklet.
The number of marks is given in brackets [ ] at the end of each question or part question.

Question number	Marks
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
Total	

This document consists of 23 printed pages and 5 blank pages.

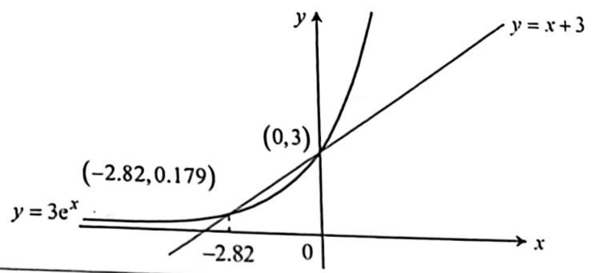
[Turn Over

1 **(a)** Sketch the graphs of $y = 3e^x$ and $y = x + 3$ on the same diagram. Indicate clearly the coordinates of the points of intersection between the 2 graphs.
Solve the inequality $3e^x > x + 3$. [3]

(b) Hence find $\int_{-2}^{2} \left|3e^x - x - 3\right| \, dx$, giving your answer in an exact form. [2]

Solution

(a)



From the graphs, $x < -2.82$ or $x > 0$.

$\int_{-2}^{2} \left|3e^x - x - 3\right| \, dx$

$= \int_{-2}^{0} -(3e^x - x - 3)\,dx + \int_{0}^{2} (3e^x - x - 3)\,dx$

$= -\left[3e^x - \frac{x^2}{2} - 3x\right]_{-2}^{0} + \left[3e^x - \frac{x^2}{2} - 3x\right]_{0}^{2}$

$= -[3 - (3e^{-2} - 2 + 6)] + [(3e^2 - 2 - 6) - 3]$

$= 3e^{-2} + 3e^2 - 10$

2 **(i)** Find $\frac{d}{dx}\left(e^{\sin^{-1}x}\sqrt{1 - x^2}\right)$. [1]

(ii) Hence using integration by parts, find $\int x \frac{e^{\sin^{-1}x}}{\sqrt{1 - x^2}} \, dx$. [3]

Solution

(i) $\frac{d}{dx}\left(e^{\sin^{-1}x}\sqrt{1 - x^2}\right) = e^{\sin^{-1}x} - \frac{xe^{\sin^{-1}x}}{\sqrt{1 - x^2}}$ ……(1)

(ii) $\int x \frac{e^{\sin^{-1}x}}{\sqrt{1 - x^2}} \, dx = xe^{\sin^{-1}x} - \boxed{\int e^{\sin^{-1}x} \, dx}$

From (1), $\boxed{e^{\sin^{-1}x}\sqrt{1 - x^2} + \int \frac{xe^{\sin^{-1}x}}{\sqrt{1 - x^2}} \, dx + c} = \int e^{\sin^{-1}x} \, dx$

So $\int x \frac{e^{\sin^{-1}x}}{\sqrt{1 - x^2}} \, dx = xe^{\sin^{-1}x} - \boxed{e^{\sin^{-1}x}\sqrt{1 - x^2} - \int \frac{xe^{\sin^{-1}x}}{\sqrt{1 - x^2}} \, dx - c}$

Commented [CKJ1]: Question Reading
Some students did not give the coordinates of the intersection point as required in the question.

Commented [CKJ2]: Common Mistake
1. Many students gave $x < -2.82$ and $x > 0$ which is incorrect instead of $x < -2.82$ or $x > 0$.
2. Some gave the incorrect range $x < -2.82$ or $x > 3$. Students should pay attention of the range of values of x in which the curve $y = 3e^x$ is higher than the line $y = x + 3$.

Commented [CKJ3]: Many students did not know how to make use of the information in (a) to remove the modulus sign in the integral.

Common Mistake
(1) $\int_{-2}^{2} \left|3e^x - x - 3\right| \, dx = \int_{-2}^{2} 3e^x - x - 3 \, dx$

Commented [CKJ4]: Common Mistake
Many students displayed good understanding from the working but were careless and gave their final answer as

(1) $e^{\sin^{-1}x} + \frac{xe^{\sin^{-1}x}}{\sqrt{1 - x^2}}$

(2) $e^{\sin^{-1}x} - \frac{e^{\sin^{-1}x}}{\sqrt{1 - x^2}}$

$$\therefore \int x \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}\,dx = \frac{1}{2}\left(xe^{\sin^{-1}x} - e^{\sin^{-1}x}\sqrt{1-x^2}\right) + D$$

3 The curve C has parametric equations

$$x = 2t - \frac{1}{t^2}, \quad y = 2t + \frac{1}{t}, \qquad t \in \mathbb{R}, t \neq 0.$$

Commented [CKJ5]: This question was well attempted.

The point P on the curve has parameter $t = 1$.

(i) Find the equation of tangent and normal to C at the point P. [4]

(ii) The tangent at P meets the y-axis at B. The normal at P meets the x-axis at A. If O is the origin, find the area of the quadrilateral $OAPB$. [2]

Solution

(i)

$$\frac{dx}{dt} = 2 + \frac{2}{t^3}$$

$$\frac{dy}{dt} = 2 - \frac{1}{t^2}$$

$$\frac{dy}{dx} = \frac{2 - \frac{1}{t^2}}{2 + \frac{2}{t^3}} = \frac{t\left(2t^2 - 1\right)}{2t^3 + 2}$$

When $t = 1$,

$\frac{dy}{dx} = \frac{1}{4}$, $x = 1$ and $y = 3$.

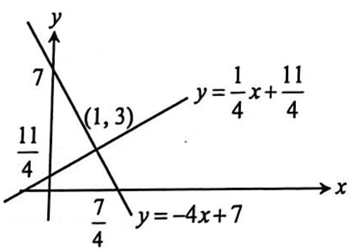
Eqn of tangent at P: $y - 3 = \frac{1}{4}(x - 1)$

$$y = \frac{1}{4}x + \frac{11}{4}$$

Eqn of normal at P: $y - 3 = -4(x - 1)$

$$y = -4x + 7$$

(ii)



Area of $OAPB = \frac{1}{2}\left(\frac{7}{4}\right)(7) - \frac{1}{2}\left(7 - \frac{11}{4}\right)(1) = 4 \text{ units}^2$

[Turn Over

4 A sequence is such that $u_0 = 2$ and $u_n = u_{n-1} + n^3 + \left(\frac{1}{2}\right)^n$ for $n \geq 1$.

(a) It is given that $\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$. By considering $\sum_{r=1}^{n}(u_r - u_{r-1})$, find a formula for u_n in terms of n. [4]

(b) Hence, using the formula of u_n found in (a), find $\sum_{r=9}^{n}\left((r+2)^3 + \left(\frac{1}{2}\right)^{r+2}\right)$ exactly. [3]

Solution

(a) $u_r - u_{r-1} = r^3 + \left(\frac{1}{2}\right)^r$

$$\sum_{r=1}^{n}(u_r - u_{r-1}) = \sum_{r=1}^{n}\left[r^3 + \left(\frac{1}{2}\right)^r\right]$$

$$\left(\begin{array}{l} \cancel{u_1} - u_0 \\ +\cancel{u_2} - \cancel{u_1} \\ +\cancel{u_3} - \cancel{u_2} \\ + \\ \vdots \\ +\cancel{u_{n-1}} - \cancel{u_{n-2}} \\ +u_n - \cancel{u_{n-1}} \end{array}\right) = \sum_{r=1}^{n} r^3 + \sum_{r=1}^{n}\left(\frac{1}{2}\right)^r$$

$$u_n - u_0 = \frac{n^2(n+1)^2}{4} + \left[1 - \left(\tfrac{1}{2}\right)^n\right]$$

$$\therefore u_n = \frac{n^2(n+1)^2}{4} + \left(1 - \left(\frac{1}{2}\right)^n\right) + 2$$

$$= \frac{n^2(n+1)^2}{4} - \left(\frac{1}{2}\right)^n + 3$$

(b) $\sum_{r=9}^{n}\left((r+2)^3 + \left(\frac{1}{2}\right)^{r+2}\right)$ replace r by r-2

$$\sum_{r=11}^{n+2}\left[r^3 + \left(\tfrac{1}{2}\right)^r\right]$$

$= u_{n+2} - u_{10}$ (using observation from (i))

$$= \left[\frac{(n+2)^2(n+3)^2}{4} - \left(\frac{1}{2}\right)^{n+2} + 3\right] - \left[\frac{(10)^2(11)^2}{4} - \left(\frac{1}{2}\right)^{10} + 3\right]$$

Commented [ABK6]: Overall improvement

Observable that for this question 4(a), a number of students have made improvement in solving this type of question which is similar to that of the question found in WA2.

However, there is still a handful of students who was not able to solve this question despite a similar one was tested in WA2.

Whenever, a question propose **"by considering"**, students should always take this course as it will be the most efficient approach. Also, this method may be compulsory as it is demanded by the statement in the question.

Commented [ABK7]: Presentation

Observable that some students are not clear about the **proper symbols** to be used in a summation.

(a) $\sum_{r=1}^{n}(u_r - u_{r-1}) = \sum_{r=1}^{n}\left[r^3 + \left(\frac{1}{2}\right)^r\right]$

versus

(b) $\sum_{r=1}^{n}(u_n - u_{n-1}) = \sum_{r=1}^{n}\left[n^3 + \left(\frac{1}{2}\right)^n\right]$.

What is wrong with (b)? Do be more careful in the way you write the summation.

Commented [ABK8]: Overall Improvement

As with WA2, a similar question was tested. Advisable to start the **"replacement"** on the question itself and NOT from the given formula. If we take the latter approach, it is achievable but the route is longer, not efficient.

Many students have managed to learn the above based on WA2. However there is still a handful of students who have not internalized this type of "replacement" question in summation despite it being covered in WA2.

	$=\dfrac{(n+2)^2(n+3)^2}{4}-\left(\dfrac{1}{2}\right)^{n+2}-3025+\dfrac{1}{1024}$	
5	**(a)** It is given that **a**, **b** and **c** are non-zero vectors. If $\lvert\mathbf{a}+\mathbf{b}\rvert=\lvert\mathbf{a}-\mathbf{b}\rvert$, show that the two vectors **a** and **b** are perpendicular to each other.	[4]
	(b) (i) Explain why the result of $$\mathbf{a}\times(\mathbf{b}+\mathbf{c})+\mathbf{b}\times(\mathbf{c}+\mathbf{a})+\mathbf{c}\times(\mathbf{a}+\mathbf{b})$$ is a vector.	[1]
	(ii) Simplify $\mathbf{a}\times(\mathbf{b}+\mathbf{c})+\mathbf{b}\times(\mathbf{c}+\mathbf{a})+\mathbf{c}\times(\mathbf{a}+\mathbf{b})$. Show your workings clearly.	[3]
	Solution	
	(a) $\lvert\mathbf{a}+\mathbf{b}\rvert=\lvert\mathbf{a}-\mathbf{b}\rvert$	
	$\Rightarrow \lvert\underset{\sim}{a}+\underset{\sim}{b}\rvert^2=\lvert\underset{\sim}{a}-\underset{\sim}{b}\rvert^2$ $\Rightarrow (\underset{\sim}{a}+\underset{\sim}{b})\cdot(\underset{\sim}{a}+\underset{\sim}{b})=(\underset{\sim}{a}-\underset{\sim}{b})\cdot(\underset{\sim}{a}-\underset{\sim}{b})$	
	$\Rightarrow \lvert\mathbf{a}\rvert^2+2\mathbf{a}\cdot\mathbf{b}+\lvert\mathbf{b}\rvert^2=\lvert\mathbf{a}\rvert^2-2\mathbf{a}\cdot\mathbf{b}+\lvert\mathbf{b}\rvert^2$ $\Rightarrow 4\mathbf{a}\cdot\mathbf{b}=0$ $\Rightarrow 4\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\theta=0$ Since **a** and **b** are non-zero vectors, then $\cos\theta=0\Rightarrow\theta=90^\circ$, thus $\mathbf{a}\perp\mathbf{b}$	
	(b)(i) Since $\mathbf{a}\times(\mathbf{b}+\mathbf{c})$, $\mathbf{b}\times(\mathbf{c}+\mathbf{a})$ and $\mathbf{c}\times(\mathbf{a}+\mathbf{b})$ are all vectors, the addition of these vectors will lead to a resultant vector.	
	(ii) $\mathbf{a}\times(\mathbf{b}+\mathbf{c})+\mathbf{b}\times(\mathbf{c}+\mathbf{a})+\mathbf{c}\times(\mathbf{a}+\mathbf{b})$	
	$=\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}+\mathbf{b}\times\mathbf{c}+\mathbf{b}\times\mathbf{a}+\mathbf{c}\times\mathbf{a}+\mathbf{c}\times\mathbf{b}$	
	$=\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}+\mathbf{b}\times\mathbf{c}-\mathbf{a}\times\mathbf{b}-\mathbf{a}\times\mathbf{c}-\mathbf{b}\times\mathbf{c}$ ($\because \mathbf{b}\times\mathbf{a}=-\mathbf{a}\times\mathbf{b}$ and $\mathbf{c}\times\mathbf{a}=-\mathbf{a}\times\mathbf{c}$)	
	$=\mathbf{0}$ will not be given if the student wrote it as a scalar quantity	
6	**(i)** The variables x and y are related by $$(x+y)\frac{dy}{dx}+ky=2 \text{ and } y=1 \text{ at } x=0,$$ where k is a constant. Show that $(x+y)\dfrac{d^2y}{dx^2}+(1+k)\dfrac{dy}{dx}+\left(\dfrac{dy}{dx}\right)^2=0$.	[1]
	(ii) Given that x is small, find the series expansion of $g(x)=\dfrac{1}{\sin^2\left(2x+\frac{\pi}{2}\right)}$ in ascending powers of x, up to and including the term in x^2.	
	If the coefficient of x^2 in the expansion of $g(x)$ is equal to twice the coefficient of x^2 in the Maclaurin series for y in **(i)**, find the value of k.	[5]

Commented [ABK9]: Presentation
For notation, in printed form, non-column vectors like these are represented in **BOLD**. When writing, the symbol must have a tilde below the letter - representing a vector. For example, when printed, the vector is typed as **b** in bold, but when written it should be in this form $\underset{\sim}{b}$.

Commented [ABK10]: Method
A number of students proposed proving this statement using geometrical approach using either square, rhombus or parallelogram. However, most of the proofs are **without** (1) proper justification (2) making unwarranted assumptions.

Commented [ABK11]: Misconception
Treating vectors like polynomials may not be true for all situations. In this case, for the dot/scalar product of two vectors, the symbol to be used is the **DOT PRODUCT**. There is no representation of POWER of a vector.

Many students incorrectly write $(\mathbf{a}+\mathbf{b})^2=(\mathbf{a}-\mathbf{b})^2$ instead of the correct representation $(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b})=(\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})$.

Commented [ABK12]: Method
From the line $4\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\theta=0$ to imply $\cos\theta=0$, we must be explicit to state that **a** and **b** are non-zero vectors as part of the working.

Commented [ABK13]: Misconception/ Presentation
As mentioned above in print form, the vector is typed as **b** but when written it should be in this form $\underset{\sim}{b}$. Thus for Q5b(ii), we must write $\underset{\sim}{0}$ to signify the zero vector and NOT 0, which signifies the scalar zero.

Commented [CSC14]: Things to Note:
Students just need to do implicit differentiation on both sides to get the result as shown.

	(iii) By further differentiation of the result found in (i), and taking $k=1$, find the Maclaurin series for y, up to and including the term in x^3.	[3]
	Solution	
	(i)	
	$(x+y)\frac{dy}{dx}+ky=2 \quad \cdots(1)$ Differentiating (1) w.r.t. x: $(x+y)\frac{d^2y}{dx^2}+\left(1+\frac{dy}{dx}\right)\frac{dy}{dx}+k\frac{dy}{dx}=0$ $(x+y)\frac{d^2y}{dx^2}+(1+k)\frac{dy}{dx}+\left(\frac{dy}{dx}\right)^2=0 \quad \cdots(2)$	
	(ii) $\sin\left(2x+\frac{\pi}{2}\right)=\sin 2x\cos\frac{\pi}{2}+\cos 2x\sin\frac{\pi}{2}=\cos 2x$	
	$\frac{1}{\sin^2\left(2x+\frac{\pi}{2}\right)}=\frac{1}{\cos^2 2x}$ $\approx\left(1-\frac{(2x)^2}{2}\right)^{-2}$	
	$=\left(1-2x^2\right)^{-2}$ $=1+4x^2+\ldots$	
	$x=0,\quad y=1:\ \frac{dy}{dx}=2-k$ $\frac{d^2y}{dx^2}=3k-6$	
	$4=2\left(\frac{3k-6}{2}\right)$ $k=\frac{10}{3}$	
	(iii) Differentiating (2) w.r.t. x: $(x+y)\frac{d^3y}{dx^3}+\left(1+\frac{dy}{dx}\right)\frac{d^2y}{dx^2}+2\frac{d^2y}{dx^2}+2\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right)=0$	
	$(x+y)\frac{d^3y}{dx^3}+\left(3+3\frac{dy}{dx}\right)\frac{d^2y}{dx^2}=0$	
	When $x=0$, $\quad y=1,\ \frac{dy}{dx}=1,\ \frac{d^2y}{dx^2}=-3,\ \frac{d^3y}{dx^3}=18$	
	$\therefore y=1+x-\frac{3}{2}x^2+3x^3+\ldots$	

Commented [CSC15]: Things to Note: Students should note that although x is given to be small, angle $\left(2x+\frac{\pi}{2}\right)$ is no longer small.

Commented [CSC16]: Things to Note: The coefficient of x^2 in the Maclaurin series is given by $\frac{f''(0)}{2!}$.

7	**(a)** State a sequence of transformations that will transform the curve with equation $y^2 - x^2 = 1$ on to the curve with equation $9y^2 - 54y - x^2 - 2x + 79 = 0$.	[4]
	(b) A curve C has equation $$9y^2 - 54y - x^2 - 2x + 79 = 0.$$	
	(i) For real values x, use a non-graphical method to determine that y cannot lie between a and b, where a and b are exact real constants to be determined.	[3]
	(ii) Sketch the curve C, indicating clearly the equations of all asymptotes and the coordinates of the turning points.	[3]
	(iii) By adding a suitable curve, determine the number of real roots of the equation, $$9\left[(x+1)^2+3\right]^2 - 54\left[(x+1)^2+3\right] - x^2 - 2x + 79 = 0.$$	[2]
	Solution	
	(a) $9y^2 - 54y - x^2 - 2x + 79 = 0$ $\Rightarrow 9(y^2 - 6y) - (x^2 + 2x) + 79 = 0$ $\Rightarrow 9\left[(y-3)^2 - 9\right] - \left[(x+1)^2 - 1\right] + 79 = 0$ $\Rightarrow 9(y-3)^2 - (x+1)^2 = 1$ $\Rightarrow (3y-9)^2 - (x+1)^2 = 1$	
	$y^2 - x^2 = 1$ Replace x by $x+1$ $\downarrow$ Translation of 1 unit in the negative x direction $y^2 - (x+1)^2 = 1$ Replace y by $y-9$ $\downarrow$ Translation of 9 units in the positive y direction $(y-9)^2 - (x+1)^2 = 1$ Replace y by $3y$ $\downarrow$ Scaling parallel to the y-axis by a factor of $\frac{1}{3}$ $(3y-9)^2 - (x+1)^2 = 1$	
	(b)(i) Consider the line $y = k$. To find the range of y where curve C cannot lie, $\Rightarrow (3k-9)^2 - (x+1)^2 = 1$ has no real roots	
	$\Rightarrow x^2 + 2x + 2 - (3k-9)^2 = 0$ has no real roots. $\Rightarrow 4 - 4\left[2 - (3k-9)^2\right] < 0$	

Commented [KSX17]: Recommendation
Complete the square and identify which standard curve this is, instead of jumping straight to guess what are the steps for transformation of curves.

Commented [KSX18]: Question Reading

Non-graphical method also mean you cannot use the 'distance method from centre' to find values of a and b as this method requires you to visualize from a graph.

Commented [KSX19]: Question Reading
You need to draw the additional curve.

Commented [KSX20]: Careless Mistakes
Many students did not complete the square correctly and hence lost marks for the description of steps. They should make use of brackets when factorizing out negative sign.

Commented [KSX21]: Things to note

1. Please read the lectures notes again to learn the proper phrasing of the transformations.
2. To obtain $9y^2$, one has to replace y in y^2 with $3y$.
3. The order of scaling and translation matters. Hence student should not jump straight to writing description. It is best to write "Replace x with $x+1$" (eg) instead as working and to track your train of thoughts.

Commented [KSX22]: Presentation

1. Students need to state that there is no real roots and hence discriminant is <0
2. Students need to write the conclusion to answer the question.

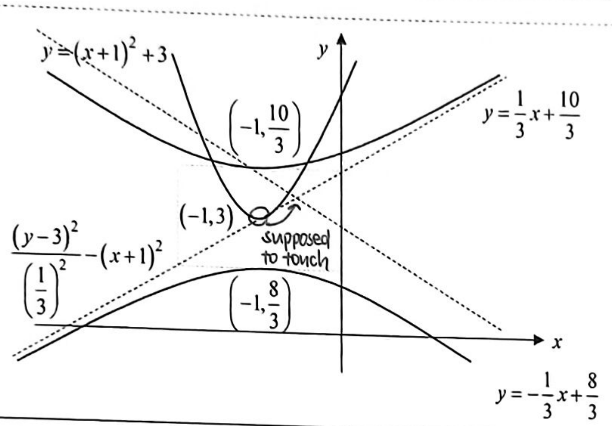
$(3k-9)^2-1<0$

$(3k-9+1)(3k-9-1)<0$

$\frac{8}{3}<k<\frac{10}{3}$

Thus y cannot lie between $\frac{8}{3}$ and $\frac{10}{3}$.

(ii)



(iii) By adding the graph of $y=(x+1)^2+3$, since there are 2 intersections between the 2 curves, so the equation

$9\left[(x+1)^2+3\right]^2-54\left[(x+1)^2+3\right]-x^2-2x+79=0$ will have 2 real roots.

8	The functions f and g are defined by $\mathrm{f}:x\mapsto\left\|4+2x-x^2\right\|,\ x\in\mathbb{R},\ x\geq 3.5,$ $\mathrm{g}:x\mapsto 4+\mathrm{e}^{ax},\ x\in\mathbb{R},\ x\geq -1,$	
	where $a>0$.	
	(a) Find $\mathrm{f}^{-1}(x)$ and state its domain.	[3]
	(b) Find the value of x for which $\mathrm{f}^{-1}(x)=\mathrm{f}(x)$.	[2]
	(c) Show that the composite function fg exists and express the exact range of fg in the form of $A+B\mathrm{e}^{-a}+C\mathrm{e}^{-2a}$, where A, B and C are real constants.	[4]
	(d) Without the use of a graphing calculator, solve the inequality $\frac{\mathrm{g}(x)}{x^2-2x-2}\geq 0$. Leave your answer in exact form.	[3]

Commented [KSX23]: Things to note

1. Coefficient of y^2 and x^2 will indicate which type of standard curve the curve is.
2. To find equation of asymptotes, student should use the "complete the square" method to change the equation of the curve into a standard form.
3. How to find equations of asymptotes:
$9(y-3)^2-(x+1)^2=0$
$9(y-3)^2=(x+1)^2$
then make y the subject.
4. Even if you do not know the equation of the asymptotes, you should still draw the asymptotes so that you can draw the hyperbola accurately.
5. Students should be aware that (-1,3) is both the turning point of the quadratic curve as well as the point of intersection of oblique asymptotes.

Common Mistake:

Students indicated that $y=\pm\frac{x}{3}$ are the equations of the asymptotes. They should however think through again as the equations does not tally with the oblique asymptotes they have drawn on the graph.

Commented [KSX24]: Presentation
You need to explain why have 2 real roots.

Solution

(a) $y=\left|4+2x-x^2\right|,\ x\in\mathbb{R},\ x\geq 3.5$

$y=-(4+2x-x^2)\ (\because 4+2x-x^2<0 \text{ for } x\geq 3.5)$

$y=x^2-2x-4$

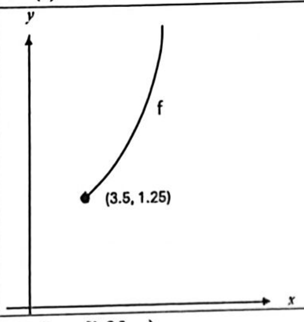
$y=(x-1)^2-1-4$

$y=(x-1)^2-5$

$x=1+\sqrt{y+5}$ or $x=1-\sqrt{y+5}$ (rej $\because x\geq 3.5$)

$x=1+\sqrt{y+5}$

$f^{-1}(x)=1+\sqrt{x+5}$



$R_f=[1.25,\infty)$

$\therefore D_{f^{-1}}=[1.25,\infty)$

(b) $f^{-1}(x)=f(x),\ x\in D_f\cap D_{f^{-1}}$

$x=f(x),\ x\in[3.5,\ \infty)$

$x=x^2-2x-4$

$x^2-3x-4=0$

$(x-4)(x+1)=0$

$x=-1$ (rej $\because x\geq 3.5$) or $x=4$

(c) $R_g = [4+e^{-a},\infty)$

$D_f=[3.5,\infty)$

Since $e^{-a}>0$, $\therefore 4+e^{-a}>3.5$

$\therefore R_g\subset D_f$, fg exists.

$R_g=\left[4+e^{-a},\infty\right)$

$f(4+e^{-a}) = (4+e^{-a})^2-2(4+e^{-a})-4$

$=16+8e^{-a}+e^{-2a}-8-2e^{-a}-4$

Commented [TCK25]: Misconceptions

$y=\left|4+2x-x^2\right|$ does not mean

- $y=\pm(4+2x-x^2)$ for we have a domain given (i.e. $x\geq 3.5$) which determines that $y=-(4+2x-x^2)$ because $4+2x-x^2<0$ when $x\geq 3.5$.
- $y^2=(4+2x-x^2)^2$ in which case you have changed the function.

Recommendation

Sketch the graph of the function within the modulus on your GC to see where the function is above or below the *x*-axis. Then check with the given domain. Finally, write down the function, this time without the modulus.

Commented [TCK26]: Things to note

Completing the square will be needed at the A levels. Make sure you are able to it.

Commented [TCK27]: Things to note

Reason for choosing the correct inverse function *always* lies in the given domain. Make sure you justify it clearly in your working.

Commented [TCK28]: Question reading

Here the question is asking you to solve for x when $f^{-1}(x)=f(x)$. It is not saying that f is a self-inverse function.

To recap, you can only call a function self-inverse if and only if $f(x)$ and $f^{-1}(x)$ are identical.

In this question, both $f(x)$ and $f^{-1}(x)$ are clearly not the same.

Things to note

Whenever $f^{-1}(x)=f(x)$, it is true that $f^{-1}(x)=f(x)=x$. So solving $f^{-1}(x)=f(x)$ is the same as solving $f(x)=x$ or $f^{-1}(x)=x$. Choose the one that is easier to solve.

Commented [TCK29]: Presentation

After writing down R_g and D_f, you need to explain why $\left[4+e^{-a},\infty\right)$ is a subset of $[3.5,\infty)$.

Misconception

The criterion for composite function fg is not $D_g\subseteq R_f$ but $R_g\subseteq D_f$.

$= 4+6e^{-a}+e^{-2a}$

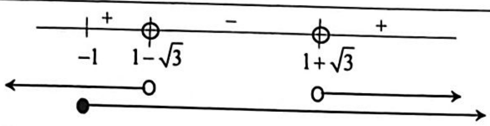
$R_{fg} = [4+6e^{-a}+e^{-2a}, \infty)$

(d) $\dfrac{g(x)}{x^2-2x-2} \geq 0$ AND $x \in D_g$

$\dfrac{4+e^{ax}}{(x-1)^2-3} \geq 0$ $\underset{\text{intersection}}{\underline{\text{AND}}}$ $x \geq -1$

Since $4+e^{ax} > 0,\ \forall x \in \mathbb{R}$

$\therefore \dfrac{1}{(x-1+\sqrt{3})(x-1-\sqrt{3})} \geq 0$ AND $x \geq -1$



$(x < 1-\sqrt{3}$ or $x > 1+\sqrt{3})$ AND $x \geq -1$

– for $x < 1-\sqrt{3}$ or $x > 1+\sqrt{3}$

$\therefore -1 \leq x < 1-\sqrt{3}$ **OR** $x > 1+\sqrt{3}$

9 **(a)** The complex numbers z_1 and z_2 are given by

$$z_1 = \frac{1+i}{1-i} \text{ and } z_2 = \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right).$$

(i) Find $z_1 + z_2$ in the form $re^{i\theta}$, where r is an exact real constant in trigonometric form such that $r > 0$, and θ is in the form $k\pi$ where k is an exact real constant such that $-1 < k \leq 1$. [3]

(ii) Find also $z_1 + z_2$ in the form $x + iy$, where x and y are exact real constant.

Hence show that $\tan\left(\dfrac{3\pi}{8}\right) = 1+\sqrt{2}$. [2]

(b) The complex number u is given by $w = \cos\theta + i\sin\theta$, where $0 < \theta < \dfrac{\pi}{2}$.

(i) Show that $1 - w^2 = -2iw\sin\theta$. [2]

(ii) Hence find the modulus and argument of $1-w^2$ in terms of θ. [2]

(iii) Given that $\left(\dfrac{1-w^2}{iw^*}\right)^n$ is real and negative and that $\theta = \dfrac{\pi}{5}$, find the three smallest positive integer values of n. [3]

Solution

(ai) $z_1 = \left(\dfrac{1+i}{1-i}\right) \times \left(\dfrac{1+i}{1+i}\right) = \dfrac{1}{2}(1+2i-1) = i = e^{i\left(\frac{\pi}{2}\right)}$

Commented [TCK30]: Things to note
Because f is an increasing function for $x \geq 3.5$ (see graph in the solution), the intermediate input $R_g = [4+e^{-a}, \infty)$ gives $R_{fg} = [f(4+e^{-a}), \infty)$

Commented [TCK31]: Question reading
This inequality question is a little different in that $g(x)$ is only defined for the domain $x \geq -1$ which must be considered when solving the inequality. This explains the method of solution.

Commented [TCK32]: Things to note
Use the number line method to get the solutions set.

Commented [TCK33]: Things to note
For more than one solution set, punctuate with the word 'or' not 'and'.

Commented [KSM34]: Misreading
1. Many disregard this and put r in the algebraic form

$$z_2 = \cos\left(\frac{\pi}{4}\right) + \mathrm{i}\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\mathrm{i}$$

$$z_1 + z_2 = \mathrm{e}^{\frac{\pi}{2}\mathrm{i}} + \mathrm{e}^{\frac{\pi}{4}\mathrm{i}} = e^{\frac{1}{2}\left(\frac{\pi}{2}i + \frac{\pi}{4}i\right)}\underbrace{\left(e^{\frac{\pi}{8}i} + e^{-\frac{\pi}{8}i}\right)}_{z + z^*}$$

$$= 2\cos\frac{\pi}{8} \times \mathrm{e}^{\frac{3\pi}{8}\mathrm{i}}$$

(ii) $z_1 + z_2 = \dfrac{1}{\sqrt{2}} + \left(1 + \dfrac{1}{\sqrt{2}}\right)\mathrm{i}$

$$\tan\frac{3\pi}{8} = \frac{1 + \frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}$$

$$= \frac{\frac{\sqrt{2}+1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = \sqrt{2} + 1 \text{ (shown)}$$

(bi) Method 1

$$w = \cos\theta + \mathrm{i}\sin\theta = \mathrm{e}^{\theta\mathrm{i}}$$

$$1 - w^2 = 1 - \mathrm{e}^{2\theta\mathrm{i}} = \mathrm{e}^{0\mathrm{i}} - \mathrm{e}^{2\theta\mathrm{i}}$$

$$= \mathrm{e}^{\theta\mathrm{i}}\left(\mathrm{e}^{-\theta\mathrm{i}} - \mathrm{e}^{\theta\mathrm{i}}\right) \qquad \text{(Special Technique)}$$

$$= w(\cos\theta - \mathrm{i}\sin\theta - \mathrm{i}\sin\theta - \cos\theta)$$

$$= w(-2\mathrm{i}\sin\theta)$$

$$= -2\mathrm{i}w\sin\theta$$

Method 2

$$1 - w^2 = 1 - (\cos\theta + \mathrm{i}\sin\theta)^2$$

$$= 1 - \cos^2\theta + \sin^2\theta - 2\mathrm{i}\sin\theta\cos\theta$$

$$= 2\sin^2\theta - 2\mathrm{i}\sin\theta\cos\theta \quad (\text{use } \sin^2\theta + \cos^2\theta = 1)$$

$$= 2\sin\theta(\sin\theta - 2\mathrm{i}\cos\theta)$$

$$= -2\mathrm{i}\sin\theta(\cos\theta + \mathrm{i}\sin\theta)$$

$$= -2\mathrm{i}w\sin\theta$$

(ii)

$$\left|1 - w^2\right| = \left|-2\mathrm{i}w\sin\theta\right| = \left|-2\sin\theta\right|\left|\,\mathrm{i}\,\right|\left|w\right|$$

$$= 2\sin\theta$$

$$\arg\left(1 - w^2\right) = \arg(-2\mathrm{i}w\sin\theta)$$

$$= \arg\left[(-2\sin\theta)\mathrm{i}\right] + \arg(w)$$

$$= -\frac{\pi}{2} + \theta$$

(iii) Method 1

Commented [KSM35]: Misconception:
Quite a number of students wrote this:
$\mathrm{e}^{\frac{\pi}{2}\mathrm{i}} + \mathrm{e}^{\frac{\pi}{4}\mathrm{i}} = \mathrm{e}^{\frac{3\pi}{4}\mathrm{i}}$ which is wrong!!

Commented [KSM36]: Misconception:
Note that it is incorrect to say
$(\cos\theta + i\sin\theta)^2 = \cos^2\theta + i^2\sin^2\theta$

	$\left(\frac{1-w^2}{\mathrm{i}w*}\right)^n=\left(\frac{2\sin\theta\, e^{\left(-\frac{\pi}{2}+\theta\right)\mathrm{i}}}{e^{\left(\frac{\pi}{2}-\theta\right)\mathrm{i}}}\right)^n=(2\sin\theta)^n e^{n(-\pi+2\theta)\mathrm{i}}$ $\because \sin\theta>0$ when $\theta=\frac{\pi}{5}$, $\therefore \arg\left(\frac{1-w^2}{\mathrm{i}w*}\right)^n=n(-\pi+2\theta)$	
	Method 2 $\arg\left(\frac{1-w^2}{\mathrm{i}w*}\right)^n=n\left[\arg(1-w^2)-\arg(\mathrm{i})-\arg w*\right]$ $=n\left(\theta-\frac{\pi}{2}-\frac{\pi}{2}-(-\theta)\right)$ $=n(2\theta-\pi)$	
	Since $\left(\frac{1-w^2}{\mathrm{i}w*}\right)^n$ is real and negative, and sub in $\theta=\frac{\pi}{5}$ $\therefore n\left[2\left(\frac{\pi}{5}\right)-\pi\right]=\pi+2k\pi$, ~~$k\in\mathbb{Z}$~~ $k\in\mathbb{Z}$ $-\frac{3}{5}n=(2k+1), k\in\mathbb{Z}$	
	$n=-\frac{5}{3}(2k+1), k\in\mathbb{Z}$ From GC tables, when $k=-2,-5,-8$, Smallest $n=5, 15, 25$	
10	A rice retailer pledges to donate a bowl of rice for every kilometre run by participants in a service-learning project. Donations will be made in complete bowls, based on the cumulative distance each individual ran by the end of the 28-day period. Distances ran by multiple individuals will not be combined. For example, if person A runs 18.8 km and person B runs 11.2 km, the retailer will donate a total of 29 bowls. Two such participants, athlete A and B, will each accumulate the distance they run for a total of 28 days via a plan each devised.	
	• Athlete A plans to run 5 km on the first day and then increase the distance by a fixed 0.65 km more than the previous day. • Athlete B plans to run 7 km on the first day and then increase the distance by 4% more than the previous day.	
	(a) Determine the least number of days required for the cumulative distance of athlete A to exceed that of athlete B.	[3]
	(b) How many bowls of rice will both athletes contribute, in total, at the end of the 28-day period?	[3]
	(c) Suppose athlete A plans to cover at least 400 km by the end of the 28-day period, what is the minimum distance he should run in day 1 if the plan to	[3]

Commented [KSM37]: Note: A real and negative complex number lies on the negative real axis, with argument π, all susequent points could be obtained by adding a multiple of 2π.

Commented [KSM38]: Presentation: The value of k must be stated and k must be integers, not real numbers.
Note: 2k+1 is odd, so one just need to sub. In odd multiples of 3 to get whole values of n!

	increase by 0.65 km more than the previous day remains the same. Give your answer to the nearest metres.	
	(d) On days where the distance athlete B is supposed to run exceeds 10 km based on his own plan, he will limit it to exactly 10 km instead. Given this change, how many bowls of rice will he contribute at the end of the 28-day period?	[3]
	Solution	
	(a) $\frac{n}{2}\left[10+(n-1)0.65\right]>\frac{7\left(1.04^n-1\right)}{1.04-1}$	
	$\frac{n(9.35+0.65n)}{2}-175\left(1.04^n-1\right)>0$	
	$n>13.396$	
	Least number of days = 14	
	(b) Total dist covered by A $=\frac{28}{2}\left[10+27(0.65)\right]=385.7$	
	Total dist covered by B $=\frac{7\left(1.04^{28}-1\right)}{1.04-1}=349.77$	
	Total number of bowls $=385+349=734$	
	(c) Let a be the distance athlete A will need to cover on the 1st day	
	$\frac{28}{2}\left[2a+27(0.65)\right]\geq 400$ $S=\frac{28}{2}[2a+(27)0.65]$	
	$a\geq 5.5107$	
	Hence athlete A will need to run at least 5511 m	
	(d) Let n be the number of days where the distance covered is at most 10 km.	
	$7(1.04)^{n-1}\leq 10$	
	$n\leq 10.094$	
	Total dist covered by B $=\frac{7\left(1.04^{10}-1\right)}{1.04-1}+10(18)=264.04$	
	Number of bowls contributed by A from the run = 264	
11	In a large town, the number of people infected by a particular virus t days after the virus was first discovered is x. It is assumed that the rate of infection is proportional to x. Initially there are 5 people who are infected by the virus, and there are 5120 people who are infected by the virus 30 days after the virus was first discovered.	
	(i) Show that $x=5(2)^{\frac{t}{3}}$.	[5]
	A cure and vaccine for the virus were discovered and administered to the population 30 days after the virus was discovered. Individuals who were cured are not at risk of reinfection. The number of people infected by the virus p days after the cure and vaccine were administered, is represented by y. It is believed that the new rate of infection from then on is proportional to $6400y-y^2$.	
	It is given that 30 days after the cure or vaccine was administered, 3200 people remain infected with the virus.	
	(ii) Show that $y=\frac{6400}{1+2^{\left(\frac{p}{15}+H\right)}}$, where H is a constant to be determined.	[6]

Commented [SH39]: Concepts/ Formulae
•Major mistake 1: Majority of students used the wrong ratio for the GP: Instead of using r=1.04, they used r=0.04!
•Major mistake 2: Did not recall the formula for sum of first n terms for the AP and GP series correctly.
•Major mistake 3: Didn't answer to the question as least number of days = 14. Phrase least n or least number of days was omitted.

Commented [SH40]: Question reading
•To calculate the correct total number of bowls donated, student must understand the example given lines 4-6.

Commented [SH41]: •Recall of AP Formula wrongly!
•Lacking of good GC Skills (many with correct formula, got wrong answers)
•Expected the answer in metres. Many gave in 5 km and 511 metres. Not accepted! 5511metres is the only accepted answer.

Commented [SH42]: Formula for nth term of GP recalled wrongly. Many used different values of n for the calculation of Sum of first n terms and this error is stemmed from using wrong Tn formula.

Commented [LT43]: Question Reading
Some misinterpret it as $\frac{dx}{dt}=kxt$.

Commented [LT44]: Question Reading
Need to know what are the variables defined in the question and use them appropriately.

One cannot use t and x in the forming of the new differential equation for they mean different things in this question.

Commented [LT45]: Question Reading
Some did not include a proportionality constant when forming the Differential Equation.

(iii) By finding the number of people in the town that will be infected by the virus in the long term, comment on the effectiveness of the cure and vaccine administered. [2]

Solution

(i) $\frac{dx}{dt} = kx$

$\int \frac{1}{x}\,dx = \int k\,dt$

$\ln|x| = kt + C$

$x = Ae^{kt},\ A = \pm e^{C}$

When $t = 0,\ x = 5$

$\Rightarrow A = 5$

When $t = 30,\ x = 5120$

$\Rightarrow 5120 = 5e^{30k}$

$k = \frac{1}{30}\ln 1024 = \frac{1}{3}\ln 2^{10} = \frac{1}{3}\ln 2$

$x = 5e^{\frac{t}{3}\ln 2} = 5\left(e^{\ln 2}\right)^{\frac{t}{3}} = 5(2)^{\frac{t}{3}}$ or $x = 5e^{\frac{t}{3}\ln 2} = 5e^{(\ln 2)^{\frac{t}{3}}} = 5(2)^{\frac{t}{3}}$

(ii) $\frac{dy}{dp} = a\left(6400y - y^2\right)$

$\int \frac{1}{6400y - y^2}\,dy = \int a\,dp$

$\int \frac{1}{3200^2 - (y-3200)^2}\,dy = \int a\,dp$

$\frac{1}{6400}\ln\left|\frac{3200+(y-3200)}{3200-(y-3200)}\right| = ap + d$

$\ln\left|\frac{y}{6400-y}\right| = 6400ap + 6400d$

$\frac{y}{6400-y} = Be^{6400ap},\ B = \pm e^{6400d}$

$y = \frac{6400Be^{6400ap}}{1+Be^{6400ap}} = \frac{6400}{1+De^{-6400ap}}$

When $p = 0,\ y = 5120$

$5120 = \frac{6400}{1+D}$

$D = \frac{1}{4}$......(1)

When $p = 30,\ y = 3200$

$3200 = \frac{6400}{1+\frac{1}{4}e^{-192000a}}$

Commented [LT46]: Presentation of Answers
It is not advisable to write it as $k\frac{dx}{dt} = x$ because it will sometimes make the subsequent steps abit more complicated.

Commented [LT47]: Presentation of Answers
During the variable separable step, it is better to keep those constant terms separated from the variable term for the ease of integration and simplication later.

Commented [LT48]: Presentation of Answer
It is important to keep the modulus at this step unless you have stated that x is bigger than zero.

It is also better to have the integration constant placed on the side that is without the logarithm term.

Commented [LT49]: Presentation of Answer
It is simpler to only substitute the values of t and x in when it is in exponential form.

Commented [LT50]: Presentation of Answer
Since it is a "show" question type, it is important to show the simplification step clearly. So one either use the property $e^{b\ln a} = e^{\ln(a^b)}$ or $e^{ab} = \left(e^{a}\right)^{b}$ in the simplication step.

Commented [LT51]: Misconception
It is important to include the modulus sign after integrating because we do not know at this point if the term inside is a negative value or positive value.

Commented [LT52]: Presentation of Answer
It is important to express one variable with respect to another after solving the differential equation.

	$a=\frac{-1}{192000}\ln 4$	
	$\therefore y=\frac{6400}{1+\frac{1}{4}e^{\frac{p}{30}\ln 4}}=\frac{6400}{1+\frac{1}{4}\left(e^{\ln 4}\right)^{\frac{p}{30}}}=\frac{6400}{1+2^{-2}\left(2^{\frac{p}{15}}\right)}$	
	$y=\frac{6400}{1+2^{\left(\frac{p}{15}-2\right)}}$, where $H=-2$	
	(iii) As $p\to\infty, 2^{\left(\frac{p}{15}-2\right)}\to\infty$ and $y\to 0$	
	Since eventually no one in the town will be infected by the virus, the vaccine produced is therefore effective.	

Commented [LT53]: Presentation of Answer
Similar to (a), it is important to show the simplification since it is an answer to be shown. Hence all workings has to be shown clearly.

Commented [LT54]: Question Reading
It is important to write down what the value of H is in order to answer to the question.

Commented [LT55]: Presentation of Answer
It is important to show clearly how y approaches 0.

Commented [LT56]: Presentation of Answer
It is important to indicate the contextual meaning of what y approaches to zero means before answering the question.

End of Paper

BLANK PAGE

ANDERSON SERANGOON JUNIOR COLLEGE

H2 MATHEMATICS **9758**

JC2 Prelim Paper 2 (100 marks) **16 Sept 2024**

3 hours

Additional Material(s): List of Formulae (MF26)

CANDIDATE NAME

CLASS

READ THESE INSTRUCTIONS FIRST

Write your name and class in the boxes above.
Please write clearly and use capital letters.
Write in dark blue or black pen. HB pencil may be used for graphs and diagrams only.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions and write your answers in this booklet.
Do not tear out any part of this booklet.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

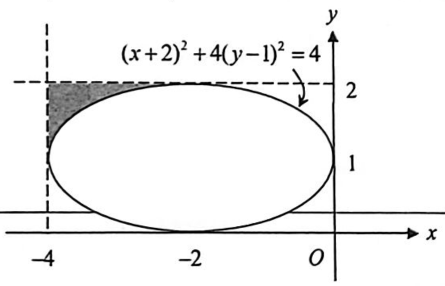
All work must be handed in at the end of the examination. If you have used any additional paper, please insert them inside this booklet.
The number of marks is given in brackets [ ] at the end of each question or part question.

Question number	Marks
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
Total	

This document consists of 13 printed pages and 3 blank pages.

[Turn Over

Section A: Pure Mathematics [40 marks]

1	By using the substitution $x = \cot\theta$, for $0 < \theta < \frac{\pi}{4}$, find $\int \frac{1}{x^2\sqrt{1+x^2}}\,dx$.	[4]
	Solution	
	(a) Let $x = \cot\theta$ $\frac{dx}{d\theta} = -\cos ec^2\theta$ $\int \frac{1}{x^2\sqrt{1+x^2}}\,dx$ $= \int \frac{1}{\cot^2\theta\sqrt{1+\cot^2\theta}} \times \left(-\cos ec^2\theta\right)\,d\theta$	
	$= \int \frac{1}{\cot^2\theta\sqrt{\cos ec^2\theta}} \times \left(-\cos ec^2\theta\right)\,d\theta$	
	$= \int (-\sin\theta)(\cos\theta)^{-2}d\theta$ **OR** $= -\int \tan\theta\sec\theta\,d\theta$	
	$= -\frac{1}{\cos\theta} + C$	
2	**(a)(i)** Find $\int \frac{9u-8}{4+9u^2}\,du$.	[3]
	(ii) The curve C is given by the parametric equations $x = u^2 + u + 1,\ y = \frac{9u}{4+9u^2}$, where $u \geq 0$. Find the exact area bounded by C, the x-axis and the line $x = 3$.	[4]
	(b) Find the volume of the solid formed when the shaded region bounded by the lines $x = -4$, $y = 2$ and the ellipse $(x+2)^2 + 4(y-1)^2 = 4$ is rotated through 2π radians about the y-axis. Give your answer correct to 1 decimal place. 	[3]

Commented [SH1]: Things to remember:
Formula for efficiency and less careless mistake.
$\frac{d}{dx}(\tan\theta) = \sec^2\theta$
$\frac{d}{dx}(\cot\theta) = -\cos ec^2\theta$

Commented [SH2]: Students made careless mistakes and the negative sign was omitted, this led to zero marks for the whole question.

Commented [SH3]: i)
$\int f'(x)\cdot[f(x)]^n\,dx = \frac{[f(x)]^{n+1}}{n+1} + C$ Using the integration formula. students to recognise, which function is f(x) and which is the derivative of f(x)
(ii) To recall the anti-derivative of $\tan\theta\sec\theta$.
Because $\frac{d}{d\theta}(\sec\theta) = \sec\theta\tan\theta$

Commented [SH4]: Things to remember:
Using substitution method and with indefinite integral, students will need to replace θ back to x.
Students should use **Toh Cah Soh** - to find the replacement in terms of x.

Solution

(ai)

$$\int \frac{9u-8}{4+9u^2}\,\mathrm{du}$$

$$= \int \frac{\frac{1}{2}(18u)}{4+9u^2} - \frac{8}{2^2+(3u)^2}\,\mathrm{du}$$

Commented [KSM5]: Many did not divide by derivative of 3u. Note you can do this $\int \frac{8}{2^2+(3u)^2}\,\mathrm{du} = \frac{8}{3}\int \frac{(3)}{2^2+(3u)^2}\,\mathrm{du}$ and adapt the MF26 formula;

or take out coefficient of u^2 to make it '+1', in order to use MF26 formula directly.

(ii) Area required $= \int_1^3 y\,\mathrm{d}x$

Commented [KSM6]: Note you need to start the integration the cartesian way, then do the parametric substitution. Remember to change the limits for x, to the limits for u !!

$$= \int_0^1 \frac{9u}{4+9u^2} \times (2u+1)\,\mathrm{d}u$$

$$= \int_0^1 \frac{18u^2+9u}{4+9u^2}\,\mathrm{d}u$$

$$= 2\int_0^1 1\,\mathrm{d}u + \int_0^1 \frac{(9u-8)}{4+9u^2}\,\mathrm{d}u$$

$$= \left[2u + \frac{1}{2}\ln(4+9u^2) - \frac{4}{3}\tan^{-1}\left(\frac{3u}{2}\right)\right]_0^1 \quad \text{(from part (i))}$$

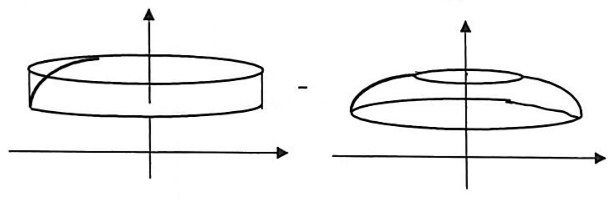
$$= \left(2+\frac{1}{2}\ln 13 - \frac{4}{3}\tan^{-1}\left(\frac{3}{2}\right)\right) - \left(0+\frac{1}{2}\ln 4 - \frac{4}{3}\tan^{-1}(0)\right)$$

$$= 2+\frac{1}{2}\ln\frac{13}{4} - \frac{4}{3}\tan^{-1}\left(\frac{3}{2}\right) \text{ units}^2$$

(b) Given $(x+2)^2 + 4(y-1)^2 = 4$

$$(x+2)^2 = 4[1-(y-1)^2] \Rightarrow x+2 = \pm 2\sqrt{1-(y-1)^2}$$

The shaded region is bounded by the section of the ellipse where $x \le -2$. Hence $x = -2-2\sqrt{1-(y-1)^2}$. ($\because$ arc has x-values <2)



Volume of solid formed

$\pi 4^2(2-1) - \pi\int_1^2 x^2\,dy$

$= 16\pi - \pi\int_1^2 \left(-2-2\sqrt{1-(y-1)^2}\right)^2 dy$

$= 9.6$ units^3 (From GC)

3 With respect to the origin O, the points A, B, C, D and E have coordinates $A(2,3,4)$, $B(6,5,7)$, $C(8,9,6)$, $D(4,7,3)$ and $E(5,6,10)$.

(a) Show that the cartesian equation of the surface containing the points A, B and E is $x-5y+2z=-5$. [2]

A line passes through the point D and the midpoint M of the edge EC.

(b) Find the vector equation of the line DM. [3]

(c) Find the exact coordinates of the foot of the perpendicular from the point M to the surface found in part **(a)**. [3]

(d) Hence find the exact shortest distance from the point M to the surface found in part **(a)**. [2]

(e) Verify the line DM intersects the surface found in part **(a)** at the point P with coordinates $(9,8,13)$. Hence find the vector equation of the reflection of the line DM about this slant surface. [4]

Solution

(a) $\overrightarrow{AB} = \begin{pmatrix}6\\5\\7\end{pmatrix} - \begin{pmatrix}2\\3\\4\end{pmatrix} = \begin{pmatrix}4\\2\\3\end{pmatrix}$ and $\overrightarrow{BE} = \begin{pmatrix}5\\6\\10\end{pmatrix} - \begin{pmatrix}6\\5\\7\end{pmatrix} = \begin{pmatrix}-1\\1\\3\end{pmatrix}$

Consider $\begin{pmatrix}4\\2\\3\end{pmatrix} \times \begin{pmatrix}-1\\1\\3\end{pmatrix} = \begin{pmatrix}3\\-15\\6\end{pmatrix} = 3\begin{pmatrix}1\\-5\\2\end{pmatrix}$

Then $\begin{pmatrix}x\\y\\z\end{pmatrix} \cdot \begin{pmatrix}1\\-5\\2\end{pmatrix} = \begin{pmatrix}2\\3\\4\end{pmatrix} \cdot \begin{pmatrix}1\\-5\\2\end{pmatrix} = 2-15+5=-5$

$\Rightarrow x(1)+y(-5)+z(2)=-5$

Thus equation of surface ABE is $x-5y+2z=-5$

(b) $\overrightarrow{OM} = \frac{1}{2}\left[\begin{pmatrix}5\\6\\10\end{pmatrix} + \begin{pmatrix}8\\9\\6\end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix}13\\15\\16\end{pmatrix}$

$\overrightarrow{DM} = \frac{1}{2}\begin{pmatrix}13\\15\\16\end{pmatrix} - \begin{pmatrix}4\\7\\3\end{pmatrix} = \frac{1}{2}\begin{pmatrix}5\\1\\10\end{pmatrix}$

Commented [KSM7]: Misconception: Some used 2D region area to minus a 3D volume, which is obviously wrong! You should consider the volume of cylinder! In the shaded region, the x values concerned satisfy $x \le -2$, so $x=-2-2\sqrt{1-(y-1)^2}$ should be used in evaluating the volume!

Commented [LT8]: Question Reading
Many did not answer to the question.

Commented [LT9]: Question Reading
Some did not know how to use the foot of perpendicular found to directly find the shortest distance needed.
Using the dot product to obtain the shortest distance is not the correct method for this part because this method can be employed using any other point on the plane. Therefore it downplay the reason why one needs to find the foot of perpendicular from point M to the plane in the previous part.

Commented [LT10]: Question Reading
It is important to make sure that the verification process is to be done on both the line and the plane.

Commented [LT11]: Misconception
It is incorrect to write it as $\begin{pmatrix}3\\-15\\6\end{pmatrix} = \begin{pmatrix}1\\-5\\2\end{pmatrix}$.

Commented [LT12]: Presentation of Answer
It is important to be clear on how the value -5 is obtained.
Similarly, one needs to show how to obtain the cartesian equation from the equation in scalar product form.

Commented [LT13]: Misconception
It is incorrect to write $\overrightarrow{OM} = \dfrac{\begin{pmatrix}5\\6\\10\end{pmatrix} - \begin{pmatrix}8\\9\\6\end{pmatrix}}{2}$. This is a wrong application of ratio theorem.

Line DM has equation: $\mathbf{r}=\begin{pmatrix}4\\7\\3\end{pmatrix}+\lambda\begin{pmatrix}5\\1\\10\end{pmatrix}, \lambda\in\mathbb{R}$

(c) Let foot of perpendicular from M to surface be N.

Thus $\overrightarrow{ON}=\frac{1}{2}\begin{pmatrix}13\\15\\16\end{pmatrix}+t\begin{pmatrix}1\\-5\\2\end{pmatrix}$ for some $t\in\mathbb{R}$

$$\Rightarrow\begin{pmatrix}\frac{13}{2}+t\\ \frac{15}{2}-5t\\ 8+2t\end{pmatrix}\cdot\begin{pmatrix}1\\-5\\2\end{pmatrix}=-5$$

$$\Rightarrow\frac{13}{2}+t-\frac{75}{2}+25t+16+4t=-5$$

$$\Rightarrow t=\frac{1}{3}$$

Thus $\overrightarrow{ON}=\begin{pmatrix}\frac{13}{2}+\frac{1}{3}\\ \frac{15}{2}-5\left(\frac{1}{3}\right)\\ 8+2\left(\frac{1}{3}\right)\end{pmatrix}=\frac{1}{6}\begin{pmatrix}41\\35\\52\end{pmatrix}$

Thus we have $N\left(\frac{41}{6},\frac{35}{6},\frac{26}{3}\right)$.

(d) $\left|\overrightarrow{MN}\right|=\left|\frac{1}{6}\begin{pmatrix}41\\35\\52\end{pmatrix}-\frac{1}{2}\begin{pmatrix}13\\15\\16\end{pmatrix}\right|=\left|\frac{1}{3}\begin{pmatrix}1\\-5\\2\end{pmatrix}\right|$

$$=\frac{1}{3}\sqrt{1+25+4}=\frac{\sqrt{30}}{3}$$

(e) Consider $\begin{pmatrix}4+5\lambda\\7+\lambda\\3+10\lambda\end{pmatrix}=\begin{pmatrix}9\\8\\13\end{pmatrix}\Rightarrow\begin{cases}\lambda=1\\\lambda=1\\\lambda=1\end{cases}$

Since the value of λ is consistent, P lines on the line DM.

Also, $9-5(8)+2(13)=-5$

Hence P also lies on the plane ABE.

Thus P is the point of intersection between the line DM and the plane ABE.

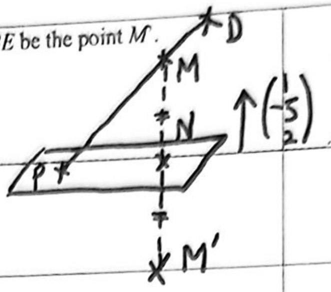
Commented [LT14]: Misconception
It is not correct to have the equation of line to be written in the form $\begin{pmatrix}4\\7\\3\end{pmatrix}+\lambda\begin{pmatrix}5\\1\\10\end{pmatrix}$ because this is not written as an equation and there is no information on what kind of values lambda will take.

Commented [LT15]: Misconception
It must not be written in the form $\overrightarrow{MN}=\frac{1}{2}\begin{pmatrix}13\\15\\16\end{pmatrix}+t\begin{pmatrix}1\\-5\\2\end{pmatrix}$ because L.H.S is a fixed vector that is parallel to the normal of the plane. The correct way to write it is $\overrightarrow{MN}=t\begin{pmatrix}1\\-5\\2\end{pmatrix}$ for some $t\in\mathbb{R}$.

Commented [LT16]: Question Reading
It is important to answer to the question.

Let the reflection of point M about the surface ABE be the point M'.
By ratio theorem (mid-point theorem),

$$\overrightarrow{ON}=\frac{\overrightarrow{OM}+\overrightarrow{OM'}}{2}$$



$$\overrightarrow{OM'}=2\overrightarrow{ON}-\overrightarrow{OM}=\frac{1}{3}\begin{pmatrix}41\\35\\52\end{pmatrix}-\frac{1}{2}\begin{pmatrix}13\\15\\16\end{pmatrix}=\frac{1}{6}\begin{pmatrix}43\\25\\56\end{pmatrix}$$

$$\overrightarrow{PM'}=\frac{1}{6}\begin{pmatrix}43\\25\\56\end{pmatrix}-\begin{pmatrix}9\\8\\13\end{pmatrix}=-\frac{1}{6}\begin{pmatrix}11\\23\\22\end{pmatrix}$$

Thus equation of the reflection of DM about the surface ABE is

$$\mathbf{r}=\begin{pmatrix}9\\8\\13\end{pmatrix}+\mu\begin{pmatrix}11\\23\\22\end{pmatrix},\mu\in\mathbb{R}$$

Commented [LT17]: Misconception
Some common misconceptions:

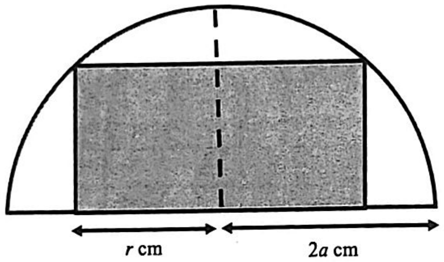
1. $\overrightarrow{OP}=\frac{\overrightarrow{OD}+\overrightarrow{OD'}}{2}$
2. $\overrightarrow{OP}=\frac{\overrightarrow{PD}+\overrightarrow{PD'}}{2}$
3. $\overrightarrow{OP}=\frac{\overrightarrow{PM}+\overrightarrow{PM'}}{2}$

To avoid making any of these misconceptions, it is important to have a drawing to see the development of the question.
Before using mid point theorem or ratio theorem, it is important to ensure that the points involved (in this case it is M, N and M') are collinear.

Commented [LT18]: Presentation of answer
It is always important to select a direction vector where the components are integer form.

Misconception
Similar to (b), it is important to have "$\mathbf{r}=$" and "$\lambda\in\mathbb{R}$" when presenting the vector equation of the line.

4 A popular toy company is designing a new water play feature for children. The toy consists of a cylindrical water container that will hold water for various playful activities. This cylindrical container is designed to be inscribed within a fixed, rigid hemispherical shell made of durable plastic of negligible thickness.



The shaded region in the diagram above shows the cross-sectional view of the upright cylindrical container that is inscribed in a hemisphere with fixed radius $2a$ cm.

(a) If the radius of the cylindrical water container is r cm, show that the volume V c of the water container is given by $V=\pi r^2\sqrt{4a^2-r^2}$. [1]

The unique feature of this toy is that the height of the cylindrical container is adjustable, allowing it to expand or contract while always touching the inner surface of the hemisphere.

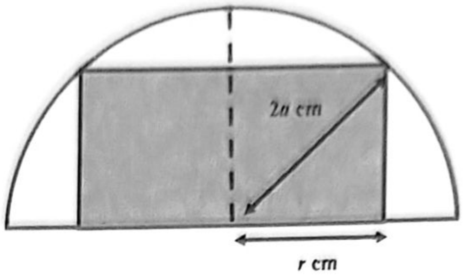
(b) Water is pumped into the container at a rate of 100π cm^{3}s^{-1} while the adjustment is taking place. If $a=50$, find the exact rate of change of the radius of the container at the instant when the height of the water container is 80 cm. [5]

(c) Using differentiation, find in terms of a, the value of r which gives a maximum value of V. Justify that this value indeed gives a maximum V. Hence write down the exact maximum volume of the cylinder in terms of a. [4]

(d) Sketch the graph showing the volume of the cylinder as the radius of the cylinder varies. [2]

Solution

(a)



Height of the cylindrical container $= \sqrt{(2a)^2 - r^2} = \sqrt{4a^2 - r^2}$

Volume of cylindrical container $V = \pi r^2 \sqrt{4a^2 - r^2}$

Need to see that $\sqrt{(2a)^2 - r^2}$ gives the height of the container

> **Commented [TCK19]: Presentation**
> There is a need to show clearly that the height of the container is derived using Pythagoras Theorem.

(b) $V = \pi r^2 \sqrt{4a^2 - r^2}$

$$\frac{dV}{dr} = \frac{d}{dr}\left(\pi r^2 \sqrt{4a^2 - r^2}\right)$$

$$= 2\pi r \sqrt{4a^2 - r^2} + \pi r^2 \left(\frac{1}{2}\right)\left(4a^2 - r^2\right)^{-\frac{1}{2}}(-2r)$$

$$= 2\pi r \sqrt{4a^2 - r^2} - \frac{\pi r^3}{\sqrt{4a^2 - r^2}}$$

$$= \frac{2\pi r\left(4a^2 - r^2\right) - \pi r^3}{\sqrt{4a^2 - r^2}}$$

$$= \frac{\pi r\left(8a^2 - 3r^2\right)}{\sqrt{4a^2 - r^2}}$$

> **Commented [TCK20]: Common mistake**
> Chain rule of differentiation was not carried out correctly - the term $(-2r)$ was missing from $\pi r^2\left(\frac{1}{2}\right)\left(4a^2 - r^2\right)^{-\frac{1}{2}}(-2r)$ in many scripts.

height = $80 = \sqrt{4(50)^2 - r^2} = \sqrt{10000 - r^2}$

$80^2 = 10000 - r^2$

$r^2 = 3600$

$r = 60 \quad (\because r > 0)$

> **Commented [TCK21]: Common mistake**
> Taking $r = 80$ instead of $r = \sqrt{4a^2 - h^2} = 60$ given $a = 50$ and $h = 80$.

> **Commented [TCK22]: Serious misconception spotted**
> $\sqrt{10000 - r^2} = 100 - r$.

When $r = 60$, $\dfrac{dV}{dr} = \dfrac{\pi(60)\left[8\left(50^2\right) - 3\left(60^2\right)\right]}{\sqrt{4\left(50^2\right) - 60^2}} = 6900\pi$

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$$

$$100\pi = 6900\pi \times \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{1}{69} \text{ cm s}^{-1}$$

(c) $\dfrac{dV}{dr} = \dfrac{\pi r\left(8a^2 - 3r^2\right)}{\sqrt{4a^2 - r^2}}$

$\frac{\mathrm{d}V}{\mathrm{d}r} = 0 \Rightarrow 8a^2 - 3r^2 = 0$ given $r > 0$

$r^2 = \frac{8}{3}a^2 \Rightarrow r = \sqrt{\frac{8}{3}}a$ given $r > 0$.

When $r = \left(\sqrt{\frac{8}{3}}a\right)^- \Rightarrow$ $(0<)r < \sqrt{\frac{8}{3}}a \Rightarrow r^2 < \frac{8}{3}a^2$

$\Rightarrow 8a^2 - 3r^2 > 0$

$\Rightarrow \frac{\mathrm{d}V}{\mathrm{d}r} = \frac{\pi r\left(8a^2 - 3r^2\right)}{\sqrt{4a^2 - r^2}} > 0$ given $r > 0$ and $\sqrt{4a^2 - r^2} > 0$.

When $r = \left(\sqrt{\frac{8}{3}}a\right)^+ \Rightarrow$ $r > \sqrt{\frac{8}{3}}a \Rightarrow r^2 > \frac{8}{3}a^2$

$\Rightarrow \cancel{3r^2}\ 8a^2 - 3r^2 < 0$

$\Rightarrow \frac{dV}{dr} = \frac{\pi r\left(8a^2 - 3r^2\right)}{\sqrt{4a^2 - r^2}} < 0.$

	$\left(\sqrt{\frac{8}{3}}a\right)^-$	$\sqrt{\frac{8}{3}}a$	$\left(\sqrt{\frac{8}{3}}a\right)^+$
$\frac{dV}{dr}$	+ve	0	– ve
Slope	/	—	\

V is maximum when $r = \sqrt{\frac{8}{3}}a$

When $r = \sqrt{\frac{8}{3}}a$,

$$V = \pi\frac{8}{3}a^2\sqrt{4a^2 - \frac{8}{3}a^2}$$
$$= \pi\frac{8}{3}a^2\sqrt{\frac{4a^2}{3}}$$
$$= \frac{16\pi a^3}{3\sqrt{3}}$$
$$= \frac{16\sqrt{3}\pi a^3}{9}\ \text{cm}^3$$

Commented [TCK23]: Question reading
Question requires answer in terms of a but a number of students do otherwise.

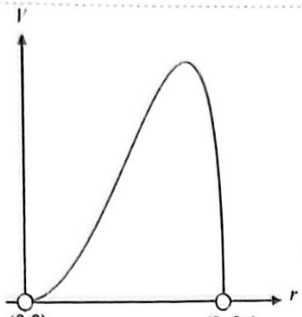
Commented [TCK24]: Things to note
The justification that V is maximum when $r = \sqrt{\frac{8}{3}}a$ is best done in the way shown rather than the $\frac{d^2V}{dr^2}$ test or substituting a value of r before and after the stationary value $\sqrt{\frac{8}{3}}a$ and determine the value of $\frac{dV}{dr}$, both of which are tedious.
Serious misconception spotted
$\sqrt{4a^2 - r^2} = 2a - r$

Alternatively

	$1.6a$	$\sqrt{\frac{8}{3}}a \approx 1.63a$	$1.7a$
$\frac{dV}{dr}$	$1.34a^2$	0	$-3.40a^2$
slope	/	—	\

Commented [TCK25]: Things to note
$\sqrt{\frac{4}{3}a^2}$ should be simplified, which many don't do.

(d)



$V = \pi r^2 \sqrt{4a^2 - r^2}$

$V > 0$

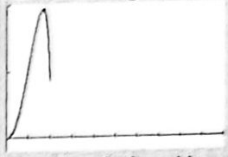
$\sqrt{4a^2 - r^2} \geq 0$

$4a^2 - r^2 \geq 0$

$r^2 \leq 4a^2$

$0 \leq r \leq 2a$

Commented [TCK26]: Things to note
By setting a=1, use the GC to sketch the curve. It is important to know that the GC has limitation in curve sketching at times.



V and r are both positive so only graph in the first quadrant should be given.

Section B: Probability and Statistics [60 marks]

5 An amateur music composer is arranging a sequence of four musical notes followed by three beats. There are 7 possible notes (labeled A to G) and 5 possible beats (labeled 1 to 5). The order of the notes and beats is important in the composition. Find the probability that a randomly chosen sequence has

(i) the third beat being a higher number than the second beat, [1]

(ii) exactly two notes the same or exactly two beats the same, but not both. [3]

$P(A \cup B) - P(A \cap B) = P(A) + P(B) - 2P(A \cap B)$

Solution

(i) Required probability $= \dfrac{4+3+2+1}{5^2} = \dfrac{2}{5}$ or $\dfrac{{}^5C_2}{5^2} = \dfrac{2}{5}$

(ii) Required probability $= \dfrac{{}^7C_3\,{}^3C_1\dfrac{4!}{2!}}{7^4} + \dfrac{{}^5C_2\,{}^2C_1\dfrac{3!}{2!}}{5^3} - 2\left(\dfrac{{}^7C_3\,{}^3C_1\dfrac{4!}{2!}}{7^4}\right)\left(\dfrac{{}^5C_2\,{}^2C_1\dfrac{3!}{2!}}{5^3}\right)$

$= 0.50099$ or $\dfrac{4296}{8575}$

$= 0.501$ (3 s.f)

6 Tetrahedral dice have four faces. Two fair tetrahedral dice, one red and one blue, have faces numbered 0, 1, 2, and 3 respectively. The dice are rolled, and the numbers faced down on the two dice are recorded. The random variable T is defined as the score on the red die multiplied by the score on the blue die.

(i) Find the probability distribution of T. [3]

(ii) Find E(T) and show that Var(T) $= \dfrac{115}{16}$. Show your workings clearly. [2]

(iii) Evaluate $P\left(|T - 2\mu| > \sigma\right)$, where $\mu = E(T)$ and $\sigma^2 = Var(T)$. [2]

Solution

(i) Table of outcomes:

	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	4	6
3	0	3	6	9

The probability distribution of T is given by:

t	0	1	2	3	4	6	9
$P(T=t)$	$\frac{7}{16}$	$\frac{1}{16}$	$\frac{2}{16}$ $=\frac{1}{8}$	$\frac{2}{16}$ $=\frac{1}{8}$	$\frac{1}{16}$	$\frac{2}{16}$ $=\frac{1}{8}$	$\frac{1}{16}$

(ii)

$$E(T)=0\left(\frac{7}{16}\right)+1\left(\frac{1}{16}\right)+2\left(\frac{1}{8}\right)+3\left(\frac{1}{8}\right)+4\left(\frac{1}{16}\right)+6\left(\frac{1}{8}\right)+9\left(\frac{1}{16}\right)=\frac{9}{4}$$

$$E(T^2)=0^2\left(\frac{7}{16}\right)+1^2\left(\frac{1}{16}\right)+2^2\left(\frac{1}{8}\right)+3^2\left(\frac{1}{8}\right)+4^2\left(\frac{1}{16}\right)+6^2\left(\frac{1}{8}\right)+9^2\left(\frac{1}{16}\right)=\frac{49}{4}$$

$$\text{Var}(T)=E(T^2)-[E(T)]^2=\frac{49}{4}-\left(\frac{9}{4}\right)^2=\frac{115}{16}$$

(iii) $P(|T-2\mu|>\sigma)=P(T-2\mu>\sigma)+P(T-2\mu<-\sigma)$

$=P(T>2\mu+\sigma)+P(T<2\mu-\sigma)$

$= P(T>7.18095)+P(T<1.81905)$

$= P(T=9)+P(T=0)+P(T=1)$

$=\frac{1}{16}+\frac{7}{16}+\frac{1}{16}=\frac{9}{16}$ or 0.5625 (cannot round off to 3.sf)

> **Commented [KSM27]: Misconception**
> Many resorted to using Normal distribution or Binomial distribution to calculate the probabilities, and lost the marks. YOU SHOULD USE VALUES OF THE PROB FROM THE TABLE in (i), that satisfy the two inequalities.

7 The masses, in grams, of the packets of semolina flour follow the distribution $N(225, 25^2)$ and the masses, in grams, of the packets of millet flour follow the distribution $N(\mu, \sigma^2)$.

(a) Find the probability that 4 times the mass of a packet of semolina flour is between 0.85 kilograms and 1.05 kilograms. [2]

(b) Let M be the mean mass of 3 packets of semolina flour and 2 packets of millet flour. Given that $P(M<125)=P(M>265)=0.02$, show that the value of μ is 150. Hence, by finding an equation involving σ, find the value of σ. [5]

> **Commented [ABK28]: Concept**
> Part (a) is obviously hinting that part (b) should be addressed differently. Many students can do part (a) well where it is required to obtain P(…<4X<…). In doing so, it is critical to form the distribution for random variable 4X.
>
> In general, if A is a random event and we want to find the P(A), we need to know what is the distribution for A. In our syllabus we have few distributions to use. In this case we are looking at the normal distribution not for X but for 4X as we are finding P(…<4X<…).

Solution

(a) Let X be the mass, in grams, of a randomly chosen packet of semolina.

$X \sim N(225, 25^2)$

$4X \sim N(4\times225,\ 4^2\times25^2)$

$4X \sim N(900,\ 100^2)$

$P(850<4X<1050)=0.62466$ (5 sf)

$=0.624$ (3 sf)

> **Commented [ABK29]: Presentation**
> Do define the random variable clearly. Also use capital letters to define the random variable and DO NOT use the reserved letters like N, Z or B to represent your random variables as these refer to Normal Distribution, Standard Normal Distribution and Binomial Distribution respectively.

(b) Let Y be the mass, in grams, of a randomly chosen packet of millet flour.

$Y \sim N(\mu, \sigma^2)$

Let $M = \boxed{\dfrac{X_1+X_2+X_3+Y_1+Y_2}{5}}$

$M \sim \text{N}\left(\dfrac{675+2\mu}{5}, \dfrac{3(25^2)+2(\sigma^2)}{25}\right)$

$\text{P}(M<125)=\text{P}(M>265)=0.02$

$\Rightarrow \dfrac{675+2\mu}{5}=\dfrac{125+265}{2}$

$\Rightarrow \dfrac{675+2\mu}{5}=195$

$\Rightarrow \mu=150$

Thus $M \sim \text{N}\left(195, \dfrac{1875+2\sigma^2}{25}\right)$

$\text{P}(M<125)=0.02$

$\text{P}\left(Z<\boxed{\dfrac{125-195}{\sqrt{\dfrac{1875+2\sigma^2}{25}}}}\right)=0.02 \quad \text{where } Z\sim \text{N}(0,1)$

$-\dfrac{70}{\sqrt{\dfrac{1875+2\sigma^2}{25}}}=-2.0537$

$1875+2\sigma^2=29042.99478$

$\sigma=116.55$ (5 s.f.) $=\cancel{116}$ 117 (3 s.f.)

Commented [ABK30]: Concept
Quite a handful of students take M as $M=\frac{3X+2Y}{5}$ which is incorrect. 3X + 2Y means 3 times the mass of ONE packet of semolina and 2 times the mass of ONE packet of millet, which is not the same as the MEAN or AVERAGE of 3 packets of semolina and 2 packets of millet flour.

Further mistakes include taking M as $M=3X+2Y$ or $M=X_1+X_2+X_3+Y_1+Y_2$ which the latter means the sum of 3 packets of semolina and 2 packets of millet which is NOT mean (average) mass at all!

Commented [ABK31]: Carelessness
Many students started off understanding that Y refers to the mass of a packet of millet i.e. $Y\sim \text{N}(\mu, \sigma^2)$. However when standardizing for the random variable M, they use μ as the mean for M i.e. E(M). E(M)=195 as we have found out when we showed $\mu = 150$.

Commented [ABK32]: Question reading
The question poses, by finding an equation involving σ, so a part of the working, we are required to show this equation before evaluating σ.

8 An office team of 10 people includes 7 men and 3 women named Anne, Beth, and Cathie. For an upcoming fire drill exercise, 5 individuals will be chosen, each assigned a unique role, to carry out the drill. Determine the number of possible ways to select 5 people from this group of 10

(i) to conduct the fire drill, [1]

(ii) such that at most 1 woman is selected to conduct the fire drill. [2]

(iii) After the fire drill exercise, the 10 people are to hold a discussion at a round table with 10 identical seats. Determine the number of ways in which Beth is seated between Anne and Cathie. [1]

(iv) A group photo of the 10 people, arranged in two rows of five, was taken after the discussion. Determine the number of ways in which Beth is not standing beside Anne or Cathie. [4]

Solution

(i) Number of ways $=\binom{10}{5}\times 5!=30240$

Note: After choosing 5 from 10 available persons, we need to permutate 5! because there are 5 unique roles to assign to the 5 chose people.

(ii) Number of ways $= \left[\binom{7}{5}\binom{3}{0}5! + \binom{7}{4}\binom{3}{1}5!\right]$

$= (21\times 1 + 35\times 3)5! = 15120$

Note: 2 cases – case 1, no women is selected out of the 3 available and case 2, one woman is selected out of the 3 available. Notice that for each case, we still need to permutate 5! Because we are still assigning 5 unique roles.

(iii) Number of ways $= (8-1)!2! = 10080$

Note: For circular cases, use the formula

no of arrangments of n unique items $= (n-1)! = \frac{n!}{n}$. In this case, there are 8 unique items, i.e. 7 men and the group A, B and C with (8-1)! using the formula. In addition, since A and C can sit on either side of B, we have 2!

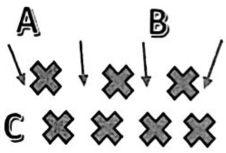
(iv) Case 1 – Beth in 1 row while Anne and Cathie are in another row

$\binom{7}{4}\times 5!\times 5!\times 2 = 1008000$

Note: Out of the 7 men, we choose 4 to be in 1st row, then 5! to arrange the 1st row and another 5! to arrange the 2nd row. Finally the case can be flipped between 1st and 2nd row, thus multiply by 2.

Case 2 – Beth and one of them in 1 row

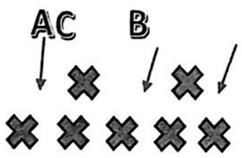
$\binom{7}{3}\times 3!\times\binom{2}{1}\times\binom{4}{2}\times 2!\times 5!\times 2 = 1209600$

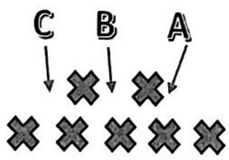


Note: Out of the 7 men, we choose 3 to be in 1st row, then permutate the 3 men. Then between of A and C, we choose one of them to be with B in 1st row. B with A/ C (2 cases) are then slotted in the 1st row (slotting method) i.e. out of the 4 available slots, we choose 2 (4 orange arrows), after slotting, these two can permutate, thus 2!. Then 5! To arrange the 2nd row. Finally the case can be flipped between 1st and 2nd row, thus multiply by 2.

Case 3 – Anne, Beth and Cathie are in the same row

$$\underbrace{\binom{7}{2}\times 2!\times\binom{3}{2}\times 2!\times 2!\times 5!\times 2}_{\text{A and C together}} + \underbrace{\binom{7}{2}\times 2!\times 3!\times 5!\times 2}_{\text{A and C separated}} = 181440$$



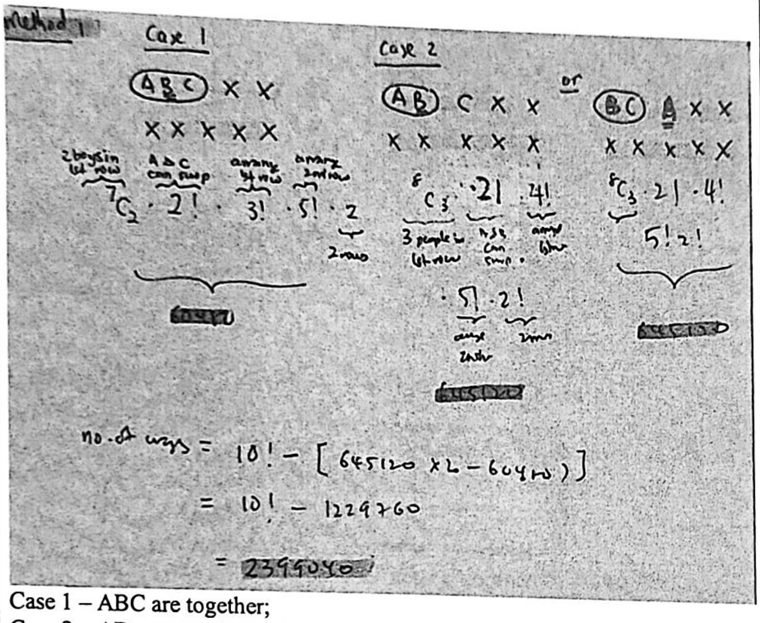


Note: When AC are together. Out of the 7 men, we choose 2 to be in 1st row, then permutate the 2 men i.e. 2!. AC together and B are then slotted in the 1st row (slotting method) i.e. out of the 3 available slots, we choose 2 (3 orange arrows), after slotting, these two groups can permutate, thus 2!. Then between A and C, they can switch places, hence another 2!. Then 5! To arrange the 2nd row. Finally the case can be flipped between 1st and 2nd row, thus multiply by 2.

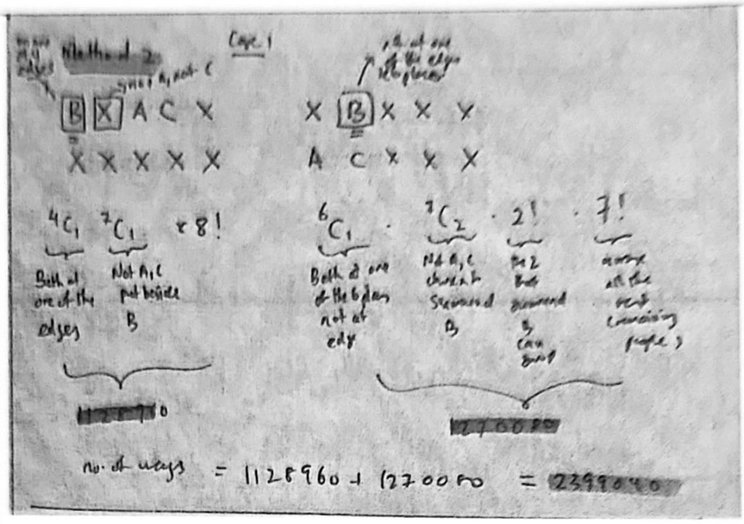
When A & C are not together, self-explanatory .

Thus number of ways = $1008000+1209600+181440=2399040$

Note that there are a few other possible methods but they are not mainstream as they require careful thinking about overlapping in counting or specific restrictions applied when counting.



Case 1 – ABC are together;
Case 2 – AB are together and
Case 3 – BC are together.
Note that case 1 will have overlap with that of case 2 and case 3.



9	A bakery produces batches of cookies. On average, the proportion of flawed cookies produced is p, where $0 < p < 1$. The cookies are packed in boxes of 20. The number of flawed cookies in a box of cookies is denoted by C.	
	(a) State, in context, one assumption needed for the number of flawed cookies in a box to be well-modelled by a binomial distribution.	[1]
	(b) Given that $P(C = 0 \text{ or } 1) = 0.15$, write down an equation for the value of p, and find this value numerically.	[2]
	For **(c)** and **(d)**, take $p = 0.08$.	
	(c) Ten boxes of cookies are randomly chosen. As part of the bakery's quality control process, a box of cookies will be accepted if it contains fewer than 4 flawed cookies, otherwise it will be rejected. Find the probability there are at least 2 but no more than 5 rejected boxes.	[4]
	(d) A random sample of 15 boxes of cookies is taken and 3 of the boxes are found to be rejected. Find the probability that the third rejected box occurs on the fifteenth box.	[3]
	Solution	
	(a) The probability of a cookie is flawed is constant at p for each cookie. OR The event that a cookie being flawed is independent of the event that another cookie being flawed.	
	(b) $C \sim B(20, p)$ $P(C=0) + P(C=1) = 0.15$ $(1-p)^{20} + \binom{20}{1} p^1(1-p)^{19} = 0.15$	
	$(1-p)^{19}(1+19p) = 0.15$	
	Using G.C, $p = 0.15891 = 0.159$	
	(c) Let C denote the number of flawed cookies in a box of 20 cookies.	

Commented [CKJ33]: Comprehension of Question
Please note that the statement on "2 possible outcomes" is not accepted as it is clear from the context that there are only 2 such outcomes, for a cookie to be flawed or not flawed.

Commented [CKJ34]: Common Mistake
Many students are not familiar with A level phrasing and interpreted the equation as $P(C = 0) = 0.15$ or $P(C = 1) = 0.15$ instead.

Commented [CKJ35]: Common Mistake
Many wrote $P(2 \le Y < 5)$ instead of $P(2 \le Y \le 5)$.

Commented [CKJ36]: Common Mistake
Many students did not realise it was a question involving conditional probability.

Commented [CKJ37]: Common Mistake
It is always about the event that is **independent** of each other but not about
- **The objects**
- **The number of something**
- **The probability of something**

The following examples are some of the WRONG statements made:
"A cookie is independent of the other cookie"

"**The number of flawed cookies is independent** of other number of flawed cookies"

"**The probability of a cookie being flawed** is independent of the probability of any other cookies being flawed."

$C \sim B(20,\ 0.08)$

$P(C<4) = P(C \le 3) = 0.92938$

Let Y be the number of rejected boxes out of 10 boxes.

$Y \sim B(10,\ 1-0.92938)$

$Y \sim B(10,\ 0.070615)$

$P(2 \le Y \le 5) = P(Y \le 5) - P(Y \le 1)$

$= 0.15388$

$= 0.154$ (to 3 sig fig)

(d) Let W be the number of rejected boxes in the first 14 boxes.

$W \sim B(14,\ 0.070615)$

Let V be the number of rejected boxes out of 15 boxes.

$V \sim B(15,\ 0.070615)$

Required Probability = P(3rd rej box is ō 15th box | V=3)

$= \dfrac{P(W=2) \times 0.070615}{P(V=3)}$

$= 0.2$

10 **(a)** Observations of 8 pairs of values (u, g), representing the hours of internet usage per week (u) and academic performance (g) in terms of Grade Point Average (GPA), are shown in the table below.

Internet usage (u)	4.0	6.0	8.0	a	12.0	16.0	18.0	20.0
GPA (g)	3.7	3.5	3.4	3.2	3.0	2.7	2.6	2.5

It is known that the equation of the linear regression line of g on u is $g = -0.0765u + 3.99$, find the value of a correct to 1 decimal place. [2]

(b) A researcher is studying the relationship between the battery life (y, in hours) of a new smartphone model and the screen brightness setting (x, in %). The following data was collected from the tests conducted at different brightness levels.

Screen Brightness (x)	10	20	30	40	50	60	70
Battery life (y)	48.2	47.4	45.5	37.3	35.6	31.1	24.3

(i) Draw a scatter diagram for these values. [2]

(ii) One of the values of y appears to be incorrect. Circle this point on your diagram and label it P. [1]

(iii) Explain why a linear model $y = a + bx$ is not a suitable model. [1]

(iv) It is thought that the battery life (y) can be modelled by one of the formulae after removing the point P.

(A) $y = a + bx^2$,

(B) $y = a + b\ln x$,

where a and b are non-zero constants.

Find, correct to 4 decimal places, the product moment correlation coefficient between y and x^2 as well as y and $\ln x$. Explain clearly which model is a better model for this set of data.

For the case identified, find the equation of a suitable regression line. [3]

Commented [CKJ38]: Students did not know how to define a random variable. A few did not define random variable in their working at all.

Presentation (Definition of random variable)
One needs to know that Binomial Distribution is a Discrete random variable and so the outcomes are countable numbers. So when we define a random variable that is Binomial in nature, the structure we adopt is:
" Let X denote the random variable "number of _(The thing of interest)_ out of _(Maximum possible outcome, n)_ ".

X, Y, W, V and A are letters that can be used for the definition of random variable(s).

Commented [SH39]: Conceptual understanding:
Important: The coordinate $(\bar{x}, \bar{y})$ must lie on the best fit line or on the regression line.
Students should use the GC to calculate the $\bar{y}$ more efficiently.
Majority of students didnt exhibit this understanding.

Commented [SH40]: Graphing skills
1. Need to label the first and the last coordinate of the data points.
2. Suitable scaling should be shown on the graph. We will need to see how the data points are arranged and reference to other data points.
3. Identify P and label it.

Commented [SH41]: Presentation
Since this part follows after the sketch of the scatter diagram, students should describe solely based on the pattern behaviour of the data points for the 1 mark.
From the scatter diagram,
Words accepted for this trend is ***1. Cuvilinear shape/behaviour hence the linear model is not suitable 2. As x increases, y decreases at the increasing rate hence the linear model is not suitable***. *No other answers are accepted.*

Commented [SH42]: Question Reading
To calculate r:
1. To remove the P (40, 37.3)
2. r value to be given/corrected to 4 decimal places
3. Equation of the regression line to be given with coefficients up to 3sf.

(v) Using the regression line found in (iv), estimate the battery life when the screen brightness is set to 80%. [1]

(vi) Comment on the reliability of your answer in part (v). [1]

Commented [SH43]: Presentation
Important to state that the linear **relationship/trend between y and x^2 may not hold.**
Students mentioning the estimate is not reliable since its outside of the data range [10, 70] and extrapolation is conducted will not be awarded the 1 mark.

Solution

(a) Using G.C, $\bar{g} = 3.075$

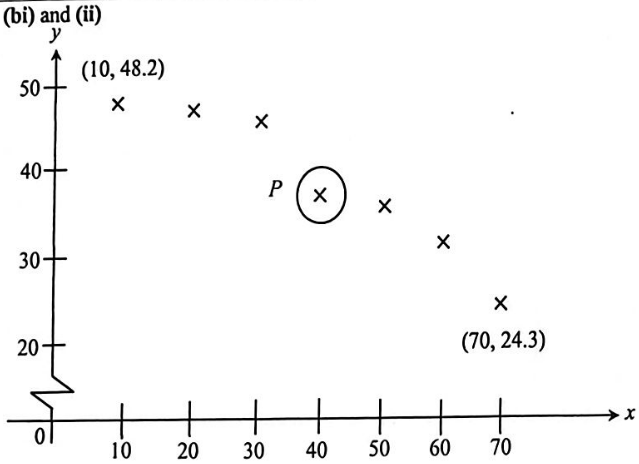
$\bar{u} = \dfrac{84 + a}{8}$

Since $(\bar{u}, \bar{g})$ lies on the regression line,

$3.075 = -0.0765\left(\dfrac{84 + a}{8}\right) + 3.99$

$a = 11.686 \approx 11.7$ (correct to 1 decimal place)

(bi) and (ii)



(iii) From the scatter diagram, as x increases, y decreases at an increasing rate. Hence a linear model is not a suitable model.

(iv) Using G.C,

$r_A = -0.9981$

$r_B = -0.8970$

Since r_A is closer to -1 than r_B, so model (A) is a better model than model (B).

From the G.C,

$y = -0.00511019x^2 + 49.2444$

$y = -0.00511x^2 + 49.2$ (3.s.f)

(v) When $x = 80$,

$y = -0.00511019(80)^2 + 49.2444 = 16.53916$

$y = 16.5$ (3.s.f)

(vi) The estimate is unreliable because the data substituted is outside the data range ($10 \le x \le 70$) and so the linear relationship between y and x^2 may not hold true.

11	**(a)** The leaves of a particular plant species have an average length of 12 cm with a standard deviation of 3.5 cm. If a random sample of 100 leaves is selected, estimate the probability that their total length is at least 1138 cm.	[2]
	(b) An operator of a public workspace at location A claims that users of its one-seater pods spend an average of 131 minutes using the facilities. To test this claim, a random sample of 64 users was observed, revealing a mean usage time of 127 minutes with a standard deviation of 16.4 minutes.	
	(i) Test at 3% level of significance whether the workspace operator's claim is overstated. You should state the hypotheses and define any symbols you use.	[5]
	(ii) Explain the meaning of 'at a 3% significance level' in the context of the question.	[1]
	(iii) The workspace operator at location B claims that the mean time spent by users of its one-seater pods is 140 minutes, with a known population standard deviation of 20.1 minutes. A new sample of 15 pod users is taken, and the sample mean usage time, $\overline{w}$, is reported. A hypothesis test is conducted at a 5% significance level, and the operator's claim is not rejected. State two necessary assumptions for the test and determine the range of values that $\overline{w}$ can take. Give your answer correct to one decimal place.	[5]

Commented [CSC44]: Question Reading: The question tells us that $n = 64$, **sample mean** = 127 and **sample variance** $= 16.4^2$.

	Solution	
	(a) Let X denote the length of a randomly chosen green leaf, in centimetres. Let L be the total lengths of 100 green leaves.	
	$L \sim N(12 \times 100, 3.5^2 \times 100)$ since $100 > 30$ (n is considered large), by Central Limit Theorem	
	$P(L \geq 1138) = 0.96175 \approx 0.962$ (3 sf)	
	(bi) Unbiased estimate of the population variance $s^2 = \frac{64}{64-1}(16.4^2) = 273.229 \approx 273.23$ (2 dp) (2 decimal places)	
	Let Y denote the time spent in minutes using the one-seater pod facilities by a randomly chosen user at location A and μ denote the population mean time spent in minutes using the one-seater pod facilities at location A.	
	To test H_0: $\mu = 131$ Against H_1: $\mu < 131$ (Workspace operator overstating the claim)	
	Conduct a one-tail test at 3% level of significance, i.e., $\alpha = 0.03$	
	Under H_0, Since $n = 64$ (> 30) is large, by Central Limit Theorem,	

Commented [CSC45]: Presentation: Students have to learn to define random variables accurately. Some students do not understand the distinction among random variables like X, $\overline{X}$, $100X$ etc.

Commented [CSC46]: Misconception: Many students thought that since n is large, Central Limit Theorem (CLT) applies and the "length" of a randomly chosen leaf will be normally distributed. However, CLT only approximates the distribution of **sample mean** or **sample sum** to be normal if sample size is large enough.

Commented [CSC47]: Since population variance is unknown, we should use the given sample variance to help us find an unbiased estimate for it using the formula

$$s^2 = \frac{n}{n-1}(\text{sample variance}).$$

Commented [CSC48]: Presentation: It is necessary to clearly define the random variable and any symbols (e.g. μ) used, as required by the question.

	$\bar{Y} \sim N\left(131, \frac{273.229}{64}\right)$ approximately.	
	$\bar{t} = 127$	
	Using GC, p-value = 0.026438 $\approx$ 0.0264 (3 sf)	
	Since p-value = 0.0264 < 0.03, we reject H_0. There is sufficient evidence at 3% level of significance to conclude that that the centre manager was overstating his claim.	
	(ii) There is a probability of 0.03 of concluding that the population average time spent using the one-seater pod facilities at location A is less than 131 minutes when in fact the population average time spent using the one-seater pod facilities in location A is 131 minutes.	
	(iii) Assume that the time spent by the users of the one-seater pods facilities in location B follows a Normal Distribution. Assume also that the time spent, on the one-seater pod facilities in location B by users, are independent of each other.	
	Let W denote the time spent in minutes using the one-seater pod facilities by a randomly chosen user at location B.	
	To test H_0: $\mu = 140$ Against H_1: $\mu \neq 140$ at 5% level of significance	
	Under H_0, $\bar{W} \sim N\left(140, \frac{20.1^2}{15}\right)$	
	Since H_0 is not rejected,	
	$-1.95996 < \frac{\bar{w}-140}{\left(\frac{20.1}{\sqrt{15}}\right)} < 1.95996$	
	$129.828 < \bar{w} < 150.1718$ $129.8 < \bar{w} < 150.2$ (1 d.p)	

Commented [CSC49]: Presentation: Students should be clear in their writing, bearing in mind the context of the question.

Commented [CSC50]: Misconception: A good number of students solve for $\frac{\bar{w}}{\frac{20.1}{\sqrt{15}}} > -1.95996$ or $\frac{\bar{w}}{\frac{20.1}{\sqrt{15}}} < 1.95996$ using "or" instead of "and", yet they can demonstrate that they are looking for the "intersection" by giving the correct final answer.

End of Paper