National Junior College
2025 – 2026 H2 Mathematics
Topic 1A Vectors I **Notes**

§1 Introduction

Key Questions:

☐ What are *scalars* and *vectors*?
☐ How can vectors be represented geometrically?
☐ What is the *magnitude* of a vector?
☐ What is a *zero vector*?
☐ What is a *unit vector*?
  ☐ How do we find the unit vector of a given vector?

1.1 Scalars and Vectors

What is a scalar?

Definition 1.1 (Scalar)
A *scalar* is a quantity with **magnitude but no direction**.

Examples of scalars Mass, distance, speed

What is a vector?

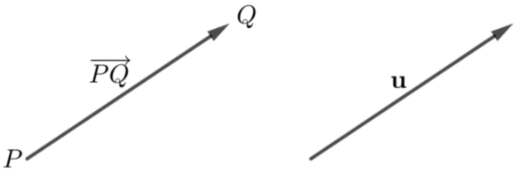
Definition 1.2 (Vector)
A *vector* is a quantity with **both magnitude and direction**.

Examples of vectors Force, displacement, velocity

1.2 Geometric Representations of Vectors

How can vectors be represented geometrically?

Geometrically, a vector can be represented by a **directed** line segment, where the arrowhead represents the **direction** of the vector.



The points P and Q are called the start and end points of the vector $\overrightarrow{PQ}$ respectively.

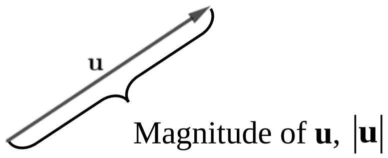
How are vectors denoted?

$\overrightarrow{PQ}$, $\mathbf{u}$ (in print form), $\underset{\sim}{u}$ (in written form)

1.3 Magnitude of a Vector

What is the magnitude of a vector?

Definition 1.3 (Magnitude)
The *magnitude* of a vector **u** is the **length** of the line segment that it is represented by $|\mathbf{u}|$.



How is the magnitude of a vector denoted?

$\left|\overrightarrow{PQ}\right|$, $|\mathbf{u}|$ (in print form), $|\underset{\sim}{u}|$ (in written form)

1.4 Zero Vectors and Unit Vectors

What is a zero vector?

Definition 1.4 (Zero Vector)
A *zero vector* is a vector whose magnitude is zero.

How is a zero vector denoted?

0 (in print form), $\underset{\sim}{0}$ (in written form)

What is a unit vector?

Definition 1.5 (Unit Vector)
A vector with magnitude 1 is called a *unit vector*.
The unit vector of a given vector **u** is the vector with the same direction as **u** and magnitude 1.

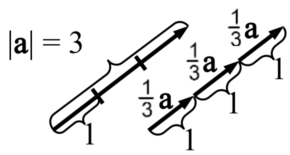
How is the unit vector of a given vector denoted?

The unit vector of a given vector **u** is denoted by $\hat{\mathbf{u}}$ (in print form) and $\underset{\sim}{\hat{u}}$ (in written form).

How do we find the unit vector of a given vector?

Example 1.1
Given that the vector **a** has magnitude 3, express the unit vector of **a** in the form $k\mathbf{a}$ for some scalar k.

Solution
Since the vector **a** has magnitude 3, one-third of **a** has magnitude 1 and shares the same direction as **a**, as shown in the diagram below.



Divide the vector by its magnitude.

Thus, the unit vector of **a** is

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{1}{3}\mathbf{a} .$$

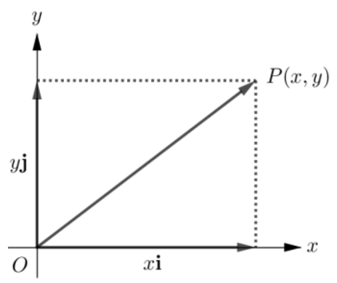
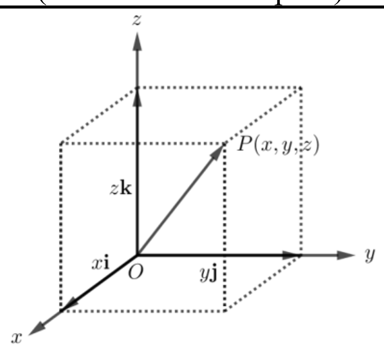
§2 Vectors in Two- and Three-Dimensions

Key Questions:

- ☐ How are 2D and 3D vectors expressed algebraically?
- ☐ What is a *position vector*?
- ☐ How do we add two vectors?
- ☐ How do we subtract one vector from another vector?
- ☐ What is a *displacement vector*?
- ☐ When are two non-zero vectors considered to be equal to each other?
- ☐ How do we multiply a vector by a scalar?
  - ☐ How do we show that two non-zero vectors are parallel?
  - ☐ How do we show that three distinct points are *collinear*?
- ☐ What are the laws of vector algebra?
- ☐ How are the magnitudes of 2D and 3D vectors calculated?
- ☐ When and how do we apply the *Ratio Theorem*?

2.1 Vectors in the Cartesian Plane & Euclidean 3D Space

How are 2D and 3D vectors expressed algebraically?

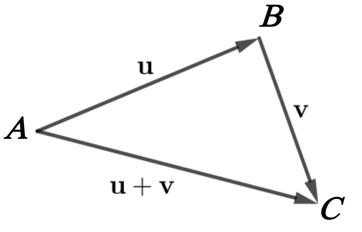
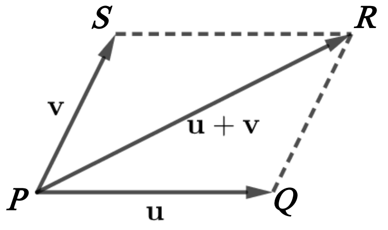
2D Vectors (in the Cartesian plane)	**3D Vectors** (in the Euclidean space)
	
$\overrightarrow{OP} = x\mathbf{i} + y\mathbf{j}$ or $\begin{pmatrix} x \\ y \end{pmatrix}$,	$\overrightarrow{OP} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ or $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$,
where **i** and **j** are the unit vectors in the positive *x*- and *y*- directions respectively, i.e.	where **i**, **j** and **k** are the unit vectors in the positive *x*-, *y*- and *z*- directions respectively, i.e.
$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$
The scalars *x*, *y* and *z* are called the *Cartesian components* of the vector $\overrightarrow{OP}$.	

What is a position vector?

Definition 2.1 (Position Vector)
With reference to a fixed point *O* (called the **origin**), the *position vector* of a point *P* relative to *O* is the vector $\overrightarrow{OP}$.

2.2 Vector Addition

How do we add two vectors geometrically?

Triangle Law of Vector Addition	Parallelogram Law of Vector Addition
	
$\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$	If $PQRS$ is a parallelogram, $\overrightarrow{PQ}+\overrightarrow{PS}=\overrightarrow{PR}$

How do we add two vectors algebraically?

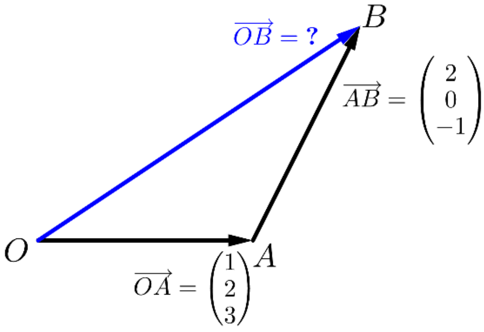
$$\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}+\begin{pmatrix}x_2\\y_2\\z_2\end{pmatrix}=\begin{pmatrix}x_1+x_2\\y_1+y_2\\z_1+z_2\end{pmatrix}$$

Example 2.1

Given that $\overrightarrow{OA}=\mathbf{i}+2\mathbf{j}+3\mathbf{k}$ and $\overrightarrow{AB}=2\mathbf{i}-\mathbf{k}$, find the vector $\overrightarrow{OB}$.

Solution

$$\overrightarrow{OB}=\overrightarrow{OA}+\overrightarrow{AB}$$
$$=\begin{pmatrix}1\\2\\3\end{pmatrix}+\begin{pmatrix}2\\0\\-1\end{pmatrix}$$
$$=\begin{pmatrix}1+2\\2+0\\3+(-1)\end{pmatrix}=\begin{pmatrix}3\\2\\2\end{pmatrix}$$

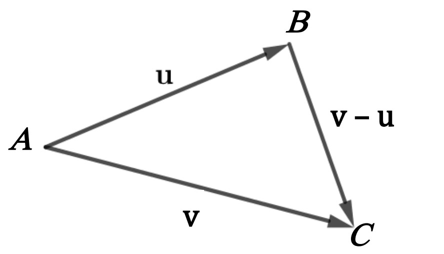


Example 2.2

Points O, A, B and P are such that $\overrightarrow{OA}=\mathbf{i}+4\mathbf{j}-3\mathbf{k}$, $\overrightarrow{OB}=5\mathbf{i}-\mathbf{j}$ and $OAPB$ is a parallelogram. Find $\overrightarrow{OP}$.

2.3 Vector Subtraction

How do we subtract one vector from another vector geometrically?



From the diagram,

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}.$$

Therefore,

$$\overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB}.$$

How do we subtract one vector from another vector algebraically?

Example 2.3

Given that $\overrightarrow{OA} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ and $\overrightarrow{OB} = 3\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$, find the vector $\overrightarrow{AB}$.

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} - \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \\ z_1 - z_2 \end{pmatrix}$$

What is a displacement vector?

See E.g. 2.3.

Definition 2.2 (Displacement Vector)

The *displacement vector* from point *A* to point *B* is the vector $\overrightarrow{AB}$ with start point *A* and end point *B*. In general,

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

2.4 Equality of Vectors

When are two non-zero vectors considered to be equal?

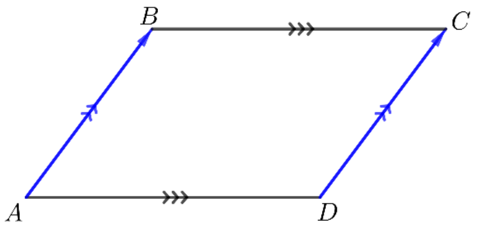
Definition 2.3 (Equality of Vectors)
Two non-zero vectors are said to be *equal* if they have the **same magnitude** and the **same direction.**

Example 2.4
The four distinct points A, B, C and D are such that

$$\overrightarrow{AB} = 3\mathbf{i} + p\mathbf{j} + 4\mathbf{k} \text{ and } \overrightarrow{DC} = (p+q)\mathbf{i} + (q-2)\mathbf{j} + (2p+q)\mathbf{k},$$

where p and q are real constants. Determine whether $ABCD$ can be a parallelogram.

Solution



For $ABCD$ to be a parallelogram, the vectors $\overrightarrow{AB}$ and $\overrightarrow{DC}$ must have the **same direction** and the **same magnitude**. Thus the two vectors have to be equal to each other. Hence

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \Leftrightarrow \begin{cases} x_1 = x_2 \\ y_1 = y_2 \\ z_1 = z_2 \end{cases}$$

$$\begin{pmatrix} 3 \\ p \\ 4 \end{pmatrix} = \begin{pmatrix} p+q \\ q-2 \\ 2p+q \end{pmatrix} \Rightarrow \begin{cases} 3 = p+q & (1) \\ p = q-2 & (2) \\ 4 = 2p+q & (3) \end{cases}$$

Solve two of the three equations simultaneously first.

(1) – (2): $3 - p = p + 2$

$2p = 1$

$p = \frac{1}{2}$.

Substituting $p = \frac{1}{2}$ into equation (2), $\frac{1}{2} = q - 2 \Rightarrow q = \frac{5}{2}$.

Substitute the values obtained into the LHS & RHS of the third equation **separately** to check if it is also satisfied.

Substituting $p = \frac{1}{2}$ and $q = \frac{5}{2}$ into equation (3),

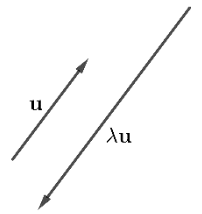
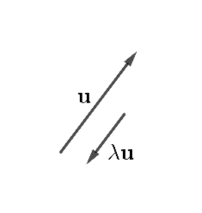
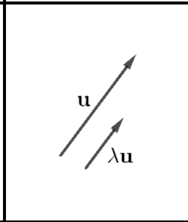
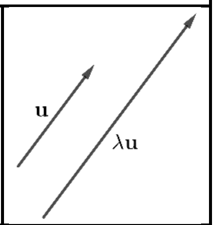
LHS of (3) $= 4$,

RHS of (3) $= 2\left(\frac{1}{2}\right) + \frac{5}{2} = \frac{7}{2} \neq$ LHS of (3).

Since there are no values of p and q that satisfy all three equations simultaneously, $ABCD$ cannot be a parallelogram.

2.5 Scalar Multiplication of a Vector

What does it mean to multiply a non-zero vector by a scalar?

<table>
<tr><td colspan="2">If $\lambda < 0$, then $\lambda\mathbf{u}$ is a vector with the opposite direction as u and $|\lambda|$ times its magnitude.</td><td colspan="2">If $\lambda > 0$, then $\lambda\mathbf{u}$ is a vector with the same direction as u and λ times its magnitude.</td></tr>
<tr><td></td><td></td><td></td><td></td></tr>
<tr><td>$\lambda < -1$</td><td>$-1 < \lambda < 0$</td><td>$0 < \lambda < 1$</td><td>$\lambda > 1$</td></tr>
</table>

How do we show that two non-zero vectors are parallel?

Result 2.4 (Parallel Vectors)
Suppose **a** and **b** are two non-zero vectors. Then

$$\mathbf{a} \text{ is parallel to } \mathbf{b} \iff \mathbf{b} = \lambda\mathbf{a} \text{ for some real scalar } \lambda.$$

Example 2.5
The vectors **a** and **b** are given by

$$\mathbf{a} = -3\mathbf{i} - \mathbf{j} + (1-p)\mathbf{k} \text{ and } \mathbf{b} = 6\mathbf{i} + (p+1)\mathbf{j} + p\mathbf{k},$$

where p is a real constant. Determine if **a** and **b** can be parallel vectors.

Apply $\mathbf{a} = \lambda\mathbf{b}$ to check if **a** and **b** can be parallel.

Solve two of the three equations simultaneously first.

Substitute the values obtained into the LHS & RHS of the third equation **separately** to check if it is also satisfied.

What is meant by collinearity and how do we show that 3 distinct points are collinear?

Definition 2.5 (Collinearity)
Two or more **points** are said to be *collinear* if they all lie on a common straight line.

Three **distinct** points A, B and C are collinear if and only if

- $\overrightarrow{AB} = \lambda \overrightarrow{BC}$ for some real scalar λ
- with a common point (in this case B).

Example 2.6
Relative to the origin O, the points A, B and C have position vectors given by

$$2\mathbf{i}+(2p-1)\mathbf{k},\ -2\mathbf{i}-2\mathbf{j}+\mathbf{k} \text{ and } 4\mathbf{i}+\mathbf{j}+(3p-2)\mathbf{k}$$

respectively, where p is a real constant. Prove that the points A, B and C are collinear.

Solution

Find a pair of displacement vectors from the 3 points.

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$
$$= \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ 2p-1 \end{pmatrix}$$
$$= \begin{pmatrix} -4 \\ -2 \\ 2-2p \end{pmatrix}$$

and

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$$
$$= \begin{pmatrix} 4 \\ 1 \\ 3p-2 \end{pmatrix} - \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}$$

Express one vector as a scalar multiple of the other vector.

$$= \begin{pmatrix} 6 \\ 3 \\ 3p-3 \end{pmatrix} = -\frac{3}{2}\begin{pmatrix} -4 \\ -2 \\ 2-2p \end{pmatrix} = -\frac{3}{2}\overrightarrow{AB},$$

Conclude that the 2 vectors are parallel and state the common point.

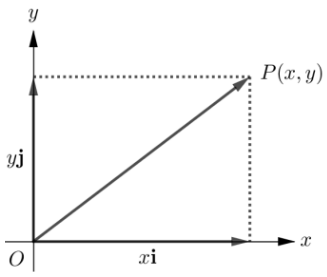
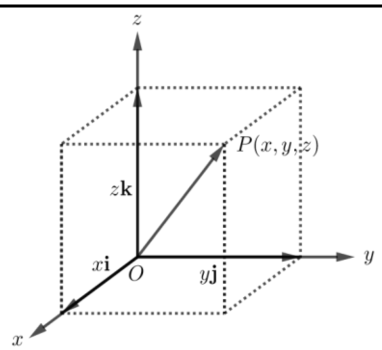
Since $\overrightarrow{BC} = -\frac{3}{2}\overrightarrow{AB}$, $\overrightarrow{AB}$ **is parallel to** $\overrightarrow{BC}$ with **common point *B***. Thus A, B and C are collinear.

2.6 Laws of Vector Algebra

What are the laws of vector algebra?

Result 2.6 (Laws of Vector Algebra) For any real scalars λ and μ, and vectors **a**, **b** and **c**,	
Commutative Law	$\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$
Associative Law (for vector addition)	$(\mathbf{a}+\mathbf{b})+\mathbf{c}=\mathbf{a}+(\mathbf{b}+\mathbf{c})$
Distributive Laws	$\lambda(\mathbf{a}+\mathbf{b})=\lambda\mathbf{a}+\lambda\mathbf{b}$
	$(\lambda+\mu)\mathbf{a}=\lambda\mathbf{a}+\mu\mathbf{a}$
Associative Law (for scalar multiplication)	$(\lambda\mu)\mathbf{a}=\lambda(\mu\mathbf{a})$
Zero Vector	$\mathbf{a}+\mathbf{0}=\mathbf{a}=\mathbf{0}+\mathbf{a}$
Negative Vector	$-\mathbf{a}+\mathbf{a}=\mathbf{0}=\mathbf{a}+(-\mathbf{a})$
Magnitude	$\lvert\lambda\mathbf{a}\rvert=\lvert\lambda\rvert\lvert\mathbf{a}\rvert$

2.7 Finding the Magnitudes of 2D and 3D Vectors

2D Vectors	**3D Vectors**
	
$\left\lvert\overrightarrow{OP}\right\rvert=\left\lvert\begin{pmatrix}x\\y\end{pmatrix}\right\rvert=\sqrt{x^2+y^2}$	$\left\lvert\overrightarrow{OP}\right\rvert=\left\lvert\begin{pmatrix}x\\y\\z\end{pmatrix}\right\rvert=\sqrt{x^2+y^2+z^2}$

Example 2.7

The vectors **a** and **b** are given by

$$\mathbf{a}=\begin{pmatrix}4p-4\\3-3p\\12p-12\end{pmatrix}\text{ and }\mathbf{b}=\begin{pmatrix}2-p\\2p-4\\4-2p\end{pmatrix},$$

where p is a real constant. Find the possible value(s) of p

(a) if **a** is a unit vector,

(b) if instead $|\mathbf{a}|=|\mathbf{b}|$.

Give your answers in exact form.

Solution

If there is a common factor among the components, factor it out first.

(a) $$\mathbf{a}=\begin{pmatrix}4p-4\\3-3p\\12p-12\end{pmatrix}=\begin{pmatrix}4(p-1)\\-3(p-1)\\12(p-1)\end{pmatrix}=(p-1)\begin{pmatrix}4\\-3\\12\end{pmatrix}$$

Recall that a unit vector is a vector with magnitude 1.

Since **a** is a unit vector, $|\mathbf{a}|=1$. Thus

$$\left|(p-1)\begin{pmatrix}4\\-3\\12\end{pmatrix}\right|=1$$

$|\lambda\mathbf{a}|=|\lambda||\mathbf{a}|$

$$|(p-1)|\left|\begin{pmatrix}4\\-3\\12\end{pmatrix}\right|=1$$

$\left|\begin{pmatrix}x\\y\\z\end{pmatrix}\right|=\sqrt{x^2+y^2+z^2}$

$$|p-1|\sqrt{4^2+3^2+12^2}=1$$

$$13|p-1|=1$$

$$|p-1|=\frac{1}{13}$$

$|a|=b\Rightarrow\begin{cases}a=b\\\text{or}\\a=-b\end{cases}$

$$p-1=\frac{1}{13}\text{ or }-\frac{1}{13}$$

$$p=\frac{14}{13}\text{ or }\frac{12}{13}.$$

If there is a common factor among the components, factorise it out first to simplify subsequent calculations.

$$|\lambda \mathbf{a}| = |\lambda||\mathbf{a}|$$

$$\left|\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right| = \sqrt{x^2 + y^2 + z^2}$$

$$|a| = b \Rightarrow \begin{cases} a = b \\ \text{or} \\ a = -b \end{cases}$$

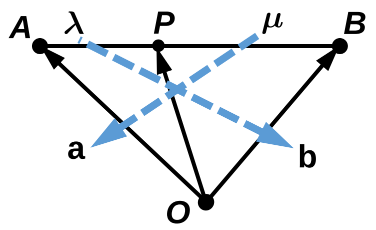
2.8 Ratio Theorem

What is the Ratio Theorem?

Result 2.7 (Ratio Theorem) [in MF26]

The point P dividing AB in the ratio $\lambda : \mu$ has position vector

$$\overrightarrow{OP} = \frac{\mu\mathbf{a} + \lambda\mathbf{b}}{\lambda + \mu}.$$



When do we apply the Ratio Theorem?

The Ratio Theorem is typically applied when we are given the position vectors of two points, say A and B, and we are required to find the position vector of a third point P, where A, B and P are collinear.

Proof of the Ratio Theorem

$$\left|\overrightarrow{AP}\right| : \left|\overrightarrow{PB}\right| = \lambda : \mu$$

$$\Rightarrow \frac{\left|\overrightarrow{AP}\right|}{\left|\overrightarrow{PB}\right|} = \frac{\lambda}{\mu}$$

$$\Rightarrow \mu\left|\overrightarrow{AP}\right| = \lambda\left|\overrightarrow{PB}\right|$$

Since $\overrightarrow{AP}$ is in the same direction as $\overrightarrow{PB}$,

$$\mu\,\overrightarrow{AP} = \lambda\,\overrightarrow{PB}.$$

Finding displacement vector from position vectors (see Definition 2.2)

Thus,

$$\mu\left(\overrightarrow{OP} - \overrightarrow{OA}\right) = \lambda\left(\overrightarrow{OB} - \overrightarrow{OP}\right)$$

$$(\lambda + \mu)\overrightarrow{OP} = \mu\,\overrightarrow{OA} + \lambda\,\overrightarrow{OB}$$

$$\overrightarrow{OP} = \frac{\mu\,\overrightarrow{OA} + \lambda\,\overrightarrow{OB}}{\lambda + \mu}$$

Example 2.8

The points A and B have position vectors $\mathbf{i}+\mathbf{j}+\mathbf{k}$ and $4\mathbf{i}+2\mathbf{j}+\mathbf{k}$ respectively, with reference to an origin O. The point C lies on the line segment AB such that $AC:CB=2:1$. Find the position vector of C.

Draw a vector diagram depicting the scenario.

Apply the Ratio Theorem.

Example 2.9

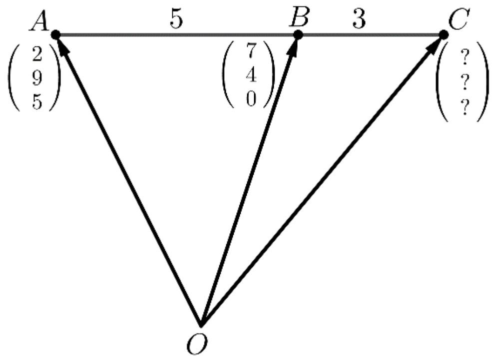
The points *A* and *B* have position vectors $2\mathbf{i}+9\mathbf{j}+5\mathbf{k}$ and $7\mathbf{i}+4\mathbf{j}$ respectively, with reference to an origin *O*. The point *C* is such that *B* lies on the line segment *AC* and $AC:CB=8:3$. Find the position vector of *C*.

Solution

Since *B* lies between *A* and *C*, calculate the ratio $AB:BC$ first.

Draw a vector diagram depicting the scenario.

$AB:BC=5:3$



By the Ratio Theorem,

Apply the Ratio Theorem.

$$\overrightarrow{OB}=\frac{3\overrightarrow{OA}+5\overrightarrow{OC}}{5+3}$$

$$\begin{pmatrix}7\\4\\0\end{pmatrix}=\frac{3\begin{pmatrix}2\\9\\5\end{pmatrix}+5\overrightarrow{OC}}{8}$$

Make $\overrightarrow{OC}$ the subject in the equation.

$$5\overrightarrow{OC}=8\begin{pmatrix}7\\4\\0\end{pmatrix}-3\begin{pmatrix}2\\9\\5\end{pmatrix}$$

$$=\begin{pmatrix}50\\5\\-15\end{pmatrix}$$

$$\overrightarrow{OC}=\frac{1}{5}\begin{pmatrix}50\\5\\-15\end{pmatrix}$$

$$=\begin{pmatrix}10\\1\\-3\end{pmatrix}$$

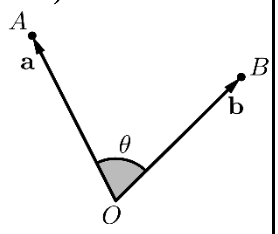
§3 Scalar (or Dot) Product

Key Questions:
- ☐ What is the *scalar product* of two vectors and how do we calculate it?
- ☐ What are the laws of scalar product?
- ☐ How do we use the scalar product to
  - ☐ find the angle between two vectors?
  - ☐ determine whether two vectors are perpendicular to each other?

3.1 Definition of Scalar (or Dot) Product

How do we define the angle between two vectors?

Definition 3.1 (Angle between Two Vectors)
Suppose **a** and **b** are the position vectors of two points A and B respectively. Then the angle between **a** and **b**, θ, is given by



$$\theta = \angle AOB.$$

By convention, we take the **non-reflex** angle i.e., $0° \le \theta \le 180°,$ unless otherwise specified.

Example 3.1
In triangle ABC, $\angle BAC$ = 30°. Find the angle between the vectors

(i) $\overrightarrow{AB}$ and $\overrightarrow{AC}$,

(ii) $\overrightarrow{BA}$ and $\overrightarrow{AC}$.

Draw a diagram to illustrate the scenario.

The angle should be at where the start-points (or end-points) of the two vectors **coincide**.

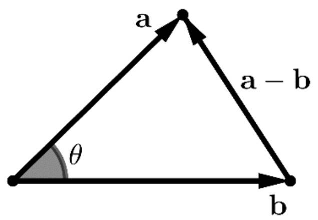
If neither the start-points nor the end-points of the given vectors coincide, construct a vector that is **equal** to the one of the two vectors so that they do.

What is the scalar product of two vectors?

Definition 3.2 (Scalar or Dot Product)	
Algebraic Definition	**Geometrical Definition**
The *scalar product* of two vectors $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$ and $\mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$ is $\mathbf{a}\cdot\mathbf{b}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\cdot\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$ $=a_1b_1+a_2b_2+a_3b_3.$	Let θ be the angle between two vectors **a** and **b**. The *scalar product* of **a** and **b** is $\mathbf{a}\cdot\mathbf{b}=\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\theta.$

Proof of Equivalence between Algebraic and Geometric Definitions of Scalar Product

Let $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$, $\mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$ and θ be the angle between **a** and **b**.



By the Cosine Rule,

$$\lvert\mathbf{a}-\mathbf{b}\rvert^2=\lvert\mathbf{a}\rvert^2+\lvert\mathbf{b}\rvert^2-2\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\theta$$

$$\Rightarrow (a_1-b_1)^2+(a_2-b_2)^2+(a_3-b_3)^2$$

$$=a_1{}^2+a_2{}^2+a_3{}^2+b_1{}^2+b_2{}^2+b_3{}^2-2\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\theta$$

$$\Rightarrow \cancel{a_1^2}-2a_1b_1+\cancel{b_1^2}+\cancel{a_2^2}-2a_2b_2+\cancel{b_2^2}+\cancel{a_3^2}-2a_3b_3+\cancel{b_3^2}$$

$$=\cancel{a_1^2}+\cancel{a_2^2}+\cancel{a_3^2}+\cancel{b_1^2}+\cancel{b_2^2}+\cancel{b_3^2}-2\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\theta$$

$$\Rightarrow -2a_1b_1-2a_2b_2-2a_3b_3=-2\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\theta$$

$$\Rightarrow -2\left(a_1b_1+a_2b_2+2a_3b_3\right)=-2\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\theta$$

$$\Rightarrow a_1b_1+a_2b_2+a_3b_3=\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\theta \text{ (shown).}$$

3.2 Laws of Scalar Product

Example 3.2

Given that $\mathbf{a}=\begin{pmatrix}1\\1\\-1\end{pmatrix}$, $\mathbf{b}=\begin{pmatrix}1\\1\\1\end{pmatrix}$ and $\mathbf{c}=\begin{pmatrix}-1\\1\\-1\end{pmatrix}$, find

(i) $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})$,

(ii) $\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}$.

What are the laws of scalar product?

Result 3.3 (Laws of Scalar Product)
For any real scalar λ and vectors **a**, **b** and **c**,

Commutative Law	$\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a}$
Associative Law	$\mathbf{a}\cdot(\lambda\mathbf{b})=\lambda(\mathbf{a}\cdot\mathbf{b})=(\lambda\mathbf{a})\cdot\mathbf{b}$
Distributive Laws	$\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})=\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}$
	$(\mathbf{b}+\mathbf{c})\cdot\mathbf{a}=\mathbf{b}\cdot\mathbf{a}+\mathbf{c}\cdot\mathbf{a}$
Magnitude	$\lvert\mathbf{a}\rvert^2=\mathbf{a}\cdot\mathbf{a}$

Proof of the Magnitude Law of Scalar Product using the…

Algebraic Definition	Geometrical Definition
Let $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$. Then $\mathbf{a}\cdot\mathbf{a}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\cdot\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$ $={a_1}^2+{a_2}^2+{a_3}^2=\lvert\mathbf{a}\rvert^2$.	The angle between any vector **a** and itself is 0°. Thus $\mathbf{a}\cdot\mathbf{a}=\lvert\mathbf{a}\rvert\lvert\mathbf{a}\rvert\cos 0°$ $=\lvert\mathbf{a}\rvert^2\times 1$ $=\lvert\mathbf{a}\rvert^2$.

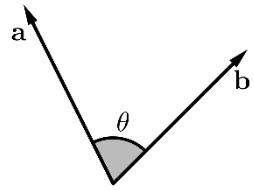
3.3 Using the Scalar Product to Find the Angle between Two Vectors

How do we use the scalar product to find the angle between two vectors?

Result 3.4 (Finding Angle between Two Vectors)

Let **a** and **b** be two vectors and θ be the angle between **a** and **b**. Then

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}.$$



Example 3.3

Relative to an origin O, A and B have position vectors $-\mathbf{i}+3\mathbf{j}-4\mathbf{k}$ and $4\mathbf{i}+3\mathbf{j}-5\mathbf{k}$ respectively.

(i) Find the angle between $\overrightarrow{OA}$ and $\overrightarrow{AB}$.

(ii) Find the size of angle OAB.

Solution

(i)

Apply the formula $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} \Rightarrow$ $\theta = \cos^{-1}\left(\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\right)$ to find the required angle.

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 5 \\ 0 \\ -1 \end{pmatrix}.$$

Angle between $\overrightarrow{OA}$ and $\overrightarrow{AB}$

$$= \cos^{-1}\frac{\begin{pmatrix} -1 \\ 3 \\ -4 \end{pmatrix}\cdot\begin{pmatrix} 5 \\ 0 \\ -1 \end{pmatrix}}{\sqrt{(-1)^2+3^2+(-4)^2}\sqrt{5^2+(-1)^2}}$$

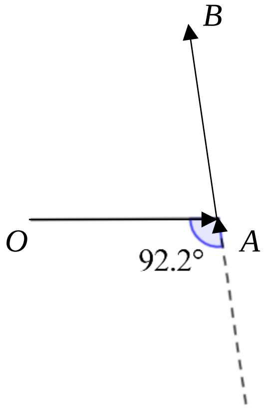
$$= \cos^{-1}\left(\frac{-5+0+4}{\sqrt{26}\sqrt{26}}\right)$$

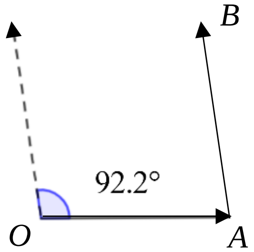
$$= \cos^{-1}\left(\frac{-1}{26}\right)$$

$$= 92.2^\circ \text{ (to 1 d.p.).}$$

How can we interpret our solution using a diagram?

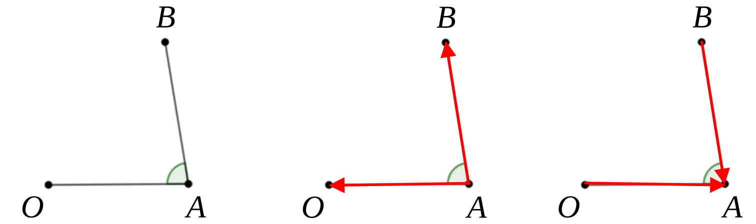
In order to identify the angle between the two vectors, arrange the two vectors such that they meet 'head-to-head' or 'tail-to-tail'.





(ii) Identify an appropriate pair of vectors.

Draw a diagram to illustrate the scenario.



Identify the two vectors for which the angle is between and calculate them.

From the diagram, angle OAB is the angle between the vectors $\overrightarrow{AO}$ and $\overrightarrow{AB}$. (Alternatively, we can also use $\overrightarrow{OA}$ and $\overrightarrow{BA}$.)

The direction of the two vectors must be chosen such that the two vectors meet 'head-to-head' or 'tail-to-tail'.

$$\overrightarrow{AO} = -\overrightarrow{OA} = \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}.$$

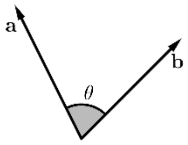
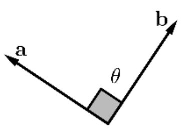
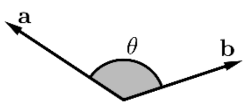
$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 5 \\ 0 \\ -1 \end{pmatrix}.$$

Apply the formula $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$ to find the required angle.

$$\cos\angle OAB = \frac{\begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}\cdot\begin{pmatrix} 5 \\ 0 \\ -1 \end{pmatrix}}{\sqrt{1^2+(-3)^2+4^2}\sqrt{5^2+(-1)^2}}$$
$$= \frac{5+0-4}{\sqrt{26}\sqrt{26}}$$
$$= \frac{1}{26}.$$

$$\angle OAB = 87.8^\circ \text{ (to 1 d.p.)}.$$

What is the geometric significance of the sign of the scalar product of two vectors?

Let $\mathbf{a}$ and $\mathbf{b}$ be two **non-zero** vectors and θ be the angle between $\mathbf{a}$ and $\mathbf{b}$.		
If $\mathbf{a}\cdot\mathbf{b} > 0$,	If $\mathbf{a}\cdot\mathbf{b} = 0$,	If $\mathbf{a}\cdot\mathbf{b} < 0$,
$\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\theta > 0$ $\cos\theta > 0$ (since $\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert > 0$) $\therefore\ \theta$ is acute 	$\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\theta = 0$ $\cos\theta = 0$ (since $\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert > 0$) $\therefore\ \theta = 90^\circ$ 	$\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\theta < 0$ $\cos\theta < 0$ (since $\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert > 0$) $\therefore\ \theta$ is obtuse 

3.4 Relationship between Scalar Product and Perpendicular Vectors

What is the geometrical significance of $\mathbf{a}\cdot\mathbf{b}=0$ *?*

Result 3.5 (Test for Perpendicularity of Vectors)
Let **a** and **b** be two vectors. Then

$$\mathbf{a}\cdot\mathbf{b}=0 \Leftrightarrow \begin{cases}\mathbf{a}=\mathbf{0} & \text{OR}\\ \mathbf{b}=\mathbf{0} & \text{OR}\\ \mathbf{a} \text{ and } \mathbf{b} \text{ are } \textbf{perpendicular}. \end{cases}$$

Example 3.4
Referred to the origin O, the position vectors of the points A and B are $\mathbf{i}-\mathbf{j}+2\mathbf{k}$ and $2\mathbf{i}+4\mathbf{j}+\mathbf{k}$ respectively. Show that $\overrightarrow{OA}$ is perpendicular to $\overrightarrow{OB}$.

Solution

Prove that

$\mathbf{a}\cdot\mathbf{b}=0$

to show that **a** and **b** are perpendicular.

$$\begin{aligned}\overrightarrow{OA}\cdot\overrightarrow{OB} &= \begin{pmatrix}1\\-1\\2\end{pmatrix}\cdot\begin{pmatrix}2\\4\\1\end{pmatrix}\\ &= 2-4+2\\ &= 0\end{aligned}$$

Since $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are non-zero vectors and $\overrightarrow{OA}\cdot\overrightarrow{OB}=0$, $\overrightarrow{OA}$ is perpendicular to $\overrightarrow{OB}$ (shown).

Example 3.5
Let **a** and **b** be two non-zero and non-parallel vectors with the same magnitude. Show that

$$(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})=0.$$

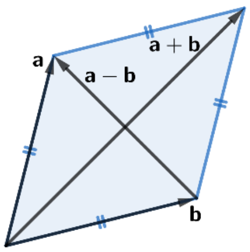
Explain the geometrical significance of this result.

Solution

Apply the laws of scalar product to expand and simplify the vector expression.

$$\begin{aligned}&(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})\\ &=\mathbf{a}\cdot(\mathbf{a}-\mathbf{b})+\mathbf{b}\cdot(\mathbf{a}-\mathbf{b}) && \text{(distributive law)}\\ &=\mathbf{a}\cdot\mathbf{a}-\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{a}-\mathbf{b}\cdot\mathbf{b} && \text{(distributive law)}\\ &=\mathbf{a}\cdot\mathbf{a}-\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{b}-\mathbf{b}\cdot\mathbf{b} && \text{(commutative law)}\\ &=\mathbf{a}\cdot\mathbf{a}-\mathbf{b}\cdot\mathbf{b}\\ &=|\mathbf{a}|^2-|\mathbf{b}|^2 && \text{(magntitude law)}\\ &=0 && (\because |\mathbf{a}|=|\mathbf{b}|)\end{aligned}$$

Draw a diagram depicting the scenario in order to infer the geometrical significance.



Since **a** and **b** are two non-parallel vectors with the same magnitude, $\mathbf{a}+\mathbf{b}$ and $\mathbf{a}-\mathbf{b}$ form the diagonals of a rhombus with sides **a** and **b**. Thus the geometrical significance of the result is that the diagonals of a rhombus are always perpendicular to each other.

Example 3.6

The points A and B have position vectors $3\mathbf{i}+\mathbf{k}$ and $\mathbf{j}-4\mathbf{k}$ respectively, with respect to an origin O. Another point C lies on the x-axis such that angle ACB is a right angle. Find the possible position vector(s) of C.

Apply the result that

$$\mathbf{a}\cdot\mathbf{b}=0$$

for two perpendicular vectors **a** and **b**.

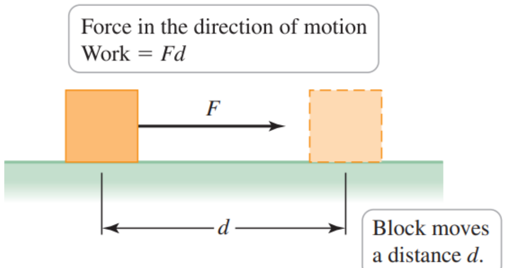
Big Idea : Models

Vectors can be used to model physical quantities such as force and displacement.

The **work** done by a force is a measurement of the energy transfer that occurs when a force causes a displacement of an object.

Displacement of object in the direction of the force

If a constant force F displaces an object a distance d in the direction of the force, what is the work W done? This is represented in diagram (a) below.



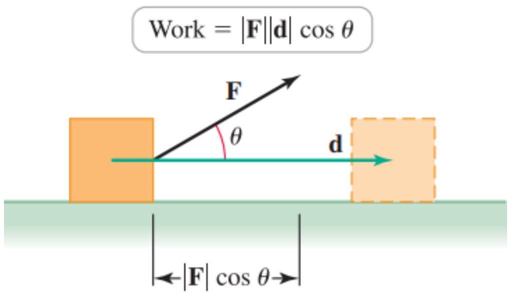
(a)

The work done is the force multiplied by the distance,

$$\text{Work} = \text{force} \times \text{distance i.e. } W = Fd\,.$$

Displacement of object not in the direction of the force

Now suppose that the force is a vector **F** applied at an angle θ to the direction of motion; the resulting displacement of the object is a vector **d**. What is the work done? This is represented in diagram (b) below.



(b)

In this case, the work done by the force is the component of the force in the direction of motion multiplied by the distance moved by the object,

$$W = \left(|\mathbf{F}|\cos\theta\right)|\mathbf{d}|\,.$$

You might recall that we call this product of the magnitudes of two vectors and the cosine of the angle between them the *dot product*. So, the formula above may be written simply as

$$W = \mathbf{F}\cdot\mathbf{d}\,.$$

§4 Vector (or Cross) Product

Key Questions:

☐ What is the *vector product* of two vectors and how do we calculate it?

☐ What are the laws of vector product?

☐ How do we use the vector product to find the area of a

- ☐ triangle?
- ☐ parallelogram?

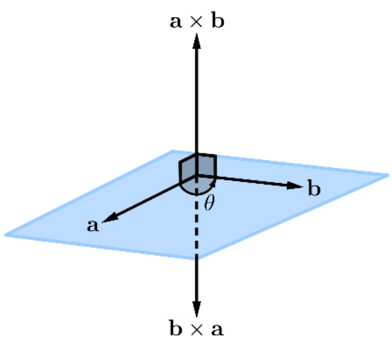
4.1 Definition of Vector (or Cross) Product

What is the vector product of two vectors?

Definition 4.1 (Vector or Cross Product)

Algebraic Definition	**Geometrical Definition**
The *vector product* of two vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ is $\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$ **(in MF26)**	Let θ be the angle between two vectors **a** and **b**. The *vector product* of **a** and **b** is $\mathbf{a} \times \mathbf{b} = \left(\lvert\mathbf{a}\rvert \lvert\mathbf{b}\rvert \sin\theta \right) \hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is a unit vector perpendicular to both **a** and **b**.

Using your **right** hand, curl your fingers (other than your thumb) inwards, first "cutting" through **a** before reaching **b**. Straighten your thumb. Then $\mathbf{a} \times \mathbf{b}$ is in the direction of your thumb, while $\mathbf{b} \times \mathbf{a}$ is in the opposite direction of $\mathbf{a} \times \mathbf{b}$.



Example 4.1

Given that $\mathbf{a}=\begin{pmatrix}3\\-4\\1\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}0\\2\\-1\end{pmatrix}$, find

(i) $\mathbf{a}\times\mathbf{b}$,

(ii) $\mathbf{b}\times\mathbf{a}$.

Find two unit vectors perpendicular to both **a** and **b**.

Solution

Apply the formula $\begin{pmatrix}a_2b_3-a_3b_2\\a_3b_1-a_1b_3\\a_1b_2-a_2b_1\end{pmatrix}$ to find the cross product of the two vectors.

(i) $\mathbf{a}\times\mathbf{b}=\begin{pmatrix}3\\-4\\1\end{pmatrix}\times\begin{pmatrix}0\\2\\-1\end{pmatrix}$

$$=\begin{pmatrix}(-1)(-4)-(2)(1)\\(0)(1)-(-1)(3)\\(2)(3)-(0)(-4)\end{pmatrix}$$

$$=\begin{pmatrix}2\\3\\6\end{pmatrix}$$

The cross product of two vectors is a vector that is perpendicular to both of them.

Use $\hat{\mathbf{n}}=\dfrac{\mathbf{n}}{|\mathbf{n}|}$ to find the unit vector of **n**.

Example 4.2

Show that the magnitude of the vector $\mathbf{a}\times\mathbf{b}$ is $|\mathbf{a}||\mathbf{b}|\sin\theta$.

Solution

$$\begin{aligned}|\mathbf{a}\times\mathbf{b}| &= \left|\left(|\mathbf{a}||\mathbf{b}|\sin\theta\right)\hat{\mathbf{n}}\right| && \text{(by geometric definition)}\\ &= |\mathbf{a}||\mathbf{b}||\sin\theta||\hat{\mathbf{n}}| && \left(|\lambda\mathbf{u}| = |\lambda||\mathbf{u}|\right)\\ &= |\mathbf{a}||\mathbf{b}||\sin\theta| && \left(\hat{\mathbf{n}}\text{ is a unit vector} \Rightarrow |\hat{\mathbf{n}}| = 1\right)\\ &= |\mathbf{a}||\mathbf{b}|\sin\theta && \left(\sin\theta \geq 0 \text{ since } 0^\circ \leq \theta \leq 180^\circ\right)\end{aligned}$$

4.2 Laws of Vector Product

What are the laws of vector product?

Result 4.2 (Laws of Vector Product)

For any real scalar λ and vectors **a**, **b** and **c**,

Anticommutative Law	$\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$
Associative Law	$\mathbf{a}\times(\lambda\mathbf{b}) = \lambda(\mathbf{a}\times\mathbf{b}) = (\lambda\mathbf{a})\times\mathbf{b}$
Distributive Laws	$\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}$
	$(\mathbf{b}+\mathbf{c})\times\mathbf{a} = \mathbf{b}\times\mathbf{a}+\mathbf{c}\times\mathbf{a}$
	$\mathbf{a}\times\mathbf{a} = \mathbf{0}$

Example 4.3

Let **a** and **b** be two non-zero and non-parallel vectors. Show that

$$(2\mathbf{a}+3\mathbf{b})\times(\mathbf{a}+4\mathbf{b}) = k(\mathbf{a}\times\mathbf{b})$$

for some real constant k to be determined.

Solution

Apply the laws of vector product to expand and simplify the vector expression.

$$\begin{aligned}&(2\mathbf{a}+3\mathbf{b})\times(\mathbf{a}+4\mathbf{b})\\ &= 2\mathbf{a}\times(\mathbf{a}+4\mathbf{b})+3\mathbf{b}\times(\mathbf{a}+4\mathbf{b})\\ &= 2(\mathbf{a}\times\mathbf{a})+8(\mathbf{a}\times\mathbf{b})+3(\mathbf{b}\times\mathbf{a})+12(\mathbf{b}\times\mathbf{b})\\ &= \mathbf{0}+8(\mathbf{a}\times\mathbf{b})-3(\mathbf{a}\times\mathbf{b})+\mathbf{0}\\ &= 5(\mathbf{a}\times\mathbf{b}) \text{ (shown)}\end{aligned}$$

4.3 Relationship between Vector Product and Parallel Vectors

What is the geometrical significance of $\mathbf{a}\times\mathbf{b}=\mathbf{0}$*?*

Result 4.3 (Vector Product of Parallel Vectors)
Let **a** and **b** be two vectors. Then

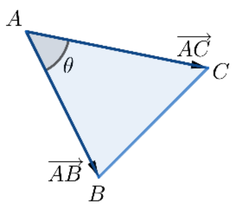
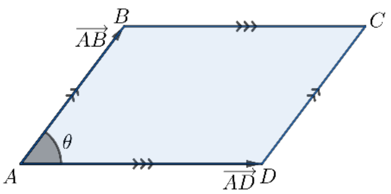
$$\mathbf{a}\times\mathbf{b}=\mathbf{0} \Leftrightarrow \begin{cases} \mathbf{a}=\mathbf{0} & \text{OR} \\ \mathbf{b}=\mathbf{0} & \text{OR} \\ \mathbf{a} \text{ and } \mathbf{b} \text{ are } \textbf{parallel}. \end{cases}$$

Proof of Result 4.3
Suppose **a** and **b** are two non-zero vectors and θ be the angle between them. Then

$$\begin{aligned} \mathbf{a}\times\mathbf{b}=\mathbf{0} &\Leftrightarrow |\mathbf{a}\times\mathbf{b}|=0 \\ &\Leftrightarrow |\mathbf{a}||\mathbf{b}|\sin\theta=0 \\ &\Leftrightarrow \sin\theta=0 \quad (\because |\mathbf{a}|\neq 0 \text{ and } |\mathbf{b}|\neq 0) \\ &\Leftrightarrow \theta=0^\circ \text{ or } \theta=180^\circ \\ &\Leftrightarrow \mathbf{a} \text{ and } \mathbf{b} \text{ are parallel.} \end{aligned}$$

4.4 Using Vector Product to find Areas of Triangles and Parallelograms

How do we use the vector product to find the area of triangles and parallelograms?

Triangle	Parallelogram
	
Area $=\frac{1}{2}\left\|\overrightarrow{AB}\times\overrightarrow{AC}\right\|$	Area $=\left\|\overrightarrow{AB}\times\overrightarrow{AD}\right\|$

Proof of Vector Product formula for Area of Triangle

$$\begin{aligned} \text{Area of } \Delta ABC &= \frac{1}{2}\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|\sin\theta \\ &= \frac{1}{2}\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right||\sin\theta||\hat{\mathbf{n}}|, \end{aligned}$$

where $\hat{\mathbf{n}}$ is a unit vector perpendicular to $\overrightarrow{AB}$ & $\overrightarrow{AC}$

$$\begin{aligned} &= \frac{1}{2}\left|\left(\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|\sin\theta\right)\hat{\mathbf{n}}\right| \\ &= \frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right| \text{ (shown).} \end{aligned}$$

Observe that
$0^\circ\le\theta\le180^\circ$
$\Rightarrow \sin\theta\ge 0$
$\Rightarrow |\sin\theta|=\sin\theta.$

Example 4.4

Referred to the origin O, the position vectors of the points A and B are $\mathbf{i}-\mathbf{j}+\mathbf{k}$ and $2\mathbf{i}+4\mathbf{j}+2\mathbf{k}$ respectively. Find the exact area of triangle OAB.

Solution

Apply the vector product formula for finding area of triangle.

$$\text{Area of } \Delta OAB = \frac{1}{2}\left|\overrightarrow{OA}\times\overrightarrow{OB}\right|$$

$$= \frac{1}{2}\left|\begin{pmatrix}1\\-1\\1\end{pmatrix}\times\begin{pmatrix}2\\4\\2\end{pmatrix}\right|$$

$$= \frac{1}{2}\left|\begin{pmatrix}(-1)(2)-(4)(1)\\(2)(1)-(1)(2)\\(1)(4)-(2)(-1)\end{pmatrix}\right|$$

$$= \frac{1}{2}\left|\begin{pmatrix}-6\\0\\6\end{pmatrix}\right|$$

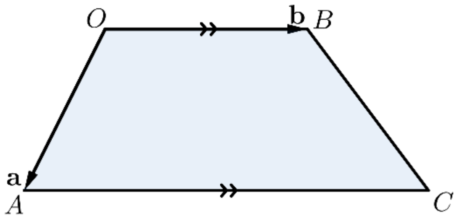
$$= \frac{1}{2}\sqrt{(-6)^2+0^2+6^2}$$

$$= \frac{1}{2}\sqrt{72}$$

$$= 3\sqrt{2} \text{ units}^2$$

Example 4.5

Referred to an origin *O*, the points *A* and *B* have position vectors **a** and **b** respectively. The trapezium *OACB* has parallel sides *OB* and *AC*, and it is given that *AC* = λ*OB*, where λ is a real scalar such that $\lambda > 1$.

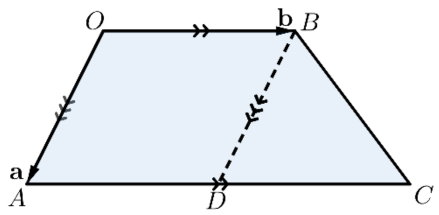


Show that the area of the trapezium *OACB* can be expressed as $k\left|\mathbf{a}\times\mathbf{b}\right|$, for some constant *k* to be determined in terms of λ.

Solution

Since the quadrilateral is not a parallelogram in this case, the shape needs to be split into smaller parallelogram(s) and/or triangle(s) in order to apply the vector product formulas for funding areas.

Let the point *D* be on the line segment *AC* such that *OA* is parallel to *BD*. Then



$\overrightarrow{BD} = \overrightarrow{OA} = \mathbf{a}$ and

$$\begin{aligned}\overrightarrow{DC} &= \overrightarrow{AC} - \overrightarrow{AD}\\ &= \lambda\overrightarrow{OB} - \overrightarrow{OB}\\ &= (\lambda - 1)\mathbf{b}\end{aligned}$$

Thus the area of trapezium *OACB*

$= \text{area of parallelogram } OADB + \text{area of } \Delta BCD$

$$= \left|\overrightarrow{OA}\times\overrightarrow{OB}\right| + \frac{1}{2}\left|\overrightarrow{BD}\times\overrightarrow{DC}\right|$$

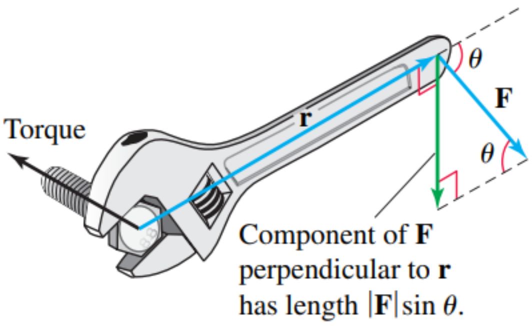
$$= \left|\mathbf{a}\times\mathbf{b}\right| + \frac{1}{2}\left|\mathbf{a}\times(\lambda-1)\mathbf{b}\right|$$

$$= \left|\mathbf{a}\times\mathbf{b}\right| + \frac{1}{2}(\lambda-1)\left|\mathbf{a}\times\mathbf{b}\right| \quad (\text{Note that } \lambda - 1 > 0 \because \lambda > 1.)$$

$$= \left(1 + \frac{1}{2}\lambda - \frac{1}{2}\right)\left|\mathbf{a}\times\mathbf{b}\right|$$

$$= \frac{1}{2}(\lambda+1)\left|\mathbf{a}\times\mathbf{b}\right| \text{ i.e., } k = \frac{1}{2}(\lambda+1) \text{ (shown)}$$

Big Ideas: Models



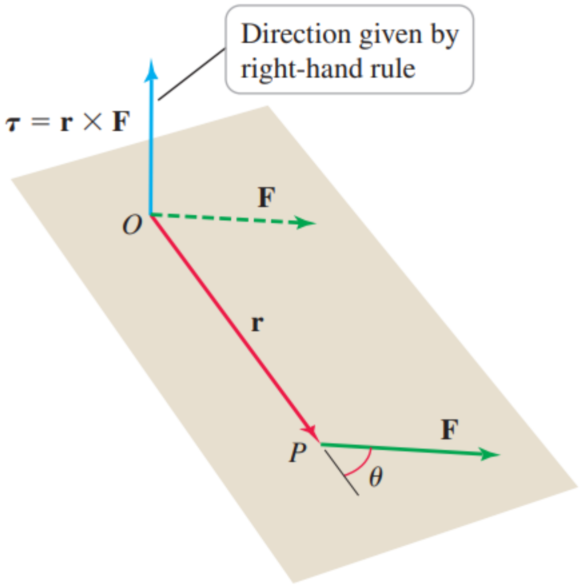
Real-world Problem

Suppose you want to loosen a bolt with a wrench. As you apply force to the end of the wrench in the plane perpendicular to the bolt, the "twisting power" you generate depends on the three variables:

- the magnitude of the force $\mathbf{F}$ applied to the wrench;
- the length $|\mathbf{r}|$ of the wrench;
- the angle at which the force is applied to the wrench.

The twisting generated by a force acting at a distance from a pivot point is called **torque**. The torque is a vector whose magnitude is proportional to $|\mathbf{F}|$, $|\mathbf{r}|$ and $\sin\theta$, where θ is the angle between $\mathbf{F}$ and $\mathbf{r}$.

How do we model the above and calculate torque?



We can model the above using the diagram on the left. Suppose a force $\mathbf{F}$ is applied to the point P at the head of a vector $\mathbf{r} = \overrightarrow{OP}$. The torque, or twisting effect, produced by the force about the point O is given by

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}.$$

The torque has a magnitude of $|\boldsymbol{\tau}| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}||\mathbf{F}|\sin\theta$, where θ is the angle between both $\mathbf{r}$ and $\mathbf{F}$.

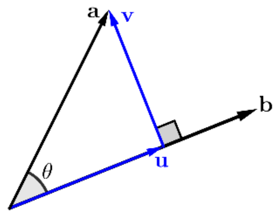
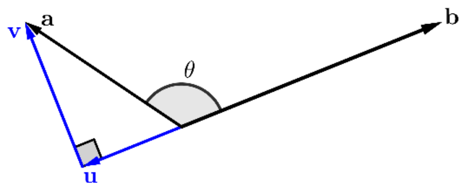
- What is the torque when $\mathbf{r}$ and $\mathbf{F}$ are parallel to each other?
- For given $\mathbf{r}$ and $\mathbf{F}$, when will the maximum torque occur?

§5 Resolving a Vector into Two Perpendicular Components

Key Questions:

- ☐ What is the *projection vector* of a vector onto another vector?
- ☐ What is the *vector component* of a vector *perpendicular* to another vector?
- ☐ How do we find
  - ☐ the *length of projection* of a vector onto another vector?
  - ☐ the *length of the vector component* of a vector *perpendicular* to another vector?
  - ☐ the *projection vector* of a vector onto another vector?
  - ☐ the *vector component* of a vector *perpendicular* to another vector?

Given any pair of non-zero vectors **a** and **b** (which bound an angle θ between them), we can always find a vector **u** parallel to **b** such that the three vectors **a**, **u** and **v** = **a** – **u** form a **right-angled triangle** with **a** as its hypotenuse, as illustrated in the diagrams below.

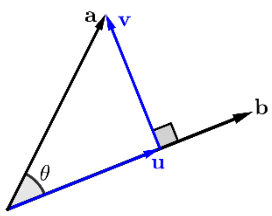
θ is acute	θ is obtuse
	

The following definition formalises this idea of "separating" a vector **a** into two perpendicular vector components relative to another vector **b**.

What is a projection vector of a vector onto another vector and the vector component of a vector perpendicular to another vector?

Definition 5.1 (Projection Vector and Perpendicular Vector Component)
For any pair of non-zero vectors **a** and **b**, let **u** and **v** be such that

- **u** is parallel to **b**,
- **u** + **v** = **a** and
- **v** is perpendicular to **u** (and thus is perpendicular to **b**).



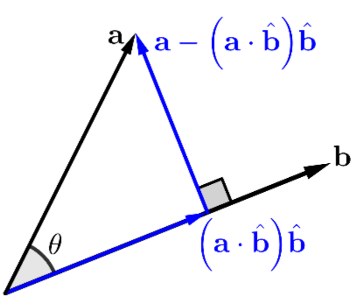
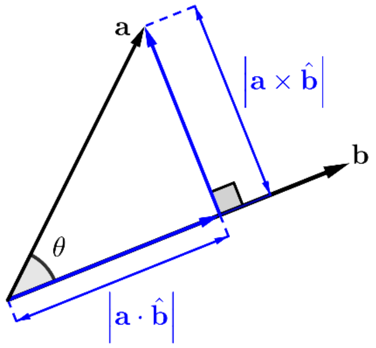
Then

- **u** is called the *projection vector* of **a** onto **b** and
- **v** is called the *vector component of* **a** *perpendicular to* **b**.

The process of finding these two vectors is known as *resolving the vector* **a** *into two perpendicular components relative to* **b**.

How do we find the projection vector and perpendicular component of a vector relative to another, as well as their lengths?

Result 5.2 (Resolving a Vector into Two Perpendicular Components)

Vectors	**Lengths (Magnitudes)**
	
Projection vector of **a** onto **b** $=(\mathbf{a}\cdot\hat{\mathbf{b}})\hat{\mathbf{b}}$	Length of projection (vector) of **a** onto **b** $=\lvert\mathbf{a}\cdot\hat{\mathbf{b}}\rvert$
Vector component of **a** perpendicular to **b** $=\mathbf{a}-(\mathbf{a}\cdot\hat{\mathbf{b}})\hat{\mathbf{b}}$	Length of vector component of **a** perpendicular to **b** $=\lvert\mathbf{a}\times\hat{\mathbf{b}}\rvert$ (1) $=\sqrt{\lvert\mathbf{a}\rvert^2-\lvert\mathbf{a}\cdot\hat{\mathbf{b}}\rvert^2}$ (2) $=\lvert\mathbf{a}-(\mathbf{a}\cdot\hat{\mathbf{b}})\hat{\mathbf{b}}\rvert$ (3)

Proof of Result 5.2

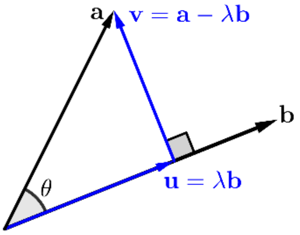
Let **u** and **v** be the projection vector of **a** onto **b** and the vector component of **a** perpendicular to **b** respectively.

By definition, since **u** is parallel to **b**,

$$\mathbf{u}=\lambda\mathbf{b}$$

for some real scalar λ. Therefore,

$$\mathbf{v}=\mathbf{a}-\mathbf{u}=\mathbf{a}-\lambda\mathbf{b}.$$



Since **v** is perpendicular to **b**, $\mathbf{v}\cdot\mathbf{b}=0$.

$$(\mathbf{a}-\lambda\mathbf{b})\cdot\mathbf{b}=0$$
$$\mathbf{a}\cdot\mathbf{b}-(\lambda\mathbf{b})\cdot\mathbf{b}=0$$
$$\mathbf{a}\cdot\mathbf{b}-\lambda(\mathbf{b}\cdot\mathbf{b})=0$$
$$\lambda(\mathbf{b}\cdot\mathbf{b})=\mathbf{a}\cdot\mathbf{b}$$
$$\lambda\lvert\mathbf{b}\rvert^2=\mathbf{a}\cdot\mathbf{b}$$
$$\lambda=\frac{\mathbf{a}\cdot\mathbf{b}}{\lvert\mathbf{b}\rvert^2}.$$

Thus,

$$\mathbf{u} = \lambda\mathbf{b} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\right)\mathbf{b} = \left(\mathbf{a}\cdot\frac{\mathbf{b}}{|\mathbf{b}|}\right)\frac{\mathbf{b}}{|\mathbf{b}|} = \left(\mathbf{a}\cdot\hat{\mathbf{b}}\right)\hat{\mathbf{b}}$$

and

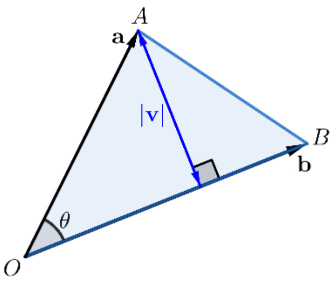
$$\begin{aligned}\mathbf{v} &= \mathbf{a} - \mathbf{u} \\ &= \mathbf{a} - \left(\mathbf{a}\cdot\hat{\mathbf{b}}\right)\hat{\mathbf{b}}.\end{aligned}$$

So,

$$\begin{aligned}|\mathbf{u}| &= \left|\left(\mathbf{a}\cdot\hat{\mathbf{b}}\right)\hat{\mathbf{b}}\right| \\ &= \left|\mathbf{a}\cdot\hat{\mathbf{b}}\right|\left|\hat{\mathbf{b}}\right| \\ &= \left|\mathbf{a}\cdot\hat{\mathbf{b}}\right| \ \left(\text{since } \hat{\mathbf{b}} \text{ is a unit vector}\right).\end{aligned}$$

Let A and B be the points with position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively, referred to an origin O.

Using the vector product formula for finding the area of a triangle,



$$\text{Area of } \Delta OAB = \frac{1}{2}|\mathbf{a}\times\mathbf{b}|$$

On the other hand,

$$\text{Area of } \Delta OAB = \frac{1}{2}\times\text{base}\times\text{height} = \frac{1}{2}|\mathbf{b}||\mathbf{v}|$$

Thus

$$\begin{aligned}\frac{1}{2}|\mathbf{b}||\mathbf{v}| = \frac{1}{2}|\mathbf{a}\times\mathbf{b}| \Rightarrow |\mathbf{b}||\mathbf{v}| &= |\mathbf{a}\times\mathbf{b}| \\ \Rightarrow |\mathbf{v}| &= \frac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{b}|} \\ &= \left|\mathbf{a}\times\frac{\mathbf{b}}{|\mathbf{b}|}\right| \\ &= \left|\mathbf{a}\times\hat{\mathbf{b}}\right|.\end{aligned}$$

Example 5.1

Relative to the origin O, two points A and B have position vectors given by $\mathbf{i}-\mathbf{j}+\mathbf{k}$ and $-\mathbf{i}+2\mathbf{j}-3\mathbf{k}$ respectively.

(i) Find the length of projection of $\overrightarrow{OB}$ onto $\overrightarrow{OA}$ exactly.

(ii) Hence, or otherwise, find the shortest distance from B to the line OA exactly.

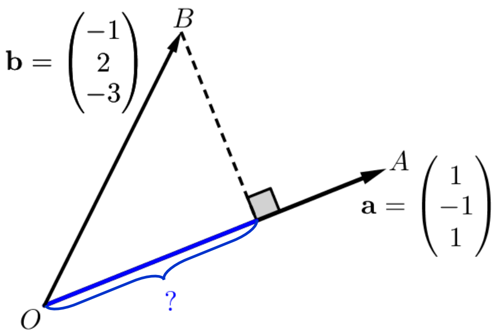
(iii) Find the projection vector of $\overrightarrow{OA}$ onto $\overrightarrow{OB}$.

(iv) Find the vector component of $\overrightarrow{OA}$ perpendicular to $\overrightarrow{OB}$.

Solution

(i) Let $\mathbf{a}$ and $\mathbf{b}$ denote the position vectors of A and B respectively.

Draw a diagram to depict the scenario.



Apply the formula $|\mathbf{b}\cdot\hat{\mathbf{a}}|$ to find the length of projection of $\overrightarrow{OB}$ onto $\overrightarrow{OA}$

Length of projection of $\overrightarrow{OB}$ onto $\overrightarrow{OA}$

$$= |\mathbf{b}\cdot\hat{\mathbf{a}}|$$

$$= \left|\begin{pmatrix}-1\\2\\-3\end{pmatrix}\cdot\frac{1}{\sqrt{1^2+(-1)^2+1^2}}\begin{pmatrix}1\\-1\\1\end{pmatrix}\right|$$

$$= \frac{1}{\sqrt{3}}\left|\begin{pmatrix}-1\\2\\-3\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\1\end{pmatrix}\right|$$

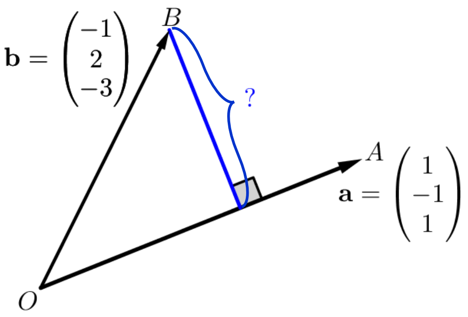
$$= \frac{1}{\sqrt{3}}|-1-2-3|$$

$$= \frac{6}{\sqrt{3}}$$

$$= 2\sqrt{3} \text{ units}$$

Draw a diagram to depict the scenario.

(ii)



Apply a suitable formula to find the length of the vector component of $\overrightarrow{OB}$ perpendicular to $\overrightarrow{OA}$

By Pythagoras' Theorem,
Shortest distance from B to OA

$$= \sqrt{|\mathbf{b}|^2 - |\mathbf{b}\cdot\hat{\mathbf{a}}|^2}$$

$$= \sqrt{\left[\sqrt{(-1)^2 + 2^2 + (-3)^2}\right]^2 - \left(2\sqrt{3}\right)^2}$$

$= \sqrt{2}$ units.

<u>Alternative method using the cross product</u>:
Shortest distance from B to OA

$$= |\mathbf{b}\times\hat{\mathbf{a}}|$$

$$= \left|\begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix} \times \frac{1}{\sqrt{1^2 + (-1)^2 + 1^2}}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\right|$$

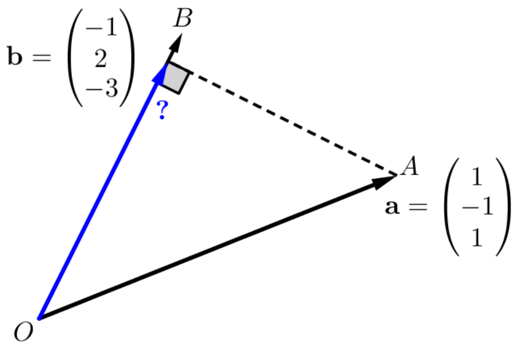
$$= \frac{1}{\sqrt{3}}\left|\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}\right|$$

$$= \frac{\sqrt{(-1)^2 + (-2)^2 + (-1)^2}}{\sqrt{3}}$$

$= \dfrac{\sqrt{6}}{\sqrt{3}} = \sqrt{2}$ units.

(iii)

Draw a diagram to depict the scenario.



Projection vector of $\overrightarrow{OA}$ onto $\overrightarrow{OB}$

Apply the formula $(\mathbf{a}\cdot\hat{\mathbf{b}})\hat{\mathbf{b}}$ to find the projection vector of $\overrightarrow{OA}$ onto $\overrightarrow{OB}$.

$$= (\mathbf{a}\cdot\hat{\mathbf{b}})\hat{\mathbf{b}} \quad \left[\text{or } \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\right)\mathbf{b}\right]$$

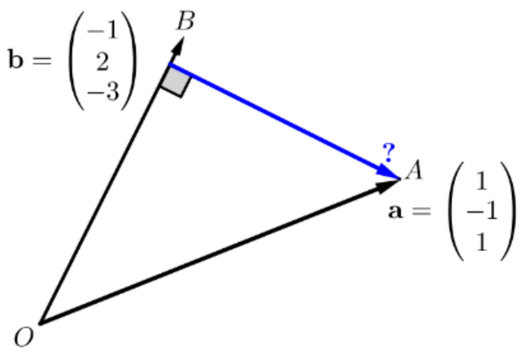
$$= \left[\frac{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\cdot\begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}}{(-1)^2 + 2^2 + (-3)^2}\right]\begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}$$

$$= \frac{-1-2-3}{14}\begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}$$

$$= -\frac{3}{7}\begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix} \text{ or } \frac{3}{7}\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$$

Draw a diagram to depict the scenario.

(iv)



Apply the formula $\mathbf{a}-(\mathbf{a}\cdot\hat{\mathbf{b}})\hat{\mathbf{b}}$ to find the vector component of $\overrightarrow{OA}$ perpendicular to $\overrightarrow{OB}$.

Vector component of $\overrightarrow{OA}$ perpendicular to $\overrightarrow{OB}$

$$=\mathbf{a}-(\mathbf{a}\cdot\hat{\mathbf{b}})\hat{\mathbf{b}}\ \left[\text{or } \mathbf{a}-\left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\right)\mathbf{b}\right]$$

$$=\begin{pmatrix}1\\-1\\1\end{pmatrix}-\frac{3}{7}\begin{pmatrix}1\\-2\\3\end{pmatrix}\ \text{(from part (\textbf{iii}))}$$

$$=\frac{1}{7}\begin{pmatrix}4\\-1\\-2\end{pmatrix}$$