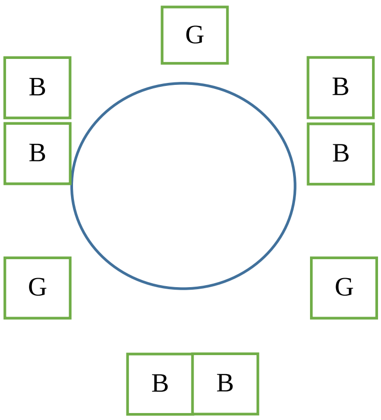
Permutations and Combinations Solutions

1(i)	Number of ways = Total number of ways without restriction - number of ways all 3 girls together $= \underbrace{(9-1)!}_{\text{9 people around a circle}} - \left[\underbrace{(7-1)!}_{\text{6 boys and 1 block of 3 girls}} \times \underset{\text{3 girls}}{3!} \right]$ $= 36000$ **<u>Alternative method</u>** Number of ways = number of ways 2 girls together, 1 separated + number of ways all 3 girls separated $= \left(\underbrace{(5-1)!}_{\text{6 boys around a circle}} \times \underset{\text{3 girls choose 2 to be in a group}}{{}^{3}C_{2}} \times \underset{\text{6 slots permute 2 groups of girls}}{{}^{6}P_{2}} \times \underbrace{2!}_{\text{permute within the group of 2 girls}} \right)$ $+ \left[\underbrace{(5-1)!}_{\text{6 boys around}} \times \underbrace{{}^{6}P_{3}}_{\text{6 slots permute 3 girls}} \right]$ $= 36000$
(ii)	Number of ways $= \underbrace{(6-1)!}_{\text{6 remaining people around a circle}} \times \underset{\text{6 slots choose 3}}{{}^{6}C_{3}} \times \underset{\text{The 3 girls}}{3!}$ $= 14400$
(iii)	Note: The arrangement must look like this  Number of ways without restrictions $= (9-1)!$ $= 40320$

	Number of ways for exactly 2 boys between any 2 girls $= \underbrace{(3-1)!}_{\text{Arranging 3 girls around a circle}} \times \underbrace{6!}_{\text{6 boys permutate within themselves}}$ $=1440$ Required prob. $=\dfrac{1440}{40320}$ $=\dfrac{1}{28}$ **<u>Alternative Method:</u>** Number of ways without restrictions $=(9-1)!$ $=40320$ Number of ways for exactly 2 boys between any 2 girls $= \underbrace{(3-1)!}_{\text{Arranging 3 girls around a circle}} \times$ $\left[\underbrace{{}^{6}C_{2}}_{\text{6boys choose 2 to slot into first slot}} \times \underbrace{{}^{4}C_{2}}_{\text{4 boys choose 2 to slot into second slot}} \times {}^{2}C_{2} \times \underbrace{(2!)^{3}}_{\text{each group of boys permute within themselves}}\right]$ $=1440$ Required prob. $=\dfrac{1440}{40320}$ $=\dfrac{1}{28}$

2(a)(i)	Case 1: 2 red + 2 green, no of ways $= {}^{5}C_{2} \times {}^{8}C_{2} = 280$ Case 2: 1 red + 3 green, no of ways $= {}^{5}C_{1} \times {}^{8}C_{3} = 280$ Case 3: 4 green, no of ways $= {}^{8}C_{4} = 70$ Total no of ways = 280 + 280 + 70 = 630
(ii)	No of ways to select at least 1 of each colour = n(any 4 of 13 balls) – n(4 of 5 red) – n(4 of 8 green) $= {}^{13}C_{4} - {}^{5}C_{4} - {}^{8}C_{4} = 640$
(b)	Case 1: blue trousers, i.e. any 2 of 5 skirts: no of ways $= {}^{5}P_{2} = 20$ Case 2: green trousers, i.e. no green skirt: no of ways $= {}^{4}P_{2} = 12$ Case 3: yellow trousers, i.e. no yellow skirt: no of ways $= {}^{4}P_{2} = 12$

	Total no of ways = 20 + 12 + 12 = 44

3	The number of seating arrangements = 4! = 24 Number of ways = 3! $\times$ 2! = 12 Number of ways = 4! $\times$ 2! $\times$ 6 = 288

4(a)	No of ways = $^{10}C_1 \times {}^{9}C_3 = 840$ **OR** $^{10}C_3 \times {}^{9}C_1 = 840$
(i)	Form CLC unit: $^{3}C_2 \times 2$ ways Seat 8 units in a round table: $(8-1)!$ ways Total number of ways = $^{3}C_2 \times 2 \times (8-1)!$ = 30240
(ii)	Total number of ways = $30240 \times 10 = 302400$
(b)	Case 1: 4 digit number, ending with 2: No of ways = 2 × 3 × 2 × 1 = 12 Case 2: 4 digit number, does not end with 2: No of ways = 2 × (2 + 1) × 1 = 6 Case 3: 5 digit number, ending with 2: No of ways = 4!× 1 = 24 Case 4: 5 digit number, does not end with 2: No of ways = $\frac{4!}{2!}$ × 2 = 24 Total number of ways = 12 + 6 + 24 + 24 = 66

5(i)	No of delegations = $\binom{4}{2}\binom{7}{3}$ = 210
(ii)	No of groupings = $\frac{\binom{11}{5}\binom{6}{3}\binom{3}{3}}{2!}$ = 4620

6(i)	No of arrangements = 10! = 3628800
(ii)	No of arrangements = 5! $\binom{6}{5}$ 5! = 86400
(iii)	No of arrangements = (7-1)! 5!(11) = 950400

7(i)	No of words = 5! $\frac{4!}{2!} \div 2$ = 720
(ii)	No of words = $\frac{6!}{2!}$ x 2 = 6! = 720
(iii)	No of words = 4! $\frac{4!}{2!}$ x $\frac{2}{2}$ = 288
(iv)	No of words = $\frac{4!}{2!}\binom{5}{4}\frac{4!}{2}$ = 720

8(a)	No of arrangements = $\frac{6!}{2!}$ = 360

(b)	No of selections = $\binom{4}{1}$ x 2 = 8

9(i)	No of 4-digit numbers = 3 x 3! = 18
(ii)	No of 4-digit numbers = 2! x 5 = 10

10(i)	Total number of possible results = $3^5 = 243$
(ii)	Number of ways of obtaining 'Windfall' = 3
(iii)	No. of ways to obtain a success = $\left(\frac{5!}{3!2!}\times 2+\frac{5!}{3!}\right)\times 3=120$ Total number of ways $=243-120\ =123$
	Reqd no. of ways = ${}^{10}C_3\times{}^{7}C_3\times{}^{4}C_4\times\frac{3!}{2!}=12600$

11(a)(i)	Required number of 8-letter code-words = 5^8 = 390625
(ii)	Required number of 8-letter code-words = $\frac{8!}{(2!)^3}$ = 5040
(iii)	First, consider that each letter occurs once, then we add 3 more letters (from *A*, *B*, *C*, *D*, *E*) to form the code-words. Case 1: 3 identical letters No. of ways = 5 Case 2: 3 different letters No. of ways = ${}^{5}C_3$ =10 Case 3: 2 identical letters and 1 different letter No. of ways = ${}^{5}C_2\times 2$ = 20 Required number of 8-letter code-words = 5 + 10 + 20 =35
(b)	Case 1: 2, 2, 5 people No. of ways = $\frac{{}^{9}C_2\times{}^{7}C_2}{2!}\times 1!\times 1!\times 4!=9072$ Case 2: 2, 3, 4 people No. of ways = ${}^{9}C_2\times{}^{7}C_3\times 1!\times 2!\times 3!=15120$ Case 3: 3, 3, 3 people No. of ways = $\frac{{}^{9}C_3\times{}^{6}C_3}{3!}\times(2!)^3$ = 2240 Total no. of ways = 26432

12(a)(i)	No. of arrangements $= \frac{4!}{2!} \times 5! = 1440$
(ii)	No. of arrangements $= 4! \times {}^3P_2 = 144$
(b)	No. of arrangements $= (10-1)! \times 2! \times 11 = 7983360$
	Case 1: Mr Lin and Mr Tan at Circular Table No. of arrangements = ${}^9C_4(5-1)! \times 2 \times 5! = 725760$ Case 2: Mr Lin and Mr Tan at Linear Table No. of arrangements = ${}^9C_3(6-1)! \times 2 \times 4! = 483840$ Total number of arrangements = 1209600

13(a)	Case 1: 4 and 5 in the second and fourth positions $2 \times 3! = 12$ Case 2: 3 and 5 in the second and fourth positions $2 \times 2 = 4$ Total number $= 12 + 4 = 16$
(b)	Number of ways $= \left[\binom{4}{1}\cdot\binom{3}{3} + \binom{4}{2}.\binom{3}{2} + \binom{4}{3}.\binom{3}{1}\right].3! = 204$ **Alternative**: ${}^7C_4.3! - 3! = 204$

14(i)	${}^6C_2 \times {}^5C_4 = 75$ ways
(ii)	Number of ways if at least one of the sisters are included = number of ways without restriction – number of ways if none of the sisters is included $= {}^{11}C_6 - {}^8C_6 = 434$ **Or** ${}^3C_1 \times {}^8C_5 + {}^3C_2 \times {}^8C_4 + {}^3C_3 \times {}^8C_3 = 434$
(iii)	Select a man to be between the 2 sisters and group the 3 of them as one unit and arrange 4 units round a table ${}^3C_1 \times 3! \times 2 = 36$
(iv)	First arrange the other 4 persons round the table. There are 4 ways to insert the sisters. $3! \times 4 = 24$ or ${}^4C_2 \times 2! \times 2! = 24$

15	For distinct gifts, 5^6 ways Now considering the distinct gifts, Case 1: 3 person get 1 gift

No of ways = $^{5}C_3 \times 5^6 = 156250$

Case 2: 1 person get 1 gift, another person gets 2 gifts

No of ways = $^{5}C_2\left(2\right)\times 5^6 = 312500$

Case 3: 1 person get 3 gifts

No of ways = $^{5}C_1 \times 5^6 = 78125$

Total number of ways
$= 156250 + 312500 + 78125 = 546875$

Alternative

<u>Stage 1: Distribute 6 distinct gifts among 5 people</u>

No of ways = 5^6

<u>Stage 2: Distribute 3 identical gifts among 5 people</u>

Case 1: 3 person get 1 gift

No of ways = $^{5}C_3$ = 10

Case 2: 1 person get 1 gift, another person gets 2 gifts

No of ways = $^{5}C_2\left(2\right)$ = 20

Case 3: 1 person get 3 gifts

No of ways = $^{5}C_1$ = 5

Total number of ways = $(10+20+5)5^6$ = 546875

16(i)	Number of ways $= 8!$ = 40320
(ii)	Number of ways $= 4!\times4!\times4$ $= 2304$ 4 = ways to arrange the boys and girls on each side $4!\times4!$ = ways the groups of 4 boys & 4 girls can arrange themselves
(iii)	Number of ways $= {}^{10}C_8\times8!$ or ${}^{10}P_8$ or $\frac{10!}{2!}$ $= 1814400$ $^{10}C_8$ = ways to choose any 8 out of the 10 seats to sit the 4 boys and 4 girls 8! = ways to arrange the boys and girls $\frac{10!}{2!}$ = arrange 10 objects in a row with 2 identical objects (empty seats)
(iv)	Number of ways $= {}^{4}C_2\times2!\times{}^{8}C_6\times6!$ or ${}^{4}P_2\times{}^{8}P_6$ $= 241920$ $^{4}C_2\times2!$ = ways to choose any 2 out of the 4 boys to sit on the 2 seats and ways to arrange the 2 boys $^{8}C_6\times6!$ = ways to choose any 6 out of the 8 seats to sit the remaining 2 boys/4 girls and ways to arrange the 2 boys/4 girls

$\frac{8!}{2}$

17(i)	Number of ways = ${}^9C_3 \times {}^6C_3 \times {}^3C_3 = 1680$
(ii)	Number of ways = $3 \times 2 \times {}^7C_2 \times {}^5C_2 \times {}^3C_3 = 1260$ (or $3 \times 2 \times {}^7C_3 \times {}^4C_2 \times {}^2C_2 = 1260$) There are 3 ways of assigning a car to *A*, 2 ways to *B*, followed by slotting the remaining people 7 people into the 3 cars. **<u>Alternative Solution</u>** When *A* and *B* are in the same car, number of ways = $3 \times {}^7C_1 \times {}^6C_3 \times {}^3C_3 = 420$ (There are 3 ways of assigning a car to both *A* and *B*, followed by slotting the remaining 7 people into the 3 cars) $\therefore$ number of ways when *A* and *B* are in different cars = $1680 - 420 = 1260$
(iii)	Number of ways = $11! - 10!2! = 32659200$ **<u>Alternative Solution</u>** Number of ways = $9! \times {}^{10}P_2 = 32659200$ (Arrange the remaining 10 people first, then look for separated slots to accommodate *A* and *B*)
(iv)	Number of ways when *A* and *B* are together (*B* on *A*'s left) = $10!$ Number of ways when there is 1 person between *A* and *B* (*B* on *A*'s left) = ${}^{10}C_1 \times 9!$ Number of ways when there are 2 people between *A* and *B* (*B* on *A*'s left) = ${}^{10}C_2 \times 2! \times 8!$ $\therefore$ number of ways = $11! - 10! - {}^{10}C_1 \times 9! - {}^{10}C_2 \times 2! \times 8! = 29030400$ **<u>Alternative Solution</u>** Number of ways = $9! \times {}^{10}C_1 \times {}^8C_1 = 29030400$ (Arrange the remaining 10 people first, then choose 1 slot to accommodate *A* or *B*, followed by choosing one slot to accommodate the last person, with at least 3 people between *A* and *B*)

18 (i)

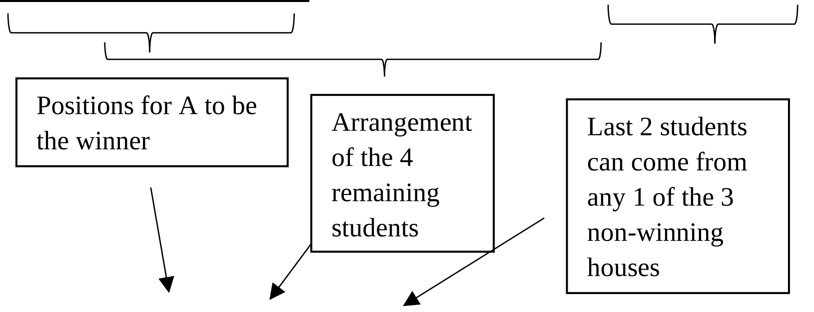
family	A		B		C		D	
	Adult	kids	Adult	kids	Adult	kids	Adult	kids
	2	2	1	2	1	3	1	1

	4 family units, No. of ways = $4!\times4!\times3!\times5!\times2!$ = 829, 440
(ii)	Case 1: 3,3,2 No. of ways = $\dfrac{{}^{8}C_3\times{}^{5}C_3\times{}^{2}C_2}{2!}$ = 280 Case 2: 2,2,2,2 No. of ways = $\dfrac{{}^{8}C_2\times{}^{6}C_2\times{}^{4}C_2\times{}^{2}C_2}{4!}$ = 105 Total no. of ways = 280 + 105 = 385
(iii)	There is only 1 way to divide the 8 children and the adults into 2 circles to satisfy all conditions. Family A and B (3 adults & 4 kids) must be in 1 circle and Family C & D are in another circle. Arrange the children in 1 circle : (4-1)! Slot in adults in between children : ${}^{4}C_3\times3!$ No. of ways = $[(4-1)!\times{}^{4}C_3\times3!]\times[(4-1)!\times{}^{4}C_3\times3!]$ = 20736

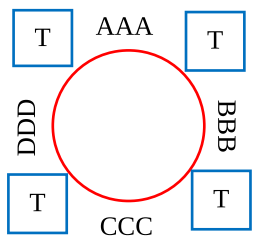
19	(i) $\dfrac{{}^{18}C_6\times{}^{12}C_6\times{}^{6}C_6}{3!}=2858856$ (Shown) (ii) Number of ways M and J in the same group $=\dfrac{{}^{16}C_4\times{}^{12}C_6\times{}^{6}C_6}{2!}=840840$ Required probability = $\dfrac{840840}{2858856}=0.294$ (3 s.f.) Case 1 : 3 particular participants stand in row 3 Number of ways = $3!\times6!=4320$ Case 2 : 3 participants stand in row 4 Number of ways = $2\times3!\times6!=8640$ Total number of ways = 12960

20(i)	No. of arrangements = $\dfrac{10!}{2!2!4!}-1=37799$
(ii)	No. of arrangements =Total no. of ways with all E's together – No. of ways all E's together and C's and L's together – No. of ways all E's together and C's but L's separated – No. of ways all E's together and L's but C's separated $=\dfrac{(7-1)!}{2!2!}-(5-1)!-(4-1)!\,{}^{4}C_2-(4-1)!\,{}^{4}C_2$ $=180-24-36-36=84$

21(i)	$(8-1)!-(7-1)!\times 2!=3600$
(ii)	$6P4=360$ **OR** $6C2\times 2!\times 4C2\times 2!=360$ **OR** $6C4\times 4C2\times 2!\times 2C2\times 2!=360$
(iii)	For there to be ties, there are 4 cases to consider for the voting: $(4,4,0,0)$, $(3,3,1,1)$, $(3,3,2,0)$ and $(2,2,2,2)$. $(4,4,0,0)$: $8C4\times 4C4\times 4C2=420$ $(3,3,1,1)$: $8C3\times 5C3\times 2C1\times 1C1\times 4C2=6720$ $(3,3,2,0)$: $8C3\times 5C3\times 2C2\times \frac{4!}{2!}=6720$ $(2,2,2,2)$: $8C2\times 6C2\times 4C2\times 2C2=2520$ Total number of ways the votes could have happened: $420+6720+6720+2520=16380$

<table>
<tr><th>Qn</th><th>Suggested Solution</th></tr>
<tr><td>22(i)</td><td>AA BB CC DD
Ways $=4!\ \times 2^4=384$
4! : arrange the 4 pairs of students
2^4 : arrange each pair of students from same house</td></tr>
<tr><td>(ii)</td><td>Example : House A is the winner
<table>
<tr><td>1^{ST}</td><td>2^{ND}</td><td>3^{RD}</td><td>4^{TH}</td><td>5^{TH}</td><td>6^{TH}</td><td>7^{TH}</td><td>8^{TH}</td></tr>
<tr><td>A</td><td>B</td><td>A</td><td>C</td><td>B</td><td>C</td><td>D</td><td>D</td></tr>
</table>


Ways $=\left({}^3C_2\times 2!\right)\times 4!\ \times\left({}^3C_1\times 2!\right)=864$</td></tr>
</table>

(iii)	Ways $=(4-1)!\ \times({}^4C_3\times 3!)^4\times 4!=47775744$ $(4-1)!$: arrange the 4 gps of 3 teachers in a circle $({}^4C_3\times 3!)^4$: arrange the 3 out of 4 selected teachers $4!$: arrange the 4 captains ABCD – teachers, T - captains	

Qn	Suggested Solution
23(a)	No. of ways $=({}^9C_2\,{}^7C_2\,{}^5C_5\times\frac{3!}{2!})+({}^9C_2\,{}^7C_3\,{}^4C_4\times 3!)+({}^9C_3\,{}^6C_3\,{}^3C_3)$ $=2268+7560+1680$ $=11508$
(b)	Case 1: Triangle formed from 1 point from each side $={}^2C_1\times{}^3C_1\times{}^4C_1=24$ Case 2: Triangle formed from 2 points on XZ + 1 other $={}^4C_2\times{}^5C_1=30$ Case 3: Triangle formed from 2 points on YZ + 1 other $={}^3C_2\times{}^6C_1=18$ Case 4: Triangle formed from 2 points on XY + 1 other $={}^2C_2\times{}^7C_1=7$ Total no. of ways $=24+30+18+7=79$ <u>Method 2 (complement)</u> No. of ways $=\begin{pmatrix}\text{no. of ways}\\\text{of choosing 3 bulbs}\\\text{without restrictions}\end{pmatrix}-\begin{pmatrix}\text{no. of ways}\\\text{of choosing 3 bulbs}\\\text{from side } YZ\end{pmatrix}-\begin{pmatrix}\text{no. of ways}\\\text{of choosing 3 bulbs}\\\text{from side } XZ\end{pmatrix}$ $={}^9C_3-{}^3C_3-{}^4C_3$ $=84-1-4$ $=79$
(c)	No. of ways $=\begin{pmatrix}\text{no. of ways}\\\text{without restrictions}\end{pmatrix}-\begin{pmatrix}\text{no. of ways}\\\text{red or blue}\\\text{is used}\end{pmatrix}-\begin{pmatrix}\text{no. of ways}\\\text{blue or green}\\\text{is used}\end{pmatrix}-\begin{pmatrix}\text{no. of ways}\\\text{red or green}\\\text{is used}\end{pmatrix}$ $+\begin{pmatrix}\text{no. of ways}\\\text{only red}\\\text{is used}\end{pmatrix}+\begin{pmatrix}\text{no. of ways}\\\text{only blue}\\\text{is used}\end{pmatrix}+\begin{pmatrix}\text{no. of ways}\\\text{only green}\\\text{is used}\end{pmatrix}$ $=3^9-3(2^9)+3$ $=19683-1536+3$ $=18150$ <u>Alternative method</u>

	No. of ways $=\begin{pmatrix}\text{no. of ways}\\ \text{without restrictions}\end{pmatrix}-\begin{pmatrix}\text{no. of ways}\\ \text{only 1 colour}\\ \text{is used}\end{pmatrix}-\begin{pmatrix}\text{no. of ways}\\ \text{exactly 2 colours}\\ \text{are used}\end{pmatrix}$ $=3^9-3-{}^3C_2(2^9-2)$ $=19683-3-1530$ $=18150$
(d)	No. of ways $=(\text{Total Area})-(\text{Area 1 and 2})-(\text{Area 2 and 3})+(\text{Area 2})$ $=\begin{pmatrix}\text{no. of ways}\\ \text{without}\\ \text{restrictions}\end{pmatrix}-\begin{pmatrix}\text{no. of ways}\\ \text{2 } A\text{'s are}\\ \text{together}\end{pmatrix}-\begin{pmatrix}\text{no. of ways}\\ \text{2 } B\text{'s are}\\ \text{together}\end{pmatrix}+\begin{pmatrix}\text{no. of ways}\\ \text{2 } A\text{'s \& 2 } B\text{'s}\\ \text{are together}\end{pmatrix}$ $=\frac{(9-1)!}{2!2!2!}-\frac{(8-1)!}{2!2!}-\frac{(8-1)!}{2!2!}+\frac{(7-1)!}{2!}$ $=5040-1260-1260+360$ $=2880$ Alternative 1 (complement) $=(\text{Total Area})-(\text{Area 1})-(\text{Area 3})-(\text{Area 2})$ $=\begin{pmatrix}\text{no. of ways}\\ \text{without}\\ \text{restrictions}\end{pmatrix}-\begin{pmatrix}\text{no. of ways}\\ \text{A's are separate}\\ \text{B's are together}\end{pmatrix}-\begin{pmatrix}\text{no. of ways}\\ \text{B's are separate}\\ \text{A's are together}\end{pmatrix}-\begin{pmatrix}\text{no. of ways}\\ \text{2 } A\text{'s \& 2 } B\text{'s}\\ \text{are together}\end{pmatrix}$ $=\frac{(9-1)!}{2!2!2!}-\left(\frac{(6-1)!}{2!}\times {}^6C_2\right)-\left(\frac{(6-1)!}{2!}\times {}^6C_2\right)-\frac{(7-1)!}{2!}$ $=5040-900-900-360$ $=2880$ Alternative 2 (complement) $=\begin{pmatrix}\text{no. of ways}\\ \text{A's are separate}\end{pmatrix}-\begin{pmatrix}\text{no. of ways}\\ \text{A's are separate and}\\ \text{2B's are together}\end{pmatrix}$ $=\left(\frac{(7-1)!}{2!2!}\times {}^7C_2\right)-\left(\frac{(6-1)!}{2!}\times {}^6C_2\right)$ $=3780-900$ $=2880$ Alternative 3 – Use the slotting method by putting in the 2 *A*'s first, followed by the 2 *B*'s.

	No. of ways $= \begin{pmatrix} \text{no. of ways} \\ \text{2 } A\text{'s are separated} \\ \text{initially} \end{pmatrix} + \begin{pmatrix} \text{no. of ways} \\ \text{2 } A\text{'s are together} \\ \text{initially} \end{pmatrix}$ $= \frac{(5-1)!}{2!} \times {}^5C_2 \times {}^7C_2 + \frac{(5-1)!}{2!} \times {}^5C_1 \times {}^6C_1$ $= 12 \times 10 \times 21 + 12 \times 5 \times 6$ $= 2880$

Qn	Suggested Solution
24(i)	No. of numbers = ${}^6C_4 \times 4! = {}^6P_4 = 360$
(ii)	Case 1: **AABC** no. of numbers = ${}^6C_1 \times {}^5C_2 \times \frac{4!}{2!} = 720$ Case 2: **AABB** no. of numbers = ${}^6C_2 \times \frac{4!}{2!2!} = 90$ Total no. of numbers = 720+90 =810 Alternatively, Total – all different – all same – 3 same and 1 different $= 1296 - 360 - 6 - {}^6C_1 \times {}^5C_1 \times \frac{4!}{3!} = 810$
(iii)	No. of numbers = $360 - 3 \times 4! = 288$