HWA CHONG INSTITUTION
2024 JC2 Preliminary Examination

MATHEMATICS
Higher 3

9820/01

Paper 1

Wednesday **18 September 2024** **3 hours**

Additional materials: 12-page Answer Booklet
List of Formula (MF26)
4-page Additional Answer Booklet (upon request)

READ THESE INSTRUCTIONS FIRST

Write your name and class on the 12-page Answer Booklet and any other additional 4-page Answer Booklets you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use paper clips, highlighters, glue or correction fluid.
Do not write anything on the List of Formula (MF26).

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

At the end of the examination, slot any additional 4-page Answer Booklets used in your 12-page Answer Booklet and indicate on the 12-page Answer Booklet the number of additional 4-page Answer Booklets used (if any).

This question paper consists of 5 printed pages and 1 blank page.

1. A sequence generated from the integer set $\{1,2,3,...,n\}$, is denoted as a special sequence if the sequence is strictly increasing and the first term is odd, the second term is even, the third term is odd, the fourth term is even, etc. For example, from the set $\{1,2,3,4,5,6\}$, some special sequences generated from the set are $\{1,2,3,6\}$, $\{1,4,5\}$, $\{3,4\}$, $\{5\}$.

Let $A(n)$ be the total number of distinct special sequences obtained from $\{1,2,3,...,n\}$.

(a) Form a recurrence relation involving $A(n), A(n-1), A(n-2)$. [4]

(b) Find the number of special sequences that can be formed from $\{1,2,...,15\}$. [2]

2. Let $\mathrm{f}(x)$ be a differentiable function such that $\mathrm{f}(x)+\mathrm{f}'(x)\le 1$ for all $x\in\mathbb{R}$, and $\mathrm{f}(0)=a$, where a is a constant with $a\ne 1$. It is given that $\mathrm{g}(x)=\mathrm{e}^x\,\mathrm{f}(x)$.

(a) Show that $\mathrm{g}'(x)\le \mathrm{e}^x$, for all $x\in\mathbb{R}$. [2]

(b) Hence, or otherwise, find the largest possible value of $\mathrm{f}(1)$ and the corresponding expression for $\mathrm{f}(x)$ in terms of a. [6]

3. For this question, it can be assumed that all the infinite series converge.

Given an infinite sequence of numbers u_0, u_1, u_2, …, the *generating function*, f, for the sequence is defined by $\mathrm{f}(x)=u_0+u_1x+u_2x^2+u_3x^3+\dots$.

(a) **(i)** Show that the sequence given by $u_n=n$, $n\ge 0$, has generating function $\mathrm{f}(x)=\dfrac{x}{(1-x)^2}$, and find the sequence that has the generating function $\mathrm{f}(x)=\dfrac{x}{(1-x)^3}$. [3]

(ii) Find the generating function for the sequence $u_n=n^2$, giving your answer simplified form. [2]

(b) The sequence u_0, u_1, u_2, … is defined by the recurrence relation

$$u_n=2u_{n-1}+1,\quad n\ge 1,$$

with $u_0=1$.

Find the generating function for this sequence, giving your answer in the form $\dfrac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials in terms of x to be determined. [3]

4. **Definition:** The function $\tau : \mathbb{Z}^+ \to \mathbb{Z}^+$ is defined as $\tau(n) = \sum_{d \mid n} 1$, where $d \mid n$ denotes that d is a positive divisor of n.

(a) Write down the values of $\tau(1)$, $\tau(3)$, $\tau(12)$. [2]

(b) Given that the prime decomposition for some $n \in \mathbb{Z}^+$ is $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$, where p_i is prime and α_i is a positive integer for every $1 \le i \le k$, write down $\tau(n)$ in terms of α_i. [2]

(c) Hence or otherwise, prove that $\tau(n)$ is odd if and only if n is a perfect square, where a perfect square is an integer that is the square of an integer. [3]

(d) Consider the following table:

n								
$\ldots$	Divisors of k							
$k=8$								
$k=7$								
$k=6$								
$k=5$								
$k=4$	1	2		4				
$k=3$	1		3					
$k=2$	1	2						
$k=1$	1							

The rows $k=1$ to $k=4$ are completely filled. Copy and fill up the table for rows $k=5$ to $k=8$. [1]

(e) Hence or otherwise, prove that $\sum_{k=1}^{n} \tau(k) = \sum_{k=1}^{n} \left\lfloor \frac{n}{k} \right\rfloor$, where $\lfloor x \rfloor$ denotes the largest positive integer less than or equals to x. [3]

5. **(a)** Using integration by parts, find the value of $\int_0^{\infty} \mathrm{e}^{-2x} \sin x \, \mathrm{d}x$. [3]

(b) Explain why $\int_0^{\infty} \mathrm{e}^{-2x} |\sin x| \, \mathrm{d}x$ exists. [2]

(c) Find the value of $\int_0^{\infty} \mathrm{e}^{-2x} |\sin x| \, \mathrm{d}x$. [7]

6. **(a)** For $n \in \mathbb{Z}$, find the possible values of the positive constant s, when $3n^2 \equiv s \pmod 5$. [4]

(b) Show that for $a, b, c \in \mathbb{Z}$ such that $3a^2 + b^2 = 5c^2$, $a \equiv 0 \pmod 5$ and $b \equiv 0 \pmod 5$. [4]

(c) Show that $(0,0,0)$ is the only solution for $3a^2 + b^2 = 5c^2$ from the set $\{(x, y, z) : x, y, z \in \mathbb{Z}\}$. [3]

(d) Is $(0,0,0)$ is the only solution for $4a^2 + 4b^2 = 5c^2$ from the set $\{(x, y, z) : x, y, z \in \mathbb{Z}\}$? Justify your answer briefly. [1]

7. The sequence of non-negative integers F_0, F_1, F_2, $\dots$ is defined by the recurrence relation

$$F_{n+2} = F_{n+1} + F_n \quad \text{for } n \geq 0,$$

with $F_0 = 0$ and $F_1 = 1$.

(a) Show that, for non-negative integers n,

$$F_{n+2}F_{n+5} - F_{n+3}F_{n+4} = F_n F_{n+3} - F_{n+1}F_{n+2}.$$ [3]

(b) Find all possible values of $F_n F_{n+3} - F_{n+1}F_{n+2}$ for $n \geq 0$. [3]

(c) Show that, for $r \in \mathbb{Z}^+$,

$$\tan^{-1}\left(\frac{1}{F_{2r}}\right) = \tan^{-1}\left(\frac{1}{F_{2r+1}}\right) + \tan^{-1}\left(\frac{1}{F_{2r+2}}\right)$$ [3]

(d) Given that the infinite series

$$\sum_{r=1}^{\infty} \tan^{-1}\left(\frac{1}{F_{2r+1}}\right)$$

converges, find the limit of this infinite series, leaving your answer in exact form.

[3]

8. **(a)** Let P be the set of 4-digit numbers *wxyz* such that $w, x, y, z \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and w, x, y, z are pairwise distinct. Find the number of members *wxyz* in P such that $w \neq 2$, $x \neq 4$, $y \neq 6$ and $z \neq 8$. [4]

(b) A school wants to distribute 99 identical notepads to four classes, Class A, Class B, Class C and Class D.

(i) Find the number of ways to distribute the notepads if Class A must have at least one notepad and Class B has lesser than nine notepads. [3]

(ii) Find the number of ways to distribute the notepads if Class A and Class B both have odd number of notepads. [2]

(iii) Each of the 4 classes contributes a unique item to present to the school. How many ways are there to put the items into identical boxes for packing where each box must contain at least one item? [2]

(c) When they first enter the school, all the students in the school are distributed into a classes. After orientation, they are redistributed into $a+b$ classes, where $a, b > 0$. Show that there are at least $b+1$ students who have fewer classmates after orientation than before orientation. [4]

9. **(a)** Using mathematical induction, or otherwise, prove that

$$\frac{1\times 3\times 5\times ...(2n-1)}{2\times 4\times 6\times ...(2n)} < \frac{1}{\sqrt{2n+1}} \text{ for } n \in \mathbb{Z}^+.$$ [5]

(b) **(i)** Given that $x_1 + x_2 ... + x_{10} = 10$ and $x_1{}^2 + x_2{}^2 + ... + x_{10}{}^2 = 10$ for $x_i \in \mathbb{R}$, where $i = 1, 2, 3...10$, by using Cauchy-Schwarz inequality or otherwise, prove that $x_1 = x_2 ... = x_{10} = 1$ is the only solution. [2]

(ii) If $x_1 + x_2 + x_3 ... + x_{10} = 10$ and $x_1{}^2 + x_2{}^2 + x_3{}^2 + ... + x_{10}{}^2 = 11$ for $x_i \in \mathbb{R}$, where $i = 1, 2, 3...10$, prove that $x_i \geq a$, where a is the greatest lower bound to be determined. Hence, find the values of $x_2, x_3 ... x_{10}$ if $x_1 = a$. [4]

(c) Given that $\frac{1}{a^2+1} + \frac{1}{b^2+1} + \frac{1}{c^2+1} = 2$ for $a, b, c \in \mathbb{R}^+$, prove that $ab + bc + ac \leq \frac{3}{2}$.

[5]

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