5. Sequences and Series (solutions)

1 **<u>MJC PROMO 2008/QN6</u>**

(a)

$$S_\infty = \frac{9}{2} = \frac{a}{1-r}, \quad -1< r <1 \text{ ------ } (1)$$

$$T_2 = ar = -2 \text{ ---------- } (2)$$

$$\frac{(2)}{(1)}: \quad r(1-r) = -\frac{4}{9} \quad \Rightarrow \quad r^2 - r - \frac{4}{9} = 0$$

$$\Rightarrow r = \frac{4}{3}(\text{n.a as } |r| < 1) \text{ or } r = \frac{-1}{3}$$

(b) <u>Method 1</u>

Sum of first k terms, $S_k = \frac{k}{2}[2a + (k-1)d]$. $T_{n-k+1} = a + (n-k)d$

The last k terms are $a + (n-k)d, a + (n-k+1)d, ..., a + (n-1)d$.

I.e. it forms another AP with first term $= a + (n-k)d$, last term $= a + (n-1)d$

number of terms $= k$

Let S_k' be the sum of the last k terms.

$$S_k' = \frac{k}{2}[a + (n-k)d + a + (n-1)d]$$

$$= \frac{k}{2}[2a + nd - kd + nd - d] = \frac{k}{2}[2a + 2nd - kd - d]$$

$$S_k' - S_k = \frac{k}{2}[2a + 2nd - kd - d] - \frac{k}{2}[2a + (k-1)d]$$

$$= \frac{k}{2}[2a + 2nd - kd - d - 2a - kd + d]$$

$$= \frac{k}{2}[2nd - 2kd] = (n-k)kd$$

<u>Method 2</u>

Sum of first k terms, $S_k = \frac{k}{2}[2a + (k-1)d]$.

Let S_k' be the sum of the last k terms

$$S_k' = S_n - S_{n-k}$$

$$= \frac{n}{2}[2a + (n-1)d] - \frac{n-k}{2}[2a + (n-k-1)d]$$

$$= \frac{1}{2}[2ak + 2knd - k^2 d - kd]$$

$$S_k\text{'} \; - \; S_k = \frac{1}{2}[2ak + 2knd - k^2d - kd] - \frac{k}{2}[2a + (k-1)d]$$

$$= \frac{1}{2}[2ak + 2knd - k^2d - kd - 2ak - k^2d + kd]$$

$$= \frac{1}{2}[2knd - 2k^2d] = (n-k)kd$$

Method 3

Sum of first k terms, $S_k = \frac{k}{2}[2a + (k-1)d]$.

(Think of the last k terms as an AP going backwards from the last term.)

$a + (n-1)d,\ a + (n-2)d,\ a + (n-3)d...,\ a + (n-k)d.$

Therefore, first term = $a + (n-1)d$

common difference = $-d$

number of terms = k

Let S_k' be the sum of the last k terms.

$$S_k\text{'} = \frac{k}{2}\{2[a + (n-1)d] + (k-1)(-d)\}$$

$$S_k\text{'} \; - \; S_k = \frac{k}{2}\{2[a + (n-1)d] + (k-1)(-d)\} - \frac{k}{2}[2a + (k-1)d]$$

$$= \frac{k}{2}[2nd - 2d - kd + d - kd + d]$$

$$= (n-k)kd$$

Method 4

Difference between the sum of last k terms and the sum of 1st k terms

$$= \sum_{r=n-k+1}^{n}\left[a + (r-1)d\right] - \sum_{r=1}^{k}\left[a + (r-1)d\right]$$

$$= \sum_{r=1}^{n}\left[a + (r-1)d\right] - \sum_{r=1}^{n-k}\left[a + (r-1)d\right] - \sum_{r=1}^{k}\left[a + (r-1)d\right]$$

$$= an + d\left[\frac{n}{2}\left(1+n\right) - n\right] - \left\{a(n-k) + d\left[\frac{n-k}{2}\left(1+n-k\right) - n + k\right]\right\} - \left\{ak + d\left[\frac{k}{2}\left(1+k\right) - k\right]\right\}$$

	$= d\left[\frac{n}{2}\left(1+n\right)-n-\frac{n-k}{2}\left(1+n-k\right)+n-k-\frac{k}{2}\left(1+k\right)+k\right]$ $= d\left[\frac{n}{2}\left(1+n\right)-\frac{n-k}{2}\left(1+n-k\right)-\frac{k}{2}\left(1+k\right)\right]$ $= \frac{d}{2}\left[\left(n+n^2-n-n^2+kn\right)+k\left(1+n-k-1-k\right)\right]$ $= \frac{d}{2}\left[kn+k\left(n-2k\right)\right]$ $= \frac{d}{2}\left[2kn-2k^2\right]$ $= dk\left(n-k\right)$ (shown)
2	**<u>HCI PROMO 2010/QN6</u>** **(a)** Sum of first 3 terms = Sum of the next 6 terms $3a+3d=6a+33d \Rightarrow d=-\frac{a}{10}$ **(b) (i)** Given $S_n=\left(a-2\right)^{-n}-1$, $T_n=S_n-S_{n-1}=\frac{1}{\left(a-2\right)^n}-\frac{1}{\left(a-2\right)^{n-1}}=\frac{1-\left(a-2\right)}{\left(a-2\right)^n}=\frac{3-a}{\left(a-2\right)^n}$ $\frac{T_n}{T_{n-1}}=\frac{\frac{3-a}{\left(a-2\right)^n}}{\frac{3-a}{\left(a-2\right)^{n-1}}}=\frac{1}{a-2}$, a constant. Thus the sequence is a GP, with common ratio $\frac{1}{a-2}$. **(ii)** For S_∞ to exist, $\left\|\frac{1}{a-2}\right\|<1 \Rightarrow \left\|a-2\right\|>1 \Rightarrow a>3$ or $a<1$ $\left\{a\in\mathbb{R}:a>3 \quad \text{or} \quad a<1\right\}$
3	**<u>SAJC PROMO 2010/QN5a</u>** **(i)** $T_n=S_n-S_{n-1}=\frac{2^{n+1}}{n!}-1-\frac{2^n}{\left(n-1\right)!}+1$ $=\frac{2^{n+1}}{n!}-\frac{2^n}{\left(n-1\right)!}=\frac{2^{n+1}-2^n n}{n!}=\frac{2^n(2-n)}{n!}$ $T_4+T_5+...T_8=S_8-S_3$ $=\frac{2^9}{8!}-1-\frac{2^4}{3!}+1$ $=\frac{4}{315}-\frac{8}{3}=-\frac{836}{315}$

4 **JJC PROMO 2009/QN3**

(a) (i) $a = 2009$, $r = -\frac{5}{7}$

$$|U_n| < \frac{1}{2009}$$

$$\left|2009\left(-\frac{5}{7}\right)^{n-1}\right| < \frac{1}{2009}$$

$$2009\left(\frac{5}{7}\right)^{n-1} < \frac{1}{2009}$$

$$(n-1)\ln\left(\frac{5}{7}\right) < \ln\left(\frac{1}{2009^2}\right)$$

$$(n-1) > \frac{-2\ln(2009)}{\ln\left(\frac{5}{7}\right)}$$

$$n > 46.2$$

Least n = 47

(ii) The negative terms of the series is U_2, U_4, U_6, …

$$2009\left(-\frac{5}{7}\right), 2009\left(-\frac{5}{7}\right)^3, 2009\left(-\frac{5}{7}\right)^5, \ldots$$

New GP with first term $2009\left(-\frac{5}{7}\right) = -1435$, Common ratio $\left(-\frac{5}{7}\right)^2 = \frac{25}{49}$

So sum to infinity exists and $S_\infty = \frac{a}{1-r} = \frac{-1435}{1-\left(\frac{25}{49}\right)} = -2929.79$

(b) 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5………., k……

$$T_k \quad \underset{1}{1} \quad \underset{2}{2\ 2} \quad \underset{3}{3\ 3\ 3} \quad \underbrace{4\ 4\ 4\ 4}_{4} \quad \underbrace{5\ 5\ 5\ 5\ 5}_{5} \; \ldots\ldots.$$

$T_k = k$

$\{T_k\}$ is an AP with $a = d = 1$ and S_k gives the position of the last term the number k appears in the given sequence.

Consider $S_k = 1000$

$$\Rightarrow \frac{k}{2}(1+k) = 1000$$

$$\Rightarrow k^2 + k - 2000 = 0$$

$$\Rightarrow k = 44.2$$

$\therefore$ The 1000th term is 45

5

HCI PROMO 2009/QN8

(i) $$\frac{u_n}{u_{n-1}} = \frac{\frac{1}{4^{n-1}}}{\frac{1}{4^{n-2}}} = \frac{1}{4} \text{ (a constant)}$$

$\therefore$ It's a GP.

(ii) $$S_n = \frac{1\left(1-\frac{1}{4^n}\right)}{1-\frac{1}{4}} \quad = \frac{4}{3}\left(1-\frac{1}{4^n}\right)$$

(iii) $$S_\infty = \frac{1}{1-\frac{1}{4}} = \frac{4}{3} \text{ or } S_\infty = \lim_{n\to\infty}\frac{4}{3}\left(1-\frac{1}{4^n}\right) = \frac{4}{3}$$

$\left|S_n - S_\infty\right| < 0.01S_\infty$

Since $S_\infty > S_{n,}$

(iv) $$\frac{4}{3}-\frac{4}{3}\left(1-\left(\frac{1}{4}\right)^n\right) < 0.01\left(\frac{4}{3}\right)$$

$$\left(\frac{1}{4}\right)^n < 0.01$$

$$n > \frac{\ln 0.01}{\ln 0.25} = 3.32$$

Alternatively,

Use GC table

$n = 3$, $\left(\frac{1}{4}\right)^n = 0.01563 > 0.01$

$n = 4$, $\left(\frac{1}{4}\right)^n = 0.00391 < 0.01$

$\therefore$ least $n = 4$

6

RVHS PROMO 2009/7

(i) In 2001, $\frac{9}{10}(1200)+100 = 1180$

In 2002, $\frac{9}{10}(1180)+100 = 1162$

(ii) In 2002, $\left(\frac{9}{10}\right)^2(1200)+\left(\frac{9}{10}\right)100+100$

In 2003, $\left(\frac{9}{10}\right)^3(1200)+\left(\frac{9}{10}\right)^2(100)+\left(\frac{9}{10}\right)100+100$

n^{th} year,

	$\left(\frac{9}{10}\right)^n(1200)+\left(\frac{9}{10}\right)^{n-1}(100)+\left(\frac{9}{10}\right)^{n-2}100+...+100$ $=\left(\frac{9}{10}\right)^n(1200)+100\left[\left(\frac{9}{10}\right)^{n-1}+\left(\frac{9}{10}\right)^{n-2}+...+\left(\frac{9}{10}\right)+1\right]$ $=\left(\frac{9}{10}\right)^n(1200)+100\left[\frac{1-\left(\frac{9}{10}\right)^n}{1-\frac{9}{10}}\right]$ $=\left(\frac{9}{10}\right)^n(1200)+1000(1-0.9^n)$ (Shown) **(iii)** As $n\to\infty,$ $\left(\frac{9}{10}\right)^n\to 0,$ Hence population $\to 1000.$
7	**MJC PROMO 2015/QN11** **(a)** $T_n=S_n-S_{n-1}$ $=n(4n-1)-(n-1)\left[4(n-1)-1\right]$ $=8n-5$ $T_n-T_{n-1}=8n-5-\left[8(n-1)-5\right]$ $=8\ (\text{constant})$ The series is arithmetic. **(b)** T_2,T_5,T_7 of an AP forms 3 consecutive terms of a GP **(i)** $\frac{a+4d}{a+d}=\frac{a+6d}{a+4d}$ $a^2+8ad+16d^2=a^2+7ad+6d^2$ $ad+10d^2=0$ since $d\neq 0, a=-10d$ common ratio, $r=\frac{-10d+4d}{-10d+d}=\frac{2}{3}$ Since $\lvert r\rvert=\frac{2}{3}<1,$ series is convergent. **(ii)** Even-numbered terms of GP: $\frac{2}{3}a,\left(\frac{2}{3}\right)^3a,\left(\frac{2}{3}\right)^5a,...$ $\therefore S=\frac{\frac{2}{3}a}{1-\left(\frac{2}{3}\right)^2}=\frac{6a}{5}$

	$S + A_n < 0$ $\frac{6a}{5} + \frac{n}{2}\left[2a + \left(n-1\right)d\right] < 0$ $\frac{6a}{5} + \frac{n}{2}\left[2a + \left(n-1\right)\left(\frac{-1}{10}a\right)\right] < 0$ since $a > 0$, $\frac{6}{5} + \frac{n}{2}\left[2 - \frac{n-1}{10}\right] < 0$ Using GC, when $n = 22$, $\frac{6}{5} + \frac{n}{2}\left[2 - \frac{n-1}{10}\right] = 0.1 > 0$ when $n = 23$, $\frac{6}{5} + \frac{n}{2}\left[2 - \frac{n-1}{10}\right] = -1.1 < 0$ $\therefore$ least $n = 23$
8	**DHS PROMOS 2010/Q8** **(i)** **Method 1 (considering the sides)** 1st term = 4 cm, common diff = 2 cm Total perimeter $S_{30} = 4\times\left[\frac{30}{2}\left[2(4)+(30-1)(2)\right]\right]$ $= 3960$ cm **Method 2 (considering the perimeter)** 1st term, a = 16 cm, common diff , d = 8 cm Total perimeter $S_{30} = \left[\frac{30}{2}\left[2(16)+(30-1)(8)\right]\right]$ $= 3960$ cm **(ii)** $S_n \le 10000$ $4\times\left[\frac{n}{2}\left[2(4)+(n-1)(2)\right]\right] \le 10000$ $\left[\frac{n}{2}\left[2(16)+(n-1)(8)\right]\right] \le 10000$ **(iii)** $4n^2 + 12n - 10000 \le 0$ $-51.5 \le n \le 48.5$ Thus largest n = 48. For largest square length, $T_{48} = 4 + (48-1)(2)$ $= 98$ cm

	Alternative presentation $10000=4\times\left[\frac{n}{2}\left[2(4)+(n-1)(2)\right]\right]$ $=\left[\frac{n}{2}\left[2(16)+(n-1)(8)\right]\right]$ $=4n^2+12n$ $4n^2+12n-10000=0.$ Fr GC, $n=48.5$ When $n=48, 4n^2+12n=4(48)^2+12(48)<10000$ When $n=49, 4n^2+12n=4(49)^2+12(49)>10000$ For largest square length, $T_{48}=4+(48-1)(2)$ take n = 48, $=98$ cm **(iii)** Total area of three circles $=(4\times 4)+(6\times 6)+(8\times 8)=116$ cm^2 Area of smallest circle, $\pi k^2=16$ cm^2 $\pi k^2+\pi(kR)^2+\pi(kR^2)^2=116$ $\pi k^2(1+R^2+R^4)=116$ $R^4+R^2+1=\frac{116}{\pi k^2}=\frac{116}{16}$ $R^4+R^2-6.25=0$ $R^2=\frac{-1\pm\sqrt{1-4(-6.25)}}{2}$ $R=\pm 1.43$ Common ratio R = 1.43 (3s.f.) (R= –1.43 rejected since R > 0)
9	**PJC Promo 2013/14(a)** At the end of first year, we have $A\left(1+\frac{R}{100}\right)$ At the end of second year, we have $\left(A+A\left(1+\frac{R}{100}\right)\right)\left(1+\frac{R}{100}\right)=A\left(1+\frac{R}{100}\right)+A\left(1+\frac{R}{100}\right)^2$

Similarly, at the end of *n*th year, we have

$$A\left(1+\frac{R}{100}\right)+A\left(1+\frac{R}{100}\right)^2+...+A\left(1+\frac{R}{100}\right)^n=A\left(1+\frac{R}{100}\right)\left(\frac{1-\left(1+\frac{R}{100}\right)^n}{1-\left(1+\frac{R}{100}\right)}\right)$$

$$=A\left(1+\frac{100}{R}\right)\left(\left(1+\frac{R}{100}\right)^n-1\right)$$

(i) $T_{10}=A\left(1+\frac{100}{8}\right)\left[\left(1+\frac{8}{100}\right)^{10}-1\right]$=50 000

A = 3195.8097≈ \$3196

(ii) $1000\left(1+12.5\right)(1.08^n-1)\ \geq\ 50000$

$1.08^n\ \geq\ 4.7037$

$n\ \geq\frac{1g4.7037}{\lg 1.08}=20.12$

She needs 21 years to reach \$50000.

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<u>SAJC PROMO 2009/QN13</u>

(a) (i) AP: $a=1200, d=50$

$T_k=3250\Rightarrow 1200+(k-1)(50)=3250\Rightarrow k=42$

$P=S_k=\tfrac{42}{2}[1200+3250]=93450$

(ii) GP: $r=0.98$

$\text{Value}=72000(0.98)^{24}=44336\text{(nearest dollar)}$

(iii) $72000(0.98)^{n-12}\leq 50\%\times 93450$

$$(0.98)^{n-12}\leq\frac{0.5\times 93450}{72000}$$

$$n-12\geq\frac{\ln\frac{0.5\times 93450}{72000}}{\ln(0.98)}$$

$$n\geq 33.4$$

Therefore, least $n=34$

(b) Given that $a_1, a_2, a_3, \ldots, a_n$ is an arithmetic progression with 1st term a and common difference d.

$$\frac{b_r}{b_{r-1}} = \left(\frac{1}{3}\right)^{a_r} \Big/ \left(\frac{1}{3}\right)^{a_{r-1}} = \left(\frac{1}{3}\right)^{a+(r-1)d-(a+(r-2)d)}$$

$$= \left(\frac{1}{3}\right)^{d} = \text{constant}$$

Given that $b_2 = 27$ and the common difference of the arithmetic progression is 2

$$b_2 = \left(\frac{1}{3}\right)^{a+d} \Rightarrow 27 = \left(\frac{1}{3}\right)^{a+2} \Rightarrow 27 = \left(\frac{1}{3}\right)^{a}\left(\frac{1}{9}\right)$$

$$\Rightarrow 243 = \left(\frac{1}{3}\right)^{a} \Rightarrow \text{a} = \frac{\ln(3)^5}{\ln(3)^{-1}} = -5$$

$$a_r = -5 + (r-1)(2) = 2r - 7$$

11(a)

SAJC/2015/Promo/10

stage	Distance run at that stage
1	2(5)
2	2[2(5)]
3	2[3(5)]
⋮	⋮
n	2[n(5)]

Total distance run after n stages

$= 2(5)(1+2+3+\ldots+n)$

$= 2(5)\left[\frac{n}{2}(1+n)\right]$

$= 5n(n+1)$

$$5n(n+1) \geq 6000$$

$$5n^2 + 5n - 6000 \geq 0$$

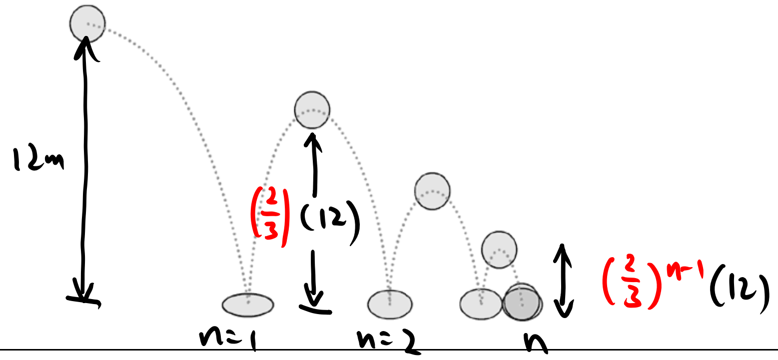
$$n^2 + n - 1200 \geq 0$$

Using the GC,

$n \leq -35.14$ or $n \geq 34.14$

Hence the least number of stages is 35.

(b)



Total distance travelled by the ball before it touches the surface for the *n*th time

$$=12+2\left[\left(\frac{2}{3}\right)(12)+\left(\frac{2}{3}\right)^2(12)+\ldots+\left(\frac{2}{3}\right)^{n-1}(12)\right]$$

$$=12+16\left[1+\frac{2}{3}+\ldots+\left(\frac{2}{3}\right)^{n-2}\right]$$

$$=12+16\left[\frac{1-\left(\frac{2}{3}\right)^{n-1}}{1-\frac{2}{3}}\right]$$

$$=12+48\left[1-\left(\frac{2}{3}\right)^{n-1}\right]$$

$$=60-48\left(\frac{3}{2}\right)\left(\frac{2}{3}\right)^n$$

$$=60-72\left(\frac{2}{3}\right)^n$$

As $n\to\infty$, $\left(\frac{2}{3}\right)^n\to 0$,

hence, total distance travelled by the ball when it comes to rest is 60 m.

12

RJC PROMO 2010/6

Let *d* be the common difference of the arithmetic progression.

u_1, u_4 and u_8 are in geometric progression, we have $\frac{u_4}{u_1}=\frac{u_8}{u_4}$

ie. $u_4^2=u_1u_8$

$(u_1+3d)^2=u_1(u_1+7d)$

$u_1^2+6u_1d+9d^2=u_1^2+7u_1d$

$u_1d-9d^2=0$

$d(u_1-9d)=0$

$d=0$ (rejected, since given A.P. is increasing)

or $u_1=9d$ -------(1)

Also $u_{10}+u_{12}+u_{14}+\ldots+u_{38}+u_{40}=1056$

$\frac{16}{2}(u_{10}+u_{40})=1056$

$(u_1+9d)+(u_1+39d)=132$

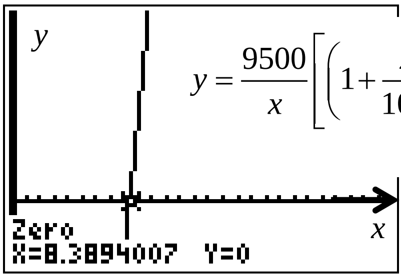
$u_1+24d=66$------(2)

	Substitute (1) into (2): $33d=66$ $d=2$ $\Rightarrow u_1=18$ $u_{108}=u_1+107d=18+107(2)=232\text{(shown)}$
13	**TJC PROMO 2010/Q4** Let *a* and *r* be the first term and common ratio of the arithmetic and geometric progression respectively. $\left(a-2\right)+\left(4r\right)=-4\Rightarrow a=-2-4r$ ------------- (1) $\left(a+4\right)+\left(a-2+4r\right)+\left(a-4+4r^2\right)=-4$ $\Rightarrow\left(a+4\right)-4+\left(a-2(2)+4(r^2)\right)=-4$ $\Rightarrow a+2r^2=0$ --------------- (2) Sub (1) into (2), $\left(-2-4r\right)+2r^2=0$ $2r^2-4r-2=0$ $r^2-2r-1=0$ $r=\dfrac{2\pm2\sqrt{2}}{2}=1\pm\sqrt{2}$ $\therefore r=1-\sqrt{2}$ (reject $r=1+\sqrt{2}$ $\because \lvert r\rvert<1$ as G.P is convergent) First term of *H* = $\left(-2-4r\right)+4 \quad =2-4r$ $=2-4\left(1-\sqrt{2}\right) \ =4\sqrt{2}-2$
14	**RI PROMO 2014/QN6** (i) For $n\le15$, $p(n)=\dfrac{n}{2}\left[2\left(9500\right)+\left(n-1\right)400\right]$ $=9500n+200n\left(n-1\right)$ $=200n^2+9300n.$ For $n>15$, $p(n)=200\left(15^2\right)+9300\left(15\right)+\left(n-15\right)15100$ $=15100n-42000.$ Hence, $p(n)=\begin{cases}200n^2+9300n, & n\le15,\\ 15100n-42000, & n>15,\end{cases}$ where $\text{A}=-42000$ (Shown).

(ii)

$$\frac{9500\left[\left(1+\frac{r}{100}\right)^{12}-1\right]}{1+\frac{r}{100}-1}=200\left(15^2\right)+9300\left(15\right) \qquad \text{...(1)}$$

$$\Rightarrow \qquad \frac{9500}{r}\left[\left(1+\frac{r}{100}\right)^{12}-1\right]=1845.$$



$$y=\frac{9500}{x}\left[\left(1+\frac{x}{100}\right)^{12}-1\right]-1845$$

Zero
X=8.3894007 Y=0

From GC, $r=8.39$ (2 dp).

15 **<u>IJC PROMO 2010/Q12</u>**

(i) **(a)** No. of integers in 1st set $=1=2^0$

No. of integers in 2nd set $=2=2^1$

No. of integers in 3rd set $=4=2^2$

No. of integers in 4th set $=8=2^3$

No. of integers in rth set $=2^{r-1}$

(b) First integer in 1st set $=2=2^1$

First integer in 2nd set $=4=2^2$

First integer in 3rd set $=8=2^3$

First integer in 4th set $=16=2^4$

First integer in rth set $=2^r$

(c) Last integer in 1st set $=2=2^2-2$

Last integer in 2nd set $=6=2^3-2$

Last integer in 3rd set $=14=2^4-2$

Last integer in 4th set $=30=2^5-2$

Last integer in rth set $=2^{r+1}-2$

(ii) No. of integers in 50th set $= 2^{49}$

First integer in 50th set $= 2^{50}$

Last integer in 50th set $= 2^{51} - 2$

$$T = \frac{2^{50}}{2}(2^{50} + 2^{51} - 2)$$
$$= 2^{48}(2^{50} + 2\times 2^{50} - 2)$$
$$= 2^{48}(3\times 2^{50} - 2) \quad \text{(shown)}$$

(iii) Total no. of terms in all *r* sets

$$= \sum_{r=1}^{n} 2^{r-1}$$
$$= \frac{1}{2}\left[\frac{2(2^n - 1)}{(2-1)}\right]$$
$$= 2^n - 1$$

$$S_n = \frac{2^n - 1}{2}(2 + 2^{n+1} - 2)$$
$$= \frac{2^n - 1}{2}(2^{n+1})$$
$$= 2^n(2^n - 1)$$

Alternatively:

No. of integers in *r*th set $= 2^{r-1}$

First integer in *r*th set $= 2^r$

Last integer in *r*th set $= 2^{r+1} - 2$

$\Rightarrow$ Sum of *r*th set $= 2^{r-2}(3\times 2^r - 2)$

$$S_n = \sum_{r=1}^{n}\left[2^{r-2}(3\times 2^r - 2)\right]$$
$$= \sum_{r=1}^{n}\left[3\times 2^{2r-2} - 2^{r-1}\right]$$
$$= \frac{3}{4}\sum_{r=1}^{n} 4^r - \frac{1}{2}\sum_{r=1}^{n} 2^r$$
$$= \frac{3}{4}\left[\frac{4(4^n - 1)}{(4-1)}\right] - \frac{1}{2}\left[\frac{2(2^n - 1)}{(2-1)}\right]$$
$$= 4^n - 2^n$$
$$= 2^n(2^n - 1)$$

	(iv) $S_n > 100\ 000$ $2^n(2^n-1) > 100\ 000$ $(2^n)^2-(2^n)-100\ 000 > 0$ $2^n > 316.72816$ or $2^n < -315.72816$ $n > 8.307$ (N.A. Since $2^n > 0$) Least *n* is 9

16	**(i)** $\sum_{r=2}^{\infty}\frac{2r+1}{r^2(r+1)^2}=\sum_{r=1}^{\infty}\frac{2r+1}{r^2(r+1)^2}-\frac{3}{4}$ As $N\to\infty$, $\frac{1}{(N+1)^2}\to 0$, thus $\sum_{r=1}^{\infty}\frac{2r+1}{r^2(r+1)^2}=1$ $\therefore \sum_{r=2}^{\infty}\frac{2r+1}{r^2(r+1)^2}=1-\frac{3}{4}=\frac{1}{4}$
	(ii) Replace *r* with *r*–1 $\sum_{r=0}^{N-2}\frac{2r+3}{(r+1)^2(r+2)^2}$ $=\sum_{r-1=0}^{r-1=N-2}\frac{2(r-1)+3}{((r-1)+1)^2((r-1)+2)^2}$ $=\sum_{r=1}^{N-1}\frac{2r+1}{r^2(r+1)^2}$ $=1-\frac{1}{N^2}$

17	**(i)** As $n \to \infty$, $\frac{3}{2(n+1)} \to 0$ and $\frac{1}{2(n+2)} \to 0$. Hence $\frac{7}{4} - \frac{3}{2(n+1)} - \frac{1}{2(n+2)} \to \frac{7}{4}$. Therefore convergence limit is $\frac{7}{4}$.
	(ii) $\frac{7}{4} - \frac{3}{2(n+1)} - \frac{1}{2(n+2)} > \frac{8}{5}$ $\Rightarrow \frac{3}{2(n+1)} + \frac{1}{2(n+2)} < \frac{3}{20} = 0.15$. From G.C., When $n = 12$, $\frac{3}{2(12+1)} + \frac{1}{2(12+2)} = 0.1511$ When $n = 13$, $\frac{3}{2(13+1)} + \frac{1}{2(13+2)} = 0.1405$ Hence least value of *n* is 13.
	(iii) $\frac{9}{3\times4\times5} + \frac{11}{4\times5\times6} + \frac{13}{5\times6\times7} + \ldots\ldots + \frac{2N+1}{N\left(N^2-1\right)}$ $= \sum_{r=3}^{N-1} \frac{2r+3}{r\left(r+1\right)\left(r+2\right)}$ $= \frac{7}{4} - \frac{3}{2N} - \frac{1}{2\left(N+1\right)} - \frac{5}{6} - \frac{7}{24}$ $= \frac{5}{8} - \frac{3}{2N} - \frac{1}{2\left(N+1\right)}$.

18 **(i)**

$$u_2 = \frac{3}{4}u_1 + 4$$

$$u_3 = \frac{3}{4}u_2 + 4 = \frac{3}{4}\left(\frac{3}{4}u_1 + 4\right) + 4$$

$$= \left(\frac{3}{4}\right)^2 u_1 + \frac{3}{4}(4) + 4$$

$$\vdots$$

$$u_n = \left(\frac{3}{4}\right)^{n-1} u_1 + \left(\frac{3}{4}\right)^{n-2} 4 + \left(\frac{3}{4}\right)^{n-3} 4 + \ldots + 4$$

$$= \left(\frac{3}{4}\right)^{n-1} u_1 + 4\left[\left(\frac{3}{4}\right)^{n-2} + \left(\frac{3}{4}\right)^{n-3} + \ldots + 1\right]$$

$$= \left(\frac{3}{4}\right)^{n-1} u_1 + 4\left[\frac{1-\left(\frac{3}{4}\right)^{n-1}}{1-\frac{3}{4}}\right]$$

$$= \left(\frac{3}{4}\right)^{n-1} u_1 + 16\left[1-\left(\frac{3}{4}\right)^{n-1}\right]$$

$$= \left(\frac{3}{4}\right)^{n-1} (3) + 16\left[1-\left(\frac{3}{4}\right)^{n-1}\right]$$

$$= 16 - 13\left(\frac{3}{4}\right)^{n-1}$$

So, $A = -13$.

(ii)

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\left[16 - 13\left(\frac{3}{4}\right)^{n-1}\right]$$

$$= 16 \qquad \text{since } \left(\frac{3}{4}\right)^{n-1} \to 0 \text{ as } n \to \infty.$$

Therefore, the sequence converges.

19	**(i)** When n is large, x_n and x_{n+1} both converge tend to α. Thus $\alpha = \dfrac{\alpha^2+6\alpha}{\alpha^2+\alpha+1}$ $\alpha^3+\alpha^2+\alpha=\alpha^2+6\alpha$ $\alpha^3-5\alpha = 0$ $\alpha(\alpha^2-5) = 0$ $\alpha = 0$ or $\pm\sqrt{5}$ **(ii)** By using a GC, $x_1 = 0.5$, $x_2 = \dfrac{13}{7}$ or 1.86 , $x_3 = 2.31$ and $x_4 = 2.22$ The sequence $\{x_n\}$ converges to $\sqrt{5}$ when n is large.
20	**(i)** As $n \to \infty$, $\dfrac{1}{n-1} \to 0$ and $\dfrac{1}{n} \to 0$. So $\displaystyle\sum_{r=3}^{n}\frac{1}{r(r-2)} \to \frac{1}{2}\left(\frac{3}{2}\right)=\frac{3}{4}$. Hence the series is convergent, and the sum to infinity is $\dfrac{3}{4}$.
	(ii) $r(r-2)=r^2-2r=(r-1)^2-1$ $(r-1)^2-1<(r-1)^2$ $\dfrac{1}{(r-1)^2-1}>\dfrac{1}{(r-1)^2}$ $\therefore \dfrac{1}{(r-1)^2}<\dfrac{1}{r(r-2)}$ $\displaystyle\sum_{r=3}^{\infty}\frac{1}{(r-1)^2}<\sum_{r=3}^{\infty}\frac{1}{r(r-2)}$ $\dfrac{1}{2^2}+\dfrac{1}{3^2}+\dfrac{1}{4^2}+\dfrac{1}{5^2}+\cdots<\dfrac{3}{4}$ $\displaystyle\frac{1}{2^2}+\sum_{r=3}^{\infty}\frac{1}{r^2}<\frac{3}{4}$ $\displaystyle\sum_{r=3}^{\infty}\frac{1}{r^2}<\frac{1}{2}$ **Alternatively,** $\displaystyle\sum_{r=3}^{\infty}\frac{1}{r^2}=\sum_{r=4}^{\infty}\frac{1}{(r-1)^2}<\sum_{r=4}^{\infty}\frac{1}{r(r-2)}=\sum_{r=3}^{\infty}\frac{1}{r(r-2)}-\frac{1}{3(3-2)}$ $=\dfrac{3}{4}-\dfrac{1}{3}=\dfrac{5}{12}<\dfrac{1}{2}$ (Shown)

21	**a) i)** Given $u_{n+1} = \dfrac{4(1+u_n)}{4+u_n}$, $u_1 = 1$ Since $u_n \to l \Rightarrow u_{n+1} \to l$ $\Rightarrow \quad l = \dfrac{4(1+l)}{4+l}$ $\Rightarrow \quad 4l + l^2 = 4 + 4l$ $\Rightarrow \quad l^2 = 4 \quad \Rightarrow l = \pm 2$ Given $u_n > 0$ for all $n \in \mathbb{Z}^+ \Rightarrow l = 2$ (ans) **ii)** To show : $u_{n+1} > u_n$ if $u_n < l \Rightarrow u_{n+1} - u_n > 0$ if $u_n < 2$ LHS = $u_{n+1} - u_n$ $= \dfrac{4(1+u_n)}{4+u_n} - u_n$ $= \dfrac{4(1+u_n) - u_n(4+u_n)}{4+u_n} = \dfrac{4-u_n^{\ 2}}{4+u_n}$ Since $0 < u_n < 2 \Rightarrow (u_n)^2 < 4 \Rightarrow 4 - (u_n)^2 > 0$ and $4 + u_n > 0$, $\dfrac{4-u_n^{\ 2}}{4+u_n} > 0$ Thus $u_{n+1} - u_n > 0 \Rightarrow u_{n+1} > u_n$ (*proven*) .