2021 JC1 H1 Maths FE level up concepts Holiday REVISION

- **Please present all your answers on this copy as it will be submitted to your tutor when Term 1 starts in 2022.**
- **If in doubt, please read your lecture notes, if still further in doubt, put a note under 'Learning Point' so that you remember to clarify your doubts with your tutor. Where possible, answers should be left as fraction.**

Question	**Solutions**	**Learning Point**
1. Permutation and Combination (Arrangement)		
(i) Three friends, Andy, Ben and Charlie, attend a graduation ceremony with their parents. How many ways can the nine people be seated in a row if **each family must be seated together**? Ans: 1296	Arrange within family = 3! Arrange 3 families = $3!\times 3!\times 3!$ Total arrangement = $3!\times 3!\times 3!\times 3! = 1296$	
(ii) How many ways can the nine people be seated in a row if **each set of parents must be seated together**? Ans: 5760	Arrange within each set of parents = 2! Arrange 3 sets of parents and 3 friends = 6! Total arrangement = $(2!)^3 \times 6! = 5760$	
(iii) How many ways can the nine people be seated in a row if **the 3 friends must be seated together**? Ans: 30240	Arrange within the 3 friends (1 unit) = 3! Arrange 1 unit and 6 parents = 7! Total arrangement = $7!\times 3! = 30240$	

(iv) How many ways can the nine people be seated in a row if **the 3 friends wants to be separated?** Ans:151200	Arrange all 6 parents first (no restriction) = 6! **SLOT** the 3 friends into the 6 parents = ${}^{7}P_{3}$ Total arrangement = ${}^{7}P_{3} \times 6! = 151200$	
(v) How many ways can the nine people be seated in a row if **each set of parents must be seated together AND the 3 friends wants to be separated?** Ans:1152	Arrange within each set of parents = $(2!)^{3}$ Arrange 3 sets of parents = 3! **SLOT** the 3 friends into the 3 sets of parents = ${}^{4}P_{3}$ (Think carefully here!) Total arrangement = $(2!)^{3} \times 3! \times {}^{4}P_{3} = 1152$	
(vi) How many ways can the nine people be seated in a row if Andy must take the first seat, Ben must take the last seat and each set of parents must be seated together? Ans:192	Arrange within each set of parents = $(2!)^{3}$ Arrange Charlie and 3 sets of parents = 4! Total arrangement = $(2!)^{3} \times 4! = 192$	

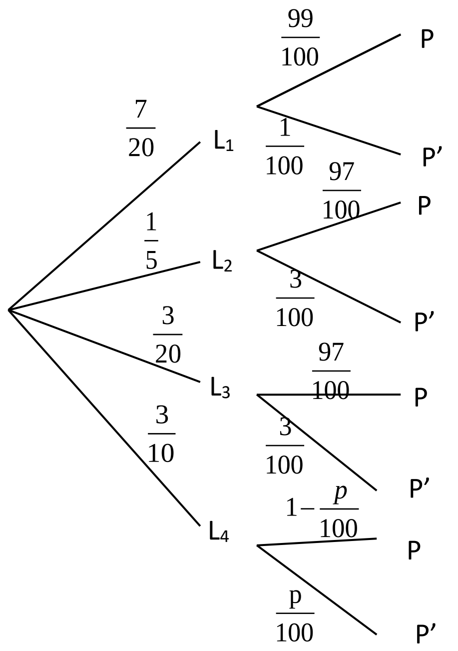
2. **Permutation and Combination (Combination)**		
In a college student committee, there are 4 Administrators, 5 Liaison officers and 6 Group Leaders. They are to form an Ex-Co that consists of a President, a Vice-President and a Secretary. (i) How many **selections** are there to form the Ex-Co? Ans 2730	Total selection (no restriction) = ${}^{15}P_3 = 2730$	
(ii) How many selections are there to form the Ex-Co if a President must be from the Administrators, a Vice-President must be from the Liaison Officers and a Secretary must be from the Group Leaders? Ans : 120	To select a President from Administrators = ${}^{4}C_1$ To select a Vice President from the Liaison Officers = ${}^{5}C_1$ To select a Secretary from the Group Leaders = ${}^{6}C_1$ Total selections = ${}^{4}C_1 \times {}^{5}C_1 \times {}^{6}C_1 = 120$	
(iii) How many selections are there to form the Ex-Co if **at least 1 of them** must be from the Administrators? Ans: 1740	<u>Method 1 (Complementary)</u> No restriction = ${}^{15}P_3 = 2730$ Nobody from the Administrators = ${}^{11}P_3 = 990$ Total selections = $2730 - 990 = 1740$ <u>Method 1 (Consider cases though not recommended)</u> Case 1: 1 from the Administrators, the other 2 from others = ${}^{4}C_1 \times {}^{11}C_2 \times 3! = 1320$ Case 2: 2 from Administrators, the other from others = ${}^{4}C_2 \times {}^{11}C_1 \times 3! = 396$ Case 3: 3 from Administrators = ${}^{4}P_3 = 24$ Total selections =1320 + 396 + 24 = 1740	

(iv) How many selections are there to form the Ex-Co if **at least 2 of them** must be from the Administrators **and none** must be from Group Leaders? Ans: 204	<u>Consider cases (only)</u> Case 1: 2 from Administrators, the other 1 from Liaison Officers = $^4C_2 \times {}^5C_1 \times 3! = 180$ Case 2: all 3 from the Administrators = $^4P_3 = 24$ Total selections = 180 + 24 = 204	
(v) How many selections are there to form the Ex-Co if **an Administrator is already made the President and only another one** must be from the Administrators? Ans: 66	one Ex-Co must be from the Administrators = $^3C_1$ 1 Ex-Co from others = $^{11}C_1$ Total selections = $^3C_1\ {}^{11}C_1 \times 2! = 66$	

3. Probability

In an egg production factory, eggs are packed into cases on 4 different production lines, L_1, L_2, L_3 and L_4. The proportion of packed eggs that come from production lines L_1, L_2, L_3 and L_4 are $\frac{7}{20}$, $\frac{1}{5}$, $\frac{3}{20}$ and $\frac{3}{10}$ respectively. Records show that a small percentage of eggs are not packed properly for sale. 1% from L_1, 3% from L_2, 3% from L_3 and p% from L_4. A case is chosen at random. Draw a tree diagram to represent this information.

(i) Given that the probability of the chosen case is faulty is 0.02. Find p.
Ans: $p = 2$



P: packed properly

P’: not packed properly

$$P(\text{a case is faulty}) = 0.02$$

$$\frac{7}{20}\left(\frac{1}{100}\right)+\frac{1}{5}\left(\frac{3}{100}\right)+\frac{3}{20}\left(\frac{3}{100}\right)+\frac{3}{10}\left(\frac{p}{100}\right)=0.02$$

$$0.014+\frac{3p}{1000}=0.02$$

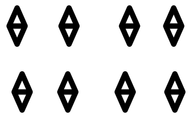
$$\frac{3p}{1000}=0.006$$

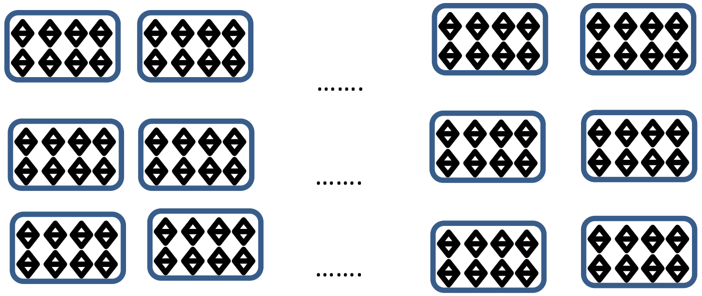
$$3p=6$$

$$p=2$$

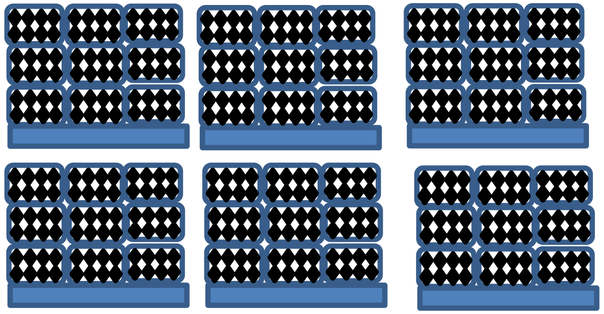
ii) Find the probability that the case is not faulty **and** is from production line L_1. Ans : $\frac{693}{2000}$	$P(\text{a case is from } L_1 \text{ and it is not faulty}) = \frac{7}{20}\left(\frac{99}{100}\right)$ $= 0.3465 \text{ or } \frac{693}{2000}$	
(iii) Find the probability that the case is not faulty **or** is from production line L_1. Ans : 0.9835	$P(\text{a case is from } L_1 \text{ or it is not faulty})$ $= \frac{7}{20} + \frac{98}{100} - \frac{693}{2000}$ $= 0.9835$	
(iv) Find the probability that a case that **is not faulty is from** L_1**.** Ans : $\frac{99}{280}$	$P(\text{a box is from } L_1 \mid \text{a box is not faulty})$ $= \frac{P(\text{a box is from } L_1 \text{ and is not faulty})}{P(\text{a box is not faulty})}$ $= \frac{0.3465}{1-0.02}$ $= \frac{99}{280}$	
(v) Ten randomly cases are chosen. Find the probability that none are faulty. Ans :0.817	Required probability = $(0.98)^{10}$ = 0.817	

4. Binomial Distribution

From a production line making toys, a fixed number of toys are inspected for faults and that 10% of the toys are faulty. (i) State, in context, two assumptions needed for the number of faulty toys found to be well modelled by a binomial distribution.	(Looking for keywords): (event+independence; probability+constant) Assumption 1: event that a faulty toy selected is independent of another faulty toy selected Assumption 2: Probability that a toy is faulty is a constant at 0.1 for each toy.	
(ii) 8 toys were selected. Find the probability that **less than 4** toys are faulty. Ans: 0.995 	Let X be no of faulty toys out of 8. X~B(___8____ , ___0.1_______) Less than 4 means: $\mathrm{P}(X<4)=\mathrm{P}(X\le 3)=0.99498\approx 0.995$	
(iii) 8 toys were selected. Find the probability that **at least 3** toys are faulty. Ans: 0.0381	**at least 3** toys are faulty means: $\mathrm{P}(X\ge 3)=1-\mathrm{P}(X\le 2)=0.0381$	
(iv) 8 toys were selected. Find the probability that between 3 to 5 **(inclusive)** toys are faulty. Ans: 0.0381	$\mathrm{P}(3\le X\le 5)=\mathrm{P}(X\le 5)-\mathrm{P}(X\le 2)=0.03807\approx 0.0381$	

(v) A box is packed with 8 toys. There are 10 such boxes. Find the probability that there are at most 8 such boxes with **less than 4** toys are faulty. Ans: 0.001104  10 boxes with 8 toys in each box	Let W be no of boxes with less than 4 toys that are faulty out of 10. W~B(___10____ , 0.99498) At most 8 such boxes means: $\mathrm{P}(X \le 8) = 0.001104 \approx 0.00110$	
(vi) A box is packed with 8 toys. There are now 30 such boxes. Find the probability that the **mean** number of toys that are faulty per box is more than 1. Ans: 0.0984  30 boxes with 8 toys in each box	Let X be__no of toys that are faulty out of 8_ X~B(____8___ , ______0.1____) Then $\overline{X}$ is the **mean** number toys that are faulty per box, out of 30 boxes. Since n = 30 is large, by Central Limit Theory $\overline{X} \sim \mathrm{N}(0.8, \frac{8 \times 0.1 \times 0.9}{30})$ approximately P($\overline{X}$ > 1) =0.09835 $\approx$ 0.0984	

(vii) Every pallet has 9 such boxes. There are 12 such pallets. Find the probability that 8 pallets contains less than 2 boxes that have at least 3 toys that are faulty per pack.
Ans: 0.00126



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12 pallets. 9 boxes in each pallet. . 8 toys in each box.

P(has at least 3 toys that are faulty per box) = P($X \geq 3$)
= 0.038092

Let M be the no boxes that have at least 3 toys that are faulty per box out of 9.

M ~ B(____9___ , __0.038092____)

P(less than 2 boxes that have at least 3 toys that are faulty per box) =

P($M < 2$) = P($M \leq 1$) = 0.956292

Let H be the no pallets that have with less than 2 boxes that have at least 3 toys that are faulty out of 12.

H ~ B(____12___ , __0.956292____)

P(8 pallets with less than 2 boxes that have at least 3 toys that are faulty per box) = P(H = 8) = 0.00126

<table>
<tr><td colspan="3">5. Normal Distribution</td></tr>
<tr><td>Organic rockmelons and watermelons are sold by weight. The weights, in grams, of Rockmelons and Watermelons are modelled as having independent normal distributions with means and standard deviations as shown in the table.

<table>
<tr><td></td><td>Mean weight</td><td>Standard deviation</td></tr>
<tr><td>Rockmelon</td><td>800</td><td>50</td></tr>
<tr><td>Watermelon</td><td>1600</td><td>180</td></tr>
</table>
(i) Two melons, one of each type, are chosen at random. Find the probability that the Rockmelon weighs between 700 grams and 900 grams, and the Watermelon weighs between 1500 grams and 1700 grams. Ans: 0.402</td>
<td>Let R and W be the random variables “weight of a randomly chosen Rockmelon and Watermelon” respectively
$R \sim \mathrm{N}(800, 50^2)$
$W \sim \mathrm{N}(1600, 180^2)$
Probability required = $\mathrm{P}(700 < R < 900)\ \mathrm{P}(1500 < W < 1700)$
$= 0.954 \times 0.421 = 0.402$</td>
<td></td></tr>
<tr><td>(ii) Find the probability that the total weight of 4 randomly chosen Rockmelons differs twice the weight of a randomly chosen Watermelon by at least 850 grams. Ans: 0.0229</td>
<td>‘differs’ :See Chapter 4 Example 12(ii)
$\mathrm{E}\left(\left(R_1+R_2+R_3+R_4\right)-2W\right)=0$
$\mathrm{Var}\left(\left(R_1+R_2+R_3+R_4\right)-2W\right)=139600$
$\left(R_1+R_2+R_3+R_4\right)-2W \sim N(0,139600)$

$\mathrm{P}(\left|\left(R_1+R_2+R_3+R_4\right)-2W\right| \geq 850)$
$=\mathrm{P}(\left(R_1+R_2+R_3+R_4\right)-2W \geq 850)$
$+\mathrm{P}(\left(R_1+R_2+R_3+R_4\right)-2W \leq -850)$
$= 0.0229$</td>
<td></td></tr>
</table>

(iii) A Rockmelon with weight exceeding m grams is considered as "Premium". Calculate the **least value of *m*** such that the probability of a randomly chosen Rockmelon being "Premium" is **0.0505**. Give your answer to the nearest gram. Ans: 882	$\mathrm{P}(R \geq m) = 0.0505$ Using GC, m = 882	
(iv) Rockmelons are sold at \$0.05 per gram and Watermelons at \$0.03 per gram. Find the probability that the **total selling price** of a randomly chosen Rockmelon and 2 randomly chosen Watermelons is more than \$150. Ans: 0.0407	Let $C = 0.05R + 0.03(W_1 + W_2)$. $\mathrm{E}(C) = 40 + 96 = 136$ $\mathrm{Var}(C) = 6.25 + 58.32 = 64.57$ $C \sim \mathrm{N}($___136____, __64.57_____$)$ $\mathrm{P}(C > 150) = 0.040732 \approx 0.0407$	
(v) 12 randomly chosen rockmelons are carefully packed into a box. Find the probability that there is at most 1 Premium rockmelon in a box. (A Rockmelon with weight exceeding 882 grams is considered as "Premium). Ans: 0.880	Let X be no of Premium rockmelon in a box out of 12. $X \sim \mathrm{B}(12, 0.0505)$ $\mathrm{P}(X \leq 1) = 0.87966 \approx 0.880$	
(vi) 20 such boxes are packed into a pallet for shipping. Find the probability that there are between 10 to 12 (inclusive) boxes with at most 1 Premium rockmelon in a box. Ans: 0.00144	Let Y be no of boxes with at most 1 Premium rockmelon in a box out of 20. $Y \sim \mathrm{B}(20, 0.879655)$ $\mathrm{P}(10 \leq Y \leq 12) = \mathrm{P}(Y \leq 12) - \mathrm{P}(Y \leq 9) = 0.0014394 \approx 0.00144$	
(vii) 18 watermelons are randomly chosen. Find the probability that the **average** weight of the watermelons are less than 1.62kg. Ans: 0.681	$\overline{W} \sim \mathrm{N}(1600, \frac{180^2}{18})$ $\mathrm{P}(\overline{W} < 1620) = 0.68132 \approx 0.681$	

6. Hypothesis Testing		
A company packs and sells brown sugar. A manager claims that the average mass of a packet of brown sugar is at least 500g. A random sample of 60 packets is examined and the mass, *x* g, of the contents of each packet is determined. The results are summarized as follows. $\sum(x-500)=-168 \qquad \sum(x-500)^2=8050$ (i) Find the unbiased estimates of the population mean and variance Ans: 497.2; $\frac{37898}{295}$	Unbiased estimate of population mean, $\bar{x}=\frac{\sum(x-500)}{60}+500$ $=\frac{-168}{60}+500=497.2$ Unbiased estimate of population variance, $s^2=\frac{1}{59}\left[8050-\frac{(-168)^2}{60}\right]$ $=128.47$ $=128$ (to 3 s.f)	
(ii) Test at the 5% level of significance whether the manager's claim is valid. Ans:p = 0.0278	Let *X* be the mass of a packet of brown sugar and μ be the mean mass of a packet of brown sugar. To test $H_0:\mu=500$ against $H_1:\mu<500$ at 5% level of significance. Since $n=60$ is large, by Central Limit Theorem: $\bar{X}\sim N\left(500, \frac{128.47}{60}\right)$ approximately under H_0. Test statistic: $Z=\frac{\bar{X}-500}{\sqrt{\frac{128.47}{60}}}\sim N(0,1)$. Using z-test, by GC, $\mu_0=500$, $s=\sqrt{128.47}$, $\bar{x}=497.2$, $n=60$, $z_{cal}=-1.91$ (to 3 s.f) *p*–value $=0.027841=0.0278$ (to 3 s.f)	

	Since p–value $= 0.02784 < 0.05$, we reject H_0 and conclude that there is sufficient evidence at 5% significance level to state that the manager's claim is not valid.	
(iii) Explain what is meant by the phrase '5% level of significance' in the context of the question.	'5% level of significance' means there is a probability of 0.05 of committing an error of rejecting the null hypothesis and concluding that the manager's claim is not valid, when the manager's claim is actually true.	
(iv) A consumer protection group took sample of 32 packets and conducts its own test. Using the same unbiased estimate of population mean and variance from (i), find the smallest level of significance of the test if the null hypothesis is rejected.	Since n = 32 is large, by Central Limit Theorem Under H_0 . $\overline{X} \sim \mathrm{N}\left(500,\ \frac{128.47}{32}\right)$ approximately p–value = 0.0811 Reject H_0 means p–value $< \frac{\alpha}{100}$ $0.0811 < \frac{\alpha}{100}$ $\alpha > 8.11$ smallest level of significance is 8.11	
(v) The consumer protection group took another sample of 20 packets and conducts its own test. Assuming that the mass of brown sugar follows a normal distribution with variance of 56, find the smallest sample mean. at 6%, level of significance, to not reject the manager's claim. Ans: $\bar{x} \approx 498$	$\overline{X} \sim \mathrm{N}\left(500,\ \frac{56}{20}\right)$ $z = \frac{\bar{x} - 500}{\sqrt{\frac{56}{20}}}$ Do not reject claim means $\frac{\bar{x} - 500}{\sqrt{\frac{56}{20}}} > -1.55477$ $\bar{x} > 497.39$ $\therefore \bar{x} \approx 498$	