Chapter S2B: Binomial Distribution

<u>SYLLABUS INCLUDES</u>

- Concept of binominal distribution $\mathrm{B}(n, p)$ as an example of a discrete probability distribution and use of $\mathrm{B}(n, p)$ as a probability model, including conditions under which the binomial situation is a suitable model
- Use of mean and variance of binomial distribution (without proof)

<u>PRE-REQUISITES</u>

- Probability
- Discrete Random Variables

<u>CONTENT</u>

INTRODUCTION

Some random variables occur so frequently in real-life that they are given special names.
In this chapter, we will look at one such probability distribution for discrete random variables, namely, the Binomial distribution.

1 BINOMIAL DISTRIBUTION

1.1 Binomial Experiment

Experiments consisting of n trials, each with two possible outcomes that may be regarded as either *success* or *failure*, are very common in the study of probability. If, in addition, the probability of getting a success (or a failure) at each trial remains constant and the trials are independent, then we called such experiments **binomial experiments**.

For example, when we toss a coin 10 times, the outcome of each toss may be a head or a tail. Let us regard getting a head as a success. Since we are using the same coin, the probability of success will remain constant. Obviously, the tosses are independent of each other. This coin-tossing experiment is then a binomial experiment.

A binomial experiment is one that possesses the following properties:

1. It consists of n independent trials.
2. The outcome of each trial is either a success or a failure.
3. The probability of success for each trial, denoted by p, remains constant.

Now consider drawing 5 cards from a deck of playing cards, one at a time and without replacement. Let us regard the outcome of each draw a success if the card drawn is a King and a failure if it is not. Then such an experiment is not a binomial experiment as we could see that it obviously violates property 1 of binomial experiment.

Question: What if the card is replaced before the next draw? Will this experiment be a binomial experiment?

Answer: Yes, it will satisfy all properties of a binomial experiment.
i.e.
(1) It consists of 5 independent trials.
(2) The outcome is either a success (if the card drawn is a King) or a failure (if the card drawn is not a King).
(3) The probability of drawing a King for each draw remains constant.

1.2 Binomial Random Variable

The random variable X denoting the number of successes in the n trials of a binomial experiment is called a **binomial random variable**. It takes values $\{0,1,2,...,n\}$.

The probability that X takes the value x, denoted by $\mathrm{P}(X=x)$, depends on the number of trials (n) as well as the probability of success (p) for each trial.
The *probability distribution* of this discrete random variable is called a **binomial distribution**.

We called n and p the *parameters* of the distribution and we write $X \sim \mathrm{B}(n, p)$.

Example 1

A coin is biased such that the probability of obtaining a head when the coin is tossed is $\frac{2}{3}$. The coin is tossed 3 times. If X represents the number of heads obtained, find $\mathrm{P}(X=x)$ for $x = 0, 1, 2, 3$ and tabulate the probability distribution of X.

Solution:

$$\mathrm{P}(X=0) = \mathrm{P}(\text{3 tails}) = \left(\frac{1}{3}\right)^3 = \frac{1}{27}$$

$$\mathrm{P}(X=1) = \mathrm{P}(\text{1 head, 2 tails}) = {}^3\mathrm{C}_1\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^2 = \frac{2}{9}$$

$$\mathrm{P}(X=2) = \mathrm{P}(\text{2 heads, 1 tail}) = {}^3\mathrm{C}_2\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right) = \frac{4}{9}$$

$\mathrm{P}(X=3)=\mathrm{P}(3\text{ heads})=\left(\frac{2}{3}\right)^3=\frac{8}{27}$

x	0	1	2	3
$\mathrm{P}(X=x)$	$\frac{1}{27}$	$\frac{2}{9}$	$\frac{4}{9}$	$\frac{8}{27}$

Note:
The random variable X representing the number of heads obtained is a binomial random variable. The number of trials (n) is 3 while the probability of success (p), i.e. the probability of obtaining a head, for each trial is $\frac{2}{3}$. We write $X \sim \mathrm{B}\left(3, \frac{2}{3}\right)$.
In general, if $X \sim \mathrm{B}(n, p)$, how do we evaluate $\mathrm{P}(X=x)$ for $x=0,1,2,\ldots,n$? We seek a formula that expresses $\mathrm{P}(X=x)$ in terms of x, n and p.

1.3 Binomial Distribution

If $X \sim \mathrm{B}(n, p)$ where X denotes the number of successes in n trials of a binomial experiment, then the probability distribution of X is given by

$$\mathrm{P}(X=x)=\binom{n}{x}p^x(1-p)^{n-x}\text{, for } x=0,1,2,\ldots,n\text{, where }\binom{n}{x}={}^nC_x=\frac{n!}{(n-x)!\,x!}$$

This is called the ***binomial distribution*** with parameters n and p.

The mean and variance of X are given by
(i) $\mathrm{E}(X)=np$
(ii) $\mathrm{Var}(X)=np(1-p)$ [Highlighted formulae can be found in formula list]

Note:
When n = 1, the binomial distribution is known as the Bernoulli distribution.

Example 2
An ordinary fair die is tossed eight times. Find the probability of obtaining at least 6 sixes.

Solution:
Let X be the number of sixes obtained in 8 tosses of the die.
Then $X \sim \mathrm{B}\left(8, \frac{1}{6}\right)$.
Probability of obtaining at least 6 sixes
$=\mathrm{P}(X \geq 6)$
$=\mathrm{P}(X=6,\ 7,\ 8)$
$={}^8C_6\left(\frac{1}{6}\right)^6\left(\frac{5}{6}\right)^2+{}^8C_7\left(\frac{1}{6}\right)^7\left(\frac{5}{6}\right)+\left(\frac{1}{6}\right)^8=0.000441\ (3\text{ s.f.})$

Example 3

A binomial random variable X has mean 1.2 and variance 1.08. Evaluate the parameters of the distribution.

Solution:

$X \sim \mathrm{B}(n, p)$.

We have $\mathrm{E}(X) = 1.2$ and $\mathrm{Var}(X) = 1.08$.

$\Rightarrow$ $np = 1.2$ and $np(1-p) = 1.08$

$\Rightarrow$ $(1.2)(1-p) = 1.08$

$\Rightarrow$ $1-p = 0.9$ $\Rightarrow$ $p = 0.1$ and $n = 12$

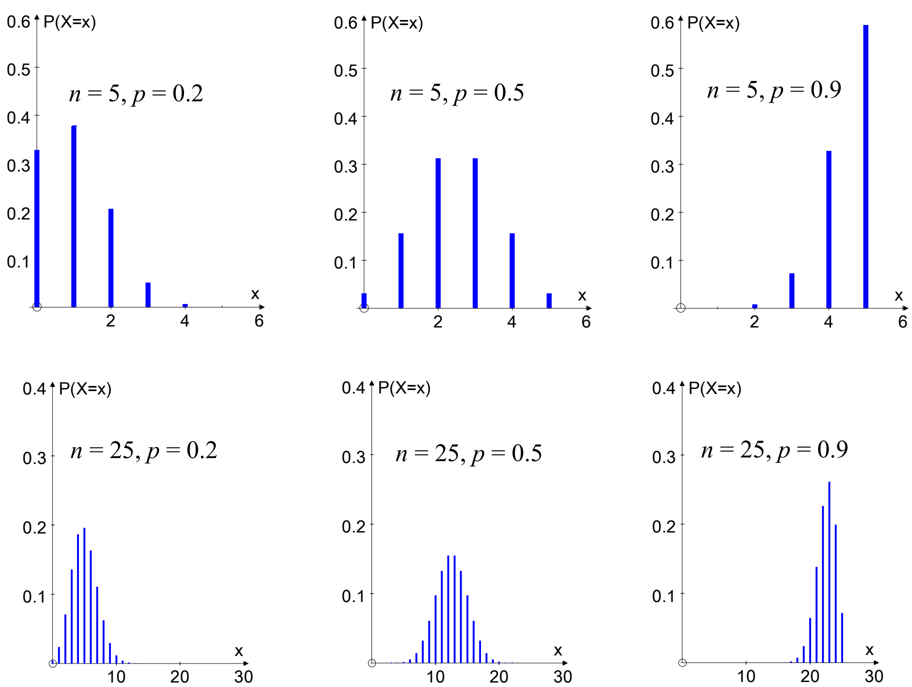
$\therefore X \sim \mathrm{B}(12, 0.1)$

Note:

For special discrete random variables such as binomial distribution, stating $X \sim \mathrm{B}(n, p)$ is equivalent to listing all probabilities of the probability distribution.

1.4 Graph of the Probability Distribution of a Binomial Random Variable

Suppose $X \sim \mathrm{B}(n, p)$. The graphs of the probability distribution of X for various values of n and p are shown below.



In general, as x increases, $\mathrm{P}(X = x)$ will increase to a maximum value and after which, it will decrease.

Example 4

(a) The probability that a sharp shooter hits a target is 0.8 and the probability that he misses is 0.2. The shots he makes are independent of each other and the probability of him hitting or missing the target remains constant. Find the probability that, in 10 shots, he will hit the target

(i) exactly 6 times,

(ii) more than 8 times.

Find also, in 10 shots, the most likely number of shots that he will hit target.

(b) A poor shooter has a probability of 0.1 of hitting the target. How many shots must he be given in order that there is at least a 90% chance that he will hit the target at least once?

Important!
Always define the random variable in the context of the question.

Solution:

(a) Let X be the number of hits, out of 10 shots, by the sharp shooter.
Then $X \sim \mathrm{B}(10, 0.8)$.

(i) Probability that he will hit the target 6 times out of 10 shots

$= \mathrm{P}(X = 6)$

$= {}^{10}\mathrm{C}_6 (0.8)^6 (0.2)^4$

$= 0.0881$ (3 s.f.)

(ii) Probability that he will hit the target more than 8 times in 10 shots

$= \mathrm{P}(X > 8)$

$= \mathrm{P}(X = 9, 10)$

$= {}^{10}\mathrm{C}_9 (0.8)^9 (0.2) + (0.8)^{10} = 0.376$ (3 s.f.)

$\mathrm{P}(X = 6) = {}^{10}\mathrm{C}_6 (0.8)^6 (0.2)^4 = 0.0881$ (3 s.f.)

$\mathrm{P}(X = 7) = {}^{10}\mathrm{C}_7 (0.8)^7 (0.2)^3 = 0.201$ (3 s.f.)

$\mathrm{P}(X = 8) = {}^{10}\mathrm{C}_8 (0.8)^8 (0.2)^2 = 0.302$ (3 s.f.)

$\mathrm{P}(X = 9) = {}^{10}\mathrm{C}_9 (0.8)^9 (0.2)^1 = 0.268$ (3 s.f.)

Since $\mathrm{P}(X = 8)$ is the greatest, he will most likely hit 8 shots on target.

(b) Let Y be the number of hits, out of n shots, by the poor shooter.
Then $Y \sim \mathrm{B}(n, 0.1)$.

We need $\mathrm{P}(Y \geq 1) \geq 0.9$

$\Rightarrow 1 - \mathrm{P}(Y = 0) \geq 0.9$

$\Rightarrow \mathrm{P}(Y = 0) \leq 0.1$

$\Rightarrow (0.9)^n \leq 0.1$

$\Rightarrow n \geq \dfrac{\lg 0.1}{\lg 0.9} \approx 21.85$

$\therefore$ The poor shooter needs at least 22 shots.

Using the Graphing Calculator:

A. $\mathrm{P}(X = k)$

We will use the **binompdf(** command to find the probability of k successes in n trials, where p is the probability of success for each trial.

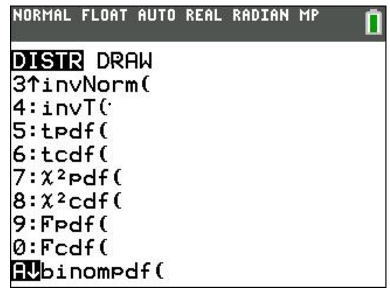
Format**: binompdf(**n, p, k**)**

Using the Graphing Calculator for Example 4a(i)
Compute the probability of 6 successes in 10 shots where the probability of a shot hitting the target is 0.8, i.e. $\mathrm{P}(X = 6)$

Given $n = 10$, $p = 0.8$ and $k = 6$,

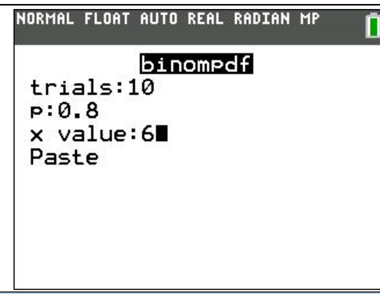
1. Press 

for **[DISTR]**
2. Scroll to select the command **binompdf(** and press A.



3. Type 10, 0.8, 6, highlight **Paste** and press enter to get

$\mathrm{P}(X = 6) = 0.0881$ (3 s.f.)



B. $\mathrm{P}(X \le k) = \mathrm{P}(X = 0, 1, \ldots, k)$

We will use the **binomcdf(** command to find the probability of k or fewer successes in n trials, where p is the probability of success for each trial.

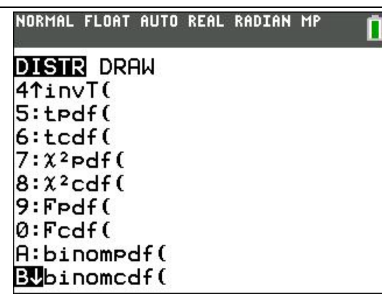
Format**: binomcdf(**n, p, k**)**

Using the Graphing Calculator for Example 4a(ii)
Compute the probability of more than 8 successes in 10 shots where the probability of a shot hitting the target is 0.8, i.e. $\mathrm{P}(X > 8) = 1 - \mathrm{P}(X \le 8)$

Given $n = 10$, $p = 0.8$ and $k = 8$,

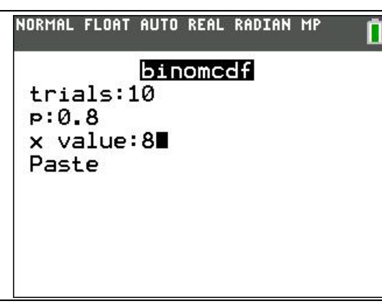
1. Press 

for **[DISTR]**
2. Scroll to select the command **binomcdf(** and press B.



3. Type 10, 0.8, 8, highlight **Paste** and press enter to get $\mathrm{P}(X \le 8) = 0.62419$ (5 s.f.). Thus, the required answer for $\mathrm{P}(X > 8) = 1 - \mathrm{P}(X \le 8)$

$\approx 1 - 0.62419$
$= 0.376$ (3sf)

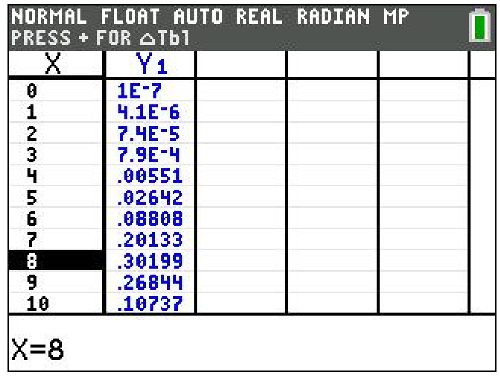


Using the Graphing Calculator for the last part in Example 4a

Find the value x such that $\mathrm{P}(X = x)$ is greatest.

1. Key $Y_1 =$ **binompdf**$(10, 0.8, x)$
2. Look at the table of values and scroll down to find value x such that $\mathrm{P}(X = x)$ is greatest.
3. Greatest $\mathrm{P}(X = x)$ occurs when value of x is 8.

Presentation:
From GC,
$\mathrm{P}(X = 7) = 0.201$ (3 s.f.)
$\mathrm{P}(X = 8) = 0.302$ (3 s.f.)
$\mathrm{P}(X = 9) = 0.268$ (3 s.f.)
Since $\mathrm{P}(X = 8)$ is the greatest, he will most likely hit 8 shots on target.



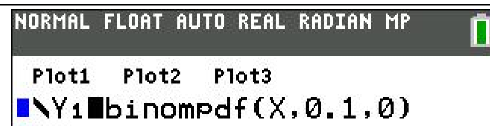
Using the Graphing Calculator for an alternative method in Example 4b

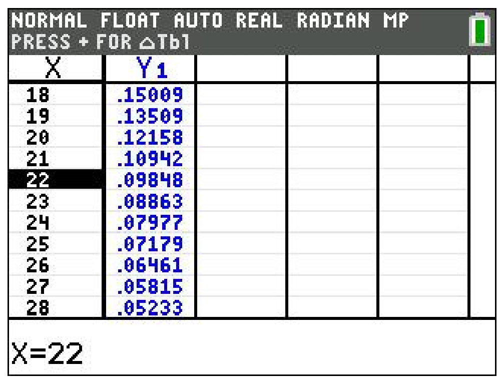
Find the least value of n such that $\mathrm{P}(Y = 0) \le 0.1$.

1. Key $Y_1 =$ **binompdf**$(X, 0.1, 0)$
2. Look at the table of values and scroll down to find least value of n such that $Y_1 = \mathrm{P}(Y = 0) \le 0.1$
3. Since 22 is the first value such that $Y_1 = \mathrm{P}(Y = 0) \le 0.1$, the least value of n is 22.

Presentation:
From GC,
When n =21, $\mathrm{P}(Y = 0) = 0.10942 > 0.1$
When n =22, $\mathrm{P}(Y = 0) = 0.09848 < 0.1$

Therefore, the least value of n is 22.





Example 5

A crossword puzzle is published in *The Times* each day of the week, except Sunday. A man is able to complete on average 8 out of 10 of the crossword puzzles.

(i) State, in context, two assumptions for the number of crossword puzzles completed in a week, except Sunday, to be well modelled by a binomial distribution.

(ii) Find the expectation and the standard deviation of the number of completed crossword puzzles in a given week.

(iii) Show that the probability that he will complete at least 5 in a given week is 0.655 to 3 significant figures.

(iv) Find the probability that in a period of four weeks, he completes fewer than 5 crossword puzzles in one of the four weeks.

Solution:

Let X be the number of crossword puzzles completed in a week, except Sunday.
Then $X \sim \mathrm{B}(6, 0.8)$.

(i) Assumptions:

(1) The event the man completes a crossword puzzle is independent from the event he completes another crossword puzzle.

(2) The probability of the man completing a crossword puzzle remains constant at 0.8.

(ii) Expectation of number of completed number puzzles in a week, $\mathrm{E}(X) = 6(0.8) = 4.8$

$\mathrm{Var}(X) = 6(0.8)(0.2) = 0.96$

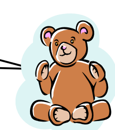
$\Rightarrow$ standard deviation of $X = \sqrt{0.96} = 0.980$ (3 s. f.)

(iii) Probability that he will complete at least 5 in a given week is

$$\begin{aligned} &\mathrm{P}(X \geq 5) \\ &= 1 - \mathrm{P}(X \leq 4) \\ &= 1 - 0.34464 \\ &= 0.65536 \\ &= 0.655 \quad \text{(3 s.f.)} \quad \text{(shown)} \end{aligned}$$

(iv) Let Y be the number of weeks, out of 4, in which he completes fewer than 5 crossword puzzles.

Note that there is a change in the random variable.



Then

$Y \sim \mathrm{B}(4,\ 1 - 0.655)$

i.e. $Y \sim \mathrm{B}(4, 0.345)$

$\mathrm{P}(Y = 1) = 0.388$ (3 s.f.)

If we are asked to show a value in the question, we can use this value (i.e. *0.655*, instead of *0.65536*) for subsequent parts in the question.

Example 6
In Singapore, 1 out of 5 primary school students are myopic. A random sample size of n is taken to study the problem of myopia among Singapore students. Let X be the number of primary school students who suffer from myopia.

State, in the context of the question, two assumptions needed to model X by a binomial distribution. Explain why one of the assumptions stated may not hold in this context.

Assume now that these assumptions do in fact hold. Find the least value of n, such that the probability of getting at least 2 myopic students in the sample is greater than 0.96.

Solution:
$X \sim \mathrm{B}(n, 0.2)$

Assumptions: (1) The event of a student is myopic is independent from the event of another student being myopic.
(2) The probability of a student being myopic remains constant at 0.2.

(1) might not hold as if two students are from the same family, factors such as genetics (parental myopia) or study habits and conditions may determine whether the student is myopic.

$\mathrm{P}(X \geq 2) > 0.96$
$\Rightarrow 1 - \mathrm{P}(X = 0) - \mathrm{P}(X = 1) > 0.96$
$\Rightarrow \mathrm{P}(X = 0) + \mathrm{P}(X = 1) < 0.04$
$\Rightarrow \mathrm{P}(X \leq 1) < 0.04$

From GC,
When $n = 22$, $\mathrm{P}(X \leq 1) = 0.04796 > 0.04$
When $n = 23$, $\mathrm{P}(X \leq 1) = 0.03984 < 0.04$

Therefore, least value of n is 23.

Example 7
A newly discovered drug is used to treat patients infected with measles. However, it was found that proportion p of these patients will suffer from an allergic reaction.

(i) In a clinical trial, a group of n patients were treated with the drug. The random variable X denotes the number of patients, out of n, who developed allergic reactions to the drug. Given that $\mathrm{E}(X)=4$ and $\mathrm{Var}(X)=3.68$, find the probability that more than 4 of the patients develop an allergic reaction.

(ii) 25 such clinical trials were performed throughout the world. The drug will be approved for use by the general population if there were at most 10 clinical trials, which have at most 4 patients developing an allergic reaction. Determine the probability of the drug being approved.

Solution:
Assumptions: (1) The event of a patient suffers from an allergic reaction is independent from the event of another patient suffering an allergic reaction.
(2) The probability of a patient suffering from an allergic reaction remains constant.

(i) $X \sim \mathrm{B}(n, p)$
$\mathrm{E}(X)=4 \Rightarrow np=4$

$\mathrm{Var}(X)=3.68 \Rightarrow np(1-p)=3.68 \Rightarrow (1-p)=\dfrac{3.68}{4}=0.92$

$\therefore p = 0.08$, $n = 50$ and $X \sim \mathrm{B}(50, 0.08)$

Probability that more than 4 of the patients develop an allergic reaction is
$=\mathrm{P}(X>4)$
$=1-\mathrm{P}(X \le 4) \approx 1-0.62895$
$=0.371$ (3 s.f.)

(ii) Let Y be the number of clinical trials, out of 25, in which there are at most four patients with allergic reactions.

Note that from (i), $\mathrm{P}(X \le 4) \approx 0.62895$.
Then $Y \sim \mathrm{B}\left(25, 0.62895\right)$
Probability of the drug being approved $=\mathrm{P}(Y \le 10)=0.0168$ (3 s.f.)

CONCLUSION

It is important that you can identify a binomial distribution. It has many important applications. For example, the binomial distribution is used in the field of quality control where industrial engineers are interested in the proportion of defectives. It is also used extensively for medical and military applications. There are many other special discrete probability distributions such as Poisson, geometric, negative binomial, etc. You can read up about them in most textbooks on Statistics.

In the next chapter, we shall look at *continuous* random variables.