H2 Mathematics 9758
Topic 10: DIFFERENTIATION TECHNIQUES
Tutorial Worksheets

1 By considering the derivative as a limit, show that the derivative of x^3 is $3x^2$.

[N00/I/4]

[Solution]

Let $\mathrm{f}(x)=x^3$.

Then $\mathrm{f}(x+\delta x)=(x+\delta x)^3=\left[x^3+3x^2(\delta x)-3x(\delta x)^2+(\delta x)^3\right]$

$$\frac{\mathrm{f}(x+\delta x)-\mathrm{f}(x)}{\delta x}=\frac{x^3+3x^2(\delta x)-3x(\delta x)^2+(\delta x)^3-x^3}{\delta x}$$

$$\lim_{\delta x\to 0}\frac{\mathrm{f}(x+\delta x)-\mathrm{f}(x)}{\delta x}=3x^2-3x\lim_{\delta x\to 0}(\delta x)+\lim_{\delta x\to 0}(\delta x)^2$$

$\mathrm{f}'(x)=3x^2$ (shown)

Watch out for mistakes with notations.

2 Differentiate each of the following with respect to x simplifying your answer.

(a) $\dfrac{x^2}{\sqrt{4-x^2}}$ **(b)** $\sqrt{1+\sqrt{x}}$ **(c)** $\left(\dfrac{x^3-1}{2x^3+1}\right)^4$

[Ans: **(a)** $\dfrac{x(8-x^2)}{(4-x^2)^{\frac{3}{2}}}$ **(b)** $\dfrac{1}{4\sqrt{x(1+\sqrt{x})}}$ **(c)** $\dfrac{36x^2(x^3-1)^3}{(2x^3+1)^5}$]

[Solution]

(a)
$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{x^2}{\sqrt{4-x^2}}\right]=\frac{\sqrt{4-x^2}(2x)-(x^2)\left(\frac{1}{2}\right)(4-x^2)^{-\frac{1}{2}}(-2x)}{4-x^2}$$
$$=\frac{2x(4-x^2)+x^3}{(4-x^2)^{\frac{3}{2}}}$$
$$=\frac{x(8-x^2)}{(4-x^2)^{\frac{3}{2}}}$$

(b)
$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\sqrt{1+\sqrt{x}}\right]=\frac{1}{2\sqrt{1+\sqrt{x}}}\left(\frac{1}{2\sqrt{x}}\right)=\frac{1}{4\sqrt{x(1+\sqrt{x})}}$$

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(c) $$\frac{d}{dx}\left[\left(\frac{x^3-1}{2x^3+1}\right)^4\right]=\frac{d}{dx}\left[\frac{\left(x^3-1\right)^4}{\left(2x^3+1\right)^4}\right]$$

$$=\frac{\left(2x^3+1\right)^4(4)\left(x^3-1\right)^3\left(3x^2\right)-\left(x^3-1\right)^4(4)\left(2x^3+1\right)^3\left(6x^2\right)}{\left(2x^3+1\right)^8}$$

$$=\frac{12x^2\left(x^3-1\right)^3\left[\left(2x^3+1\right)-2\left(x^3-1\right)\right]}{\left(2x^3+1\right)^5}$$

$$=\frac{36x^2\left(x^3-1\right)^3}{\left(2x^3+1\right)^5}$$

(Alternative, apply chain rule first)

$$\frac{d}{dx}\left[\left(\frac{x^3-1}{2x^3+1}\right)^4\right]=4\left(\frac{x^3-1}{2x^3+1}\right)^3\left(\frac{d}{dx}\left(\frac{x^3-1}{2x^3+1}\right)\right)$$

$$=4\left(\frac{x^3-1}{2x^3+1}\right)^3\left(\frac{d}{dx}\left(\frac{x^3-1}{2x^3+1}\right)\right)$$

$$=4\left(\frac{x^3-1}{2x^3+1}\right)^3\left(\frac{1}{2}\frac{d}{dx}\left(\frac{2x^3-2}{2x^3+1}\right)\right)$$

$$=4\left(\frac{x^3-1}{2x^3+1}\right)^3\left(\frac{1}{2}\frac{d}{dx}\left(1-\frac{3}{2x^3+1}\right)\right)$$

$$=2\left(\frac{x^3-1}{2x^3+1}\right)^3\left(\frac{d}{dx}\left(-3\left(2x^3+1\right)^{-1}\right)\right)$$

$$=2\left(\frac{x^3-1}{2x^3+1}\right)^3\left(3\left(2x^3+1\right)^{-2}\left(6x^2\right)\right)$$

$$=\frac{36x^2\left(x^3-1\right)^3}{\left(2x^3+1\right)^5}$$

3 Find the derivative with respect to x of

(a) $\cos x^\circ$, **(b)** $\cot(1-2x^2)$,

(c) $\tan^3(5x)$, **(d)** $\frac{\sec x}{1+\tan x}$.

[Ans: **(a)** $-\frac{\pi}{180}\sin x^\circ$ **(b)** $4x\operatorname{cosec}^2\left(1-2x^2\right)$ **(c)** $15\tan^2(5x)\sec^2(5x)$ **(d)** $\frac{\sec x(\tan x-1)}{(1+\tan x)^2}$]

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[Solution]

(a) Let $y = \cos x^\circ = \cos\dfrac{\pi x}{180}$

$$\frac{dy}{dx} = -\frac{\pi}{180}\sin\frac{\pi x}{180} = -\frac{\pi}{180}\sin x^\circ$$

(b) $$\frac{d}{dx}\left[\cot(1-2x^2)\right] = -\operatorname{cosec}^2\left(1-2x^2\right)\times\left(-4x\right) = 4x\operatorname{cosec}^2\left(1-2x^2\right)$$

(c) $$\frac{d}{dx}\left[\tan^3\left(5x\right)\right] = 3\tan^2\left(5x\right)\sec^2\left(5x\right)\times 5 = 15\tan^2\left(5x\right)\sec^2\left(5x\right)$$

(d) $$\frac{d}{dx}\left[\frac{\sec x}{1+\tan x}\right] = \frac{\left(1+\tan x\right)\left(\sec x\tan x\right)-\left(\sec x\right)\left(\sec^2 x\right)}{\left(1+\tan x\right)^2}$$

$$= \frac{\sec x\left(\tan x+\tan^2 x-\sec^2 x\right)}{\left(1+\tan x\right)^2} \qquad \text{but } 1+\tan^2 x \equiv \sec^2 x$$

$$= \frac{\sec x\left(\tan x-1\right)}{\left(1+\tan x\right)^2}$$

4 Find the derivative with respect to x of

(a) $y = e^{1+\sin 3x}$ **(b)** $y = x^2 e^{\frac{1}{x}}$

(c) $y = \ln\left[\dfrac{1-x}{\sqrt{1+x^2}}\right]$ **(d)** $y = \dfrac{\ln\left(2x\right)}{x}$

(e) $y = \log_2\left(3x^4 - e^x\right)$ **(f)** $y = 3^{\ln(\sin x)}$

[Ans: **(a)** $3e^{1+\sin 3x}\cos 3x$ **(b)** $e^{\frac{1}{x}}\left(2x-1\right)$ **(c)** $-\dfrac{1+x}{(1-x)\left(1+x^2\right)}$ **(d)** $\dfrac{1-\ln(2x)}{x^2}$

(e) $\dfrac{12x^3-e^x}{\left(3x^4-e^x\right)\ln 2}$ **(f)** $3^{\ln(\sin x)}\cot x\ln 3$]

[Solution]

(a) $y = e^{1+\sin 3x}$

$$\frac{dy}{dx} = e^{1+\sin 3x}\left(3\cos 3x\right) = 3e^{1+\sin 3x}\cos 3x$$

(b) $y = x^2 e^{\frac{1}{x}}$

(d) $y = \dfrac{\ln\left(2x\right)}{x}$

$$\frac{dy}{dx} = \frac{(x)\left(\dfrac{2}{2x}\right)-[\ln(2x)](1)}{x^2} = \frac{1-\ln(2x)}{x^2}$$

(e) $y = \log_2\left(3x^4-e^x\right) = \dfrac{\ln\left(3x^4-e^x\right)}{\ln 2}$

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$$\frac{dy}{dx}=(2x)\left(e^{\frac{1}{x}}\right)+\left(x^2\right)\left(-\frac{1}{x^2}e^{\frac{1}{x}}\right)$$
$$=e^{\frac{1}{x}}(2x-1)$$

(c) $y=\ln\left[\frac{1-x}{\sqrt{1+x^2}}\right]=\ln(1-x)-\frac{1}{2}\ln\left(1+x^2\right)$

$$\frac{dy}{dx}=\frac{1}{1-x}\times(-1)-\frac{1}{2}\left(\frac{1}{1+x^2}\right)(2x)$$
$$=\frac{-\left(1+x^2\right)-x(1-x)}{(1-x)\left(1+x^2\right)}$$
$$=-\frac{1+x}{(1-x)\left(1+x^2\right)}$$

$$\frac{dy}{dx}=\frac{1}{\ln 2}\left(\frac{12x^3-e^x}{3x^4-e^x}\right)$$
$$=\frac{12x^3-e^x}{\left(3x^4-e^x\right)\ln 2}$$

(f) $y=3^{\ln(\sin x)}$ i.e. $\ln y=\ln(\sin x)\ln 3$

$$\frac{1}{y}\frac{dy}{dx}=\left(\frac{\cos x}{\sin x}\right)\ln 3$$
$$=\cot x\ln 3$$
$$\frac{dy}{dx}=y\cot x\ln 3$$
$$=3^{\ln(\sin x)}\cot x\ln 3$$

5 Find $\frac{dy}{dx}$ in terms of x and y for each of the following:

(a) $y^3-3x^2y+2x^3=1$ **(b)** $(yx)^2=x^2 2^x$

(c) $e^{x+y}=e^{2x}+y$ **(d)** $y^2=x^2+\sin xy$

[Ans: **(a)** $\frac{2x}{y+x}$ **(b)** $\frac{y}{2}\ln 2$ **(c)** $\frac{2e^{2x}-e^{x+y}}{e^{x+y}-1}$ **(d)** $\frac{2x+y\cos xy}{2y-x\cos xy}$]

[Solution]

(a) $y^3-3x^2y+2x^3=1$

$$\Rightarrow\quad 3y^2\frac{dy}{dx}-\left(6xy+3x^2\frac{dy}{dx}\right)+6x^2=0$$

$$\Rightarrow\quad \left(3y^2-3x^2\right)\frac{dy}{dx}=6xy-6x^2$$

$$\Rightarrow\quad \frac{dy}{dx}=\frac{6x(y-x)}{3\left(y^2-x^2\right)}=\frac{2x}{y+x}$$

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(b) $(yx)^2 = x^2 2^x$ $y^2 = 2^x \quad (\text{for } x \neq 0)$ $2y\frac{dy}{dx} = 2^x \ln 2$ $\frac{dy}{dx} = \frac{y^2}{2y}\ln 2$ $= \frac{y}{2}\ln 2$	Alternatively, $(yx)^2 = x^2 2^x$ $2\ln(xy) = \ln x^2 + \ln 2^x \quad (\text{for } x \neq 0)$ $2\ln x + 2\ln y = 2\ln x + x\ln 2$ $\ln y = \frac{1}{2}x\ln 2$ Differentiate w.r.t. x, $\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\ln 2$ $\frac{dy}{dx} = \frac{y}{2}\ln 2$

(c) $e^{x+y} = e^{2x} + y$

$$\Rightarrow \quad e^{x+y}\left(1+\frac{dy}{dx}\right) = 2e^{2x} + \frac{dy}{dx}$$

$$\Rightarrow \quad \left(e^{x+y}-1\right)\frac{dy}{dx} = 2e^{2x} - e^{x+y}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2e^{2x} - e^{x+y}}{\left(e^{x+y}-1\right)}$$

(d) $y^2 = x^2 + \sin xy$

$$\Rightarrow \quad 2y\frac{dy}{dx} = 2x + \cos(xy)\times\left(y + x\frac{dy}{dx}\right)$$

$$\Rightarrow \quad 2y\frac{dy}{dx} - x\frac{dy}{dx}\cos(xy) = 2x + y\cos(xy)$$

$$\Rightarrow \quad [2y - x\cos(xy)]\frac{dy}{dx} = 2x + y\cos(xy)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2x + y\cos(xy)}{2y - x\cos(xy)}$$

6 Differentiate each of the following with respect to x:

(a) $\tan^{-1}\sqrt{x}$ **(b)** $5\sin^{-1}\left(\frac{x}{10}\right)$

(c) $e^{\cos^{-1}2x}$ **(d)** $x\tan^{-1}(3x) - \ln\frac{1+9x^2}{1-9x^2}$

[Ans: **(a)** $\frac{1}{2\sqrt{x}(1+x)}$ **(b)** $\frac{5}{\sqrt{100-x^2}}$ **(c)** $-\frac{2e^{\cos^{-1}2x}}{\sqrt{1-4x^2}}$ **(d)** $\tan^{-1}3x-\frac{15x}{1+9x^2}-\frac{18x}{1-9x^2}$]

[Solution]

(a) $$\frac{d}{dx}\left(\tan^{-1}\sqrt{x}\right)=\frac{1}{1+(\sqrt{x})^2}\left(\frac{1}{2}x^{-\frac{1}{2}}\right)=\frac{1}{2\sqrt{x}(1+x)}$$

(b) $$\frac{d}{dx}\left[5\sin^{-1}\left(\frac{x}{10}\right)\right]=\frac{5}{\sqrt{1-\left(\frac{x}{10}\right)^2}}\left(\frac{1}{10}\right)=\frac{5}{\sqrt{100-x^2}}$$

(c) $$\frac{d}{dx}\left(e^{\cos^{-1}2x}\right)=e^{\cos^{-1}2x}\left(-\frac{1}{\sqrt{1-(2x)^2}}\right)(2)=-\frac{2e^{\cos^{-1}2x}}{\sqrt{1-4x^2}}$$

(d) $$\frac{d}{dx}\left(x\tan^{-1}3x-\ln\frac{1+9x^2}{1-9x^2}\right)$$

$$=\frac{d}{dx}\left\{x\tan^{-1}3x-\left[\ln(1+9x^2)-\ln(1-9x^2)\right]\right\}$$

$$=\tan^{-1}3x+x\frac{3}{1+(3x)^2}-\left(\frac{18x}{1+9x^2}-\frac{-18x}{1-9x^2}\right)$$

$$=\tan^{-1}3x+\frac{3x}{1+9x^2}-\frac{18x}{1+9x^2}-\frac{18x}{1-9x^2}$$

$$=\tan^{-1}3x-\frac{15x}{1+9x^2}-\frac{18x}{1-9x^2}$$

7 Find an expression for $\frac{dy}{dx}$ for the following in terms of x and/or y:

(a) $y^3 = x\sin^{-1}x$

(b) $y=a^{2\log_a x}$

(c) $y=(\ln x)^x$

(d) $y=\sqrt[3]{\frac{e^x(x+1)}{x^2+1}}$, $x>0$

[Ans: **(a)** $\frac{1}{3y^2}\left(\sin^{-1}x+\frac{x}{\sqrt{1-x^2}}\right)$ **(b)** $2x$ **(c)** $y\ln(\ln x)+\frac{y}{\ln x}$

(d) $\frac{y}{3}\left(1+\frac{1}{x+1}-\frac{2x}{x^2+1}\right)$]

[Solution]

(a)
$$3y^2\frac{\mathrm{d}y}{\mathrm{d}x}=\sin^{-1}x+\frac{x}{\sqrt{1-x^2}}$$
$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{3y^2}\left(\sin^{-1}x+\frac{x}{\sqrt{1-x^2}}\right)$$

(b)
$$y=a^{\log_a x^2}=x^2$$
$$\frac{\mathrm{d}y}{\mathrm{d}x}=2x$$

(c)
$$\ln y=\ln\left[(\ln x)^x\right]=x\ln(\ln x)$$
$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}=\ln(\ln x)+x\left(\frac{\frac{1}{x}}{\ln x}\right)=\ln(\ln x)+\frac{1}{\ln x}$$
$$\frac{\mathrm{d}y}{\mathrm{d}x}=y\ln(\ln x)+\frac{y}{\ln x}$$

(d)
$$\ln y=\ln\left(\sqrt[3]{\frac{\mathrm{e}^x(x+1)}{x^2+1}}\right)=\frac{1}{3}\left[\ln\mathrm{e}^x+\ln(x+1)-\ln(x^2+1)\right]$$
$$\ln y=\frac{1}{3}\left[x+\ln(x+1)-\ln(x^2+1)\right]$$
$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{3}\left(1+\frac{1}{x+1}-\frac{2x}{x^2+1}\right)$$
$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y}{3}\left(1+\frac{1}{x+1}-\frac{2x}{x^2+1}\right)$$

8 If $\ln y=\tan^{-1}t$, prove that $y\frac{\mathrm{d}^2y}{\mathrm{d}t^2}+(2t-1)\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2=0$.

[Solution]
$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}t}=\frac{1}{1+t^2}$$
$$(1+t^2)\frac{\mathrm{d}y}{\mathrm{d}t}-y=0$$

$$(1+t^2)\frac{\mathrm{d}^2y}{\mathrm{d}t^2}+2t\frac{\mathrm{d}y}{\mathrm{d}t}-\frac{\mathrm{d}y}{\mathrm{d}t}=0$$

$$\frac{\mathrm{d}y}{\mathrm{d}t}(1+t^2)\frac{\mathrm{d}^2y}{\mathrm{d}t^2}+\frac{\mathrm{d}y}{\mathrm{d}t}\left(2t\frac{\mathrm{d}y}{\mathrm{d}t}-\frac{\mathrm{d}y}{\mathrm{d}t}\right)=0$$

$$y\frac{\mathrm{d}^2y}{\mathrm{d}t^2}+(2t-1)\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2=0 \quad \text{(shown)}$$

9 If $y^2+ay+b=x$ where a and b are constants, show that $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^3=0$.

[Solution]

$$2y\frac{\mathrm{d}y}{\mathrm{d}x}+a\frac{\mathrm{d}y}{\mathrm{d}x}=1$$

$$(2y+a)\frac{\mathrm{d}y}{\mathrm{d}x}=1$$

$$(2y+a)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(2\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}y}{\mathrm{d}x}=0$$

$$\frac{1}{\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=0$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^3=0 \quad \text{(shown)}$$

10 For each of the following curves, find the gradient at the specified point:

(a) $x^3+y^3+3xy-1=0$ at the point $(2,-1)$

(b) $y^4+x^2y^2=4a^3(x+4a)$, where a is a constant, at the point $(a,2a)$

[Ans: **(a)** -1 **(b)** $-\frac{1}{9}$]

[Solution]

(a) Differentiating w.r.t. x,

$$3x^2+3y^2\frac{\mathrm{d}y}{\mathrm{d}x}+3x\frac{\mathrm{d}y}{\mathrm{d}x}+3y=0$$

At $(2,-1)$, $$2^2+(-1)^2\frac{\mathrm{d}y}{\mathrm{d}x}+2\frac{\mathrm{d}y}{\mathrm{d}x}+(-1)=0$$

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$$3\frac{\mathrm{d}y}{\mathrm{d}x}+3=0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-1$$

(b) Differentiating w.r.t. x,

$$4y^3\frac{\mathrm{d}y}{\mathrm{d}x}+x^2(2y)\frac{\mathrm{d}y}{\mathrm{d}x}+2xy^2=4a^3$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}\left(2y^3+x^2y\right)+xy^2=2a^3$$

At $(a,2a)$, $\frac{\mathrm{d}y}{\mathrm{d}x}\left(2(2a)^3+a^2(2a)\right)+4a(2a)^2=2a^3$

$$\frac{\mathrm{d}y}{\mathrm{d}x}(16a^3+2a^3)+4a^3=2a^3$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-2a^3}{18a^3}=-\frac{1}{9}$$

11 N14/I/2

The curve C has equation $x^2y+xy^2+54=0$. Without using a calculator, find the coordinates of the point on C at which the gradient is -1, showing that there is only one such point.

[Ans: $(-3,-3)$]

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[Solution]

$x^2y+xy^2+54=0$

Differentiate w.r.t x

$2xy+x^2\frac{\mathrm{d}y}{\mathrm{d}x}+y^2+2xy\frac{\mathrm{d}y}{\mathrm{d}x}=0$

When $\frac{\mathrm{d}y}{\mathrm{d}x}=-1$,

$2xy-x^2+y^2-2xy=0$

$x^2=y^2$

$x=\pm y$

Substitute $x=y$ into C

$y^3+y^3+54=0$

$2y^3=-54$

$y^3=-27$

$y=-3$

Substitute $x=-y$ into C

$y^3-y^3+54=0$

(no solution)

$\therefore$ Coordinates of the point at which the gradient is -1 is $(-3,-3)$

Hence there is only one such point.

12 It is given that x and y satisfy the equation $\tan^{-1}x+\tan^{-1}y+\tan^{-1}(xy)=\frac{7}{12}\pi$.

Find the value of y when $x = 1$.

(i) Express $\frac{\mathrm{d}}{\mathrm{d}x}\tan^{-1}(xy)$ in terms of x, y and $\frac{\mathrm{d}y}{\mathrm{d}x}$.

(ii) Show that, when $x = 1$, $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{1}{3}-\frac{1}{2\sqrt{3}}$. [N00/I/11]

[Ans: $\frac{1}{\sqrt{3}}$ **(i)** $\frac{1}{1+(xy)^2}\left(x\frac{\mathrm{d}y}{\mathrm{d}x}+y\right)$]

[Solution]

Given $\tan^{-1}x+\tan^{-1}y+\tan^{-1}(xy)=\frac{7}{12}\pi$

(i) When $x = 1$:

$\tan^{-1}1+\tan^{-1}y+\tan^{-1}y=\frac{7}{12}\pi$

$\frac{\pi}{4}+2\tan^{-1}y=\frac{7\pi}{12} \quad\Rightarrow\quad \tan^{-1}y=\frac{\pi}{6} \quad\Rightarrow\quad y=\tan\frac{\pi}{6} \quad\Rightarrow\quad y=\frac{1}{\sqrt{3}}$

(ii) $\frac{\mathrm{d}}{\mathrm{d}x}\left[\tan^{-1}(xy)\right]=\frac{1}{1+(xy)^2}\left(y+x\frac{\mathrm{d}y}{\mathrm{d}x}\right)$

(iii) $\tan^{-1}x+\tan^{-1}y+\tan^{-1}(xy)=\frac{7}{12}\pi$

$$\frac{1}{1+x^2}+\left(\frac{1}{1+y^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{y+x\frac{\mathrm{d}y}{\mathrm{d}x}}{1+(xy)^2}=0$$

When $x=1,\ y=\frac{1}{\sqrt{3}}$:

$$\frac{1}{2}+\left(\frac{3}{4}\right)\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{\frac{1}{\sqrt{3}}+\frac{\mathrm{d}y}{\mathrm{d}x}}{4/3}=0 \quad\Rightarrow\quad \frac{1}{2}+\left(\frac{3}{4}\right)\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{3}{4\sqrt{3}}+\left(\frac{3}{4}\right)\frac{\mathrm{d}y}{\mathrm{d}x}=0$$

$$\left(\frac{6}{4}\right)\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{1}{2}-\frac{3}{4\sqrt{3}} \quad\Rightarrow\quad \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2}{3}\left(-\frac{1}{2}-\frac{3}{4\sqrt{3}}\right)=-\frac{1}{3}-\frac{1}{2\sqrt{3}} \quad \text{(shown)}$$

13 Find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t.

(a) $x=\frac{1}{1+t^2},\ y=\frac{t}{1+t^2}$ **(b)** $x=\frac{1}{2}\left(\mathrm{e}^t-\mathrm{e}^{-t}\right),\ y=\frac{1}{2}\left(\mathrm{e}^t+\mathrm{e}^{-t}\right)$

(c) $x=a\sec t, y=a\tan t$ **(d)** $x=\mathrm{e}^{3t}\cos 3t,\ y=\mathrm{e}^{3t}\sin 3t$

[Ans: **(a)** $\frac{t^2-1}{2t}$ **(b)** $\frac{\mathrm{e}^{2t}-1}{\mathrm{e}^{2t}+1}$ **(c)** $\operatorname{cosec} t$ **(d)** $\frac{\sin 3t+\cos 3t}{\cos 3t-\sin 3t}$]

[Solution]

(a) $x=\frac{1}{1+t^2}$, $y=\frac{t}{1+t^2}$

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{-2t}{\left(1+t^2\right)^2} \text{ and } \frac{\mathrm{d}y}{\mathrm{d}t}=\frac{\left(1+t^2\right)(1)-(t)(2t)}{\left(1+t^2\right)^2}=\frac{1-t^2}{\left(1+t^2\right)^2}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}=\frac{t^2-1}{2t}$$

(b) $x=\frac{1}{2}\left(\mathrm{e}^t-\mathrm{e}^{-t}\right)$, $y=\frac{1}{2}\left(\mathrm{e}^t+\mathrm{e}^{-t}\right)$

$$\frac{dx}{dt}=\frac{1}{2}\left(e^{t}+e^{-t}\right) \quad \text{and} \quad \frac{dy}{dt}=\frac{1}{2}\left(e^{t}-e^{-t}\right)$$

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{e^{t}-e^{-t}}{e^{t}+e^{-t}}=\frac{e^{2t}-1}{e^{2t}+1}$$

(c) $x=a\sec t \quad , \quad y=a\tan t$

$$\frac{dx}{dt}=a\sec t\tan t \quad \text{and} \quad \frac{dy}{dt}=a\sec^{2} t$$

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{a\sec^{2} t}{a\sec t\tan t}=\operatorname{cosec} t$$

(d) $x=e^{3t}\cos 3t,\ y=e^{3t}\sin 3t$

$$\frac{dx}{dt}=\left(3e^{3t}\right)\left(\cos 3t\right)+\left(e^{3t}\right)\left(-3\sin 3t\right)=3e^{3t}\left(\cos 3t-\sin 3t\right)$$

$$\frac{dy}{dt}=\left(3e^{3t}\right)\left(\sin 3t\right)+\left(e^{3t}\right)\left(3\cos 3t\right)=3e^{3t}\left(\sin 3t+\cos 3t\right)$$

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{\sin 3t+\cos 3t}{\cos 3t-\sin 3t}$$

Supplementary Questions

14 Differentiate the following with respect to x:

(a) $\ln\left(x+\sqrt{x^2-4}\right)$ **(b)** $\sin^{-1}\left(\sqrt{1-x^4}\right)$ **(c)** $\left(x+x^2\right)^x$

[Ans: **(a)** $\frac{1}{\sqrt{x^2-4}}$ **(b)** $\frac{-2x}{\sqrt{1-x^4}}$ **(c)** $(x+x^2)^x\left(\frac{1+2x}{1+x}+\ln(x+x^2)\right)$]

Solution	
(a) $\frac{d}{dx}\ln\left(x+\sqrt{x^2-4}\right)=\frac{1+\frac{2x}{2\sqrt{x^2-4}}}{x+\sqrt{x^2-4}}$ $=\frac{\sqrt{x^2-4}+x}{\left(x+\sqrt{x^2-4}\right)\sqrt{x^2-4}}=\frac{1}{\sqrt{x^2-4}}$ (b $\frac{d}{dx}\left[\sin^{-1}(\sqrt{1-x^4})\right]$ $=\frac{\frac{1}{2}\left(\frac{-4x^3}{\sqrt{1-x^4}}\right)}{\sqrt{1-(1-x^4)}}$ $=\frac{-2x^3}{x^2\sqrt{1-x^4}}=\frac{-2x}{\sqrt{1-x^4}}$	(c) $y=\left(x+x^2\right)^x$ $\ln y=\ln\left(x+x^2\right)^x$ $=x\ln\left(x+x^2\right)$ $\frac{1}{y}\frac{dy}{dx}=x\frac{1+2x}{x+x^2}+\ln(x+x^2)$ $\frac{dy}{dx}=(x+x^2)^x\left(\frac{1+2x}{1+x}+\ln(x+x^2)\right)$

15 Find $\frac{dy}{dx}$ in terms of x and y for the following equations:

(a) $\sin y+x=xy$ **(b)** $\ln\left(1+y\right)=\tan^{-1}x$ **(c)** $y=\sin\left(x+y\right)^2$

[Ans: **(a)** $\frac{y-1}{\cos y-x}$ **(b)** $\frac{1+y}{1+x^2}$ **(c)** $\frac{2\left(x+y\right)\cos\left(x+y\right)^2}{1-2\left(x+y\right)\cos\left(x+y\right)^2}$]

(a) $\sin y+x=xy$ $\frac{dy}{dx}\cos y+1=x\frac{dy}{dx}+y$ $\frac{dy}{dx}\left(\cos y-x\right)=y-1$ $\frac{dy}{dx}=\frac{y-1}{\cos y-x}$	(b) $\ln\left(1+y\right)=\tan^{-1}x$ $\frac{1}{1+y}\frac{dy}{dx}=\frac{1}{1+x^2}$ $\frac{dy}{dx}=\frac{1+y}{1+x^2}$

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(c) $y=\sin(x+y)^2$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\cos(x+y)^2\,2(x+y)\left(1+\frac{\mathrm{d}y}{\mathrm{d}x}\right)$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2(x+y)\cos(x+y)^2}{1-2(x+y)\cos(x+y)^2}$	

16 If $x^2+3xy-y^2=3$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ at the point (1, 1). [Ans: -5, 78]

[Solution] $x^2+3xy-y^2=3$ $2x+3x\frac{\mathrm{d}y}{\mathrm{d}x}+3y-2y\frac{\mathrm{d}y}{\mathrm{d}x}=0$ $2x+3y+(3x-2y)\frac{\mathrm{d}y}{\mathrm{d}x}=0\cdots(1)$ $\frac{\mathrm{d}y}{\mathrm{d}x}(3x-2y)=-2x-3y$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2x+3y}{2y-3x}$ Differentiating (1) wrt x, $2+3\frac{\mathrm{d}y}{\mathrm{d}x}+(3x-2y)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(3-2\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}y}{\mathrm{d}x}=0$	When $x=1$, $y=1$, $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2+3}{2-3}$, $=-5$ $2+3(-5)+(3-2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+(3-2(-5))(-5)=0$ $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=78$

17 If $y=\mathrm{e}^{kt}\cos pt$, prove that $\frac{\mathrm{d}^2y}{\mathrm{d}t^2}-2k\frac{\mathrm{d}y}{\mathrm{d}t}+\left(k^2+p^2\right)y=0$. If $\frac{\mathrm{d}y}{\mathrm{d}t}=2p$ and $\frac{\mathrm{d}^2y}{\mathrm{d}t^2}=3p$ when $t=\frac{3\pi}{2p}$, calculate k and prove that $p=\frac{9\pi}{8\ln 2}$.

[Ans: $k=\frac{3}{4}$]

[Solution] Differentiate with respect to t, $\frac{\mathrm{d}y}{\mathrm{d}t}=\mathrm{e}^{kt}(-\sin pt)(p)+(\cos pt)\left(\mathrm{e}^{kt}\right)k$ $=k\mathrm{e}^{kt}\cos pt-p\mathrm{e}^{kt}\sin pt$	$\frac{\mathrm{d}y}{\mathrm{d}t}=ky-p\mathrm{e}^{kt}\sin pt$

Since $e^{kt}\cos pt = y$,

$\frac{dy}{dt} = ky - pe^{kt}\sin pt$ ---(1)

Differentiate with respect to t,

$$\frac{d^2y}{dt^2} = k\frac{dy}{dt} - p\left[e^{kt}(\cos pt)(p) + (\sin pt)e^{kt}k\right]$$

$$= k\frac{dy}{dt} - p^2 \underbrace{e^{kt}(\cos pt)}_{y} - k\underbrace{pe^{kt}\sin pt}_{\text{from(1) } ky - \frac{dy}{dt}}$$

$$\frac{d^2y}{dt^2} = k\frac{dy}{dt} - p^2y + k\left(\frac{dy}{dt} - ky\right)$$

$$= k\frac{dy}{dt} - p^2y + k\frac{dy}{dt} - k^2y$$

$$= 2k\frac{dy}{dt} - \left(k^2 + p^2\right)y$$

$$\frac{d^2y}{dt^2} - 2k\frac{dy}{dt} + \left(k^2 + p^2\right)y = 0 \text{ (shown)}$$

When $t = \frac{3\pi}{2p}$, $\frac{dy}{dt} = 2p$, $\frac{d^2y}{dt^2} = 3p$

$$y = e^{\frac{3\pi}{2p}k}\cos\left(\frac{3\pi}{2p}\times p\right)$$

$$= e^{\frac{3\pi}{2p}k}\cos\left(\frac{3}{2}\pi\right)$$

$$= 0$$

$$2p = (0)k - pe^{k\frac{3\pi}{2p}}\sin\left(\frac{3\pi}{2p}\times p\right)$$

$$2p = 0 - pe^{k\frac{3\pi}{2p}}\sin\left(\frac{3}{2}\pi\right)$$

$$2p = pe^{k\frac{3\pi}{2p}} \quad \ldots (1)$$

$$3p - 2k(2p) + \left(k^2 + p^2\right)(0) = 0$$

$$3p - 4kp = 0$$

$$3p = 4kp$$

$$k = \frac{3}{4}$$

Substitute $k = \frac{3}{4}$ into (1)

$$2p = pe^{\frac{3\pi}{2p}\times\frac{3}{4}}$$

$$2p = pe^{\frac{9\pi}{8p}}$$

Taking ln on both sides

$$\ln 2p = \ln pe^{\frac{9\pi}{8p}}$$

$$\ln 2 + \ln p = \ln p + \ln e^{\frac{9\pi}{8p}}$$

$$\ln 2 = \frac{9\pi}{8p}$$

$$p = \frac{9\pi}{8\ln 2}$$

18 Find, by the first principles, the first derivative of $f(x) = \cos x$, given that $\lim_{x\to 0}\frac{\sin x}{x} = 1$.

(a) $\mathrm{f}(x+\delta x)=\cos(x+\delta x)=\cos x\cos(\delta x)-\sin x\sin(\delta x)$

$\mathrm{f}(x+\delta x)-\mathrm{f}(x)=\cos(x+\delta x)=\cos x\cos(\delta x)-\sin x\sin(\delta x)-\cos x$

$$\frac{\mathrm{f}(x+\delta x)-\mathrm{f}(x)}{\delta x}=\frac{\cos x\cos(\delta x)-\sin x\sin(\delta x)-\cos x}{\delta x}$$

$$=\frac{\cos x\left[\cos(\delta x)-1\right]-\sin x\sin(\delta x)}{\delta x}=\frac{\cos x\left[-2\sin^2\left(\frac{\delta x}{2}\right)\right]-\sin x\sin(\delta x)}{\delta x}$$

$$\lim_{\delta x\to 0}\frac{\mathrm{f}(x+\delta x)-\mathrm{f}(x)}{\delta x}=\lim_{\delta x\to 0}\frac{\cos x\left[-2\sin^2\left(\frac{\delta x}{2}\right)\right]-\sin x\sin(\delta x)}{\delta x}$$

$$=(\cos x)\lim_{\delta x\to 0}\frac{\left[-2\sin^2\left(\frac{\delta x}{2}\right)\right]}{\delta x}-(\sin x)\lim_{\delta x\to 0}\frac{\sin(\delta x)}{\delta x}$$

$$=-(\cos x)\lim_{\delta x\to 0}\left[\frac{\sin\left(\frac{\delta x}{2}\right)}{\left(\frac{\delta x}{2}\right)}\sin\left(\frac{\delta x}{2}\right)\right]-\sin x(1)$$

$$\mathrm{f}'(x)=-(\cos x)(1)(0)-\sin x=-\sin x$$

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