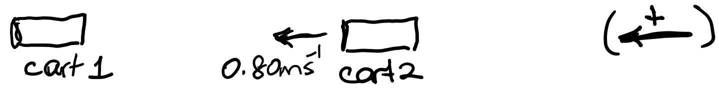
Suggested Solutions for **Worked Examples**

1	**a)**	$V_{\text{Car,Truck}} = V_{\text{Car,Earth}} + V_{\text{Earth,Truck}}$ $V_{\text{Car,Truck}} = 25 - 30 = -5 \text{ m s}^{-1}$ The observer in the truck observes that the car moves in the opposite direction at a constant speed of 5 m s^{-1}.
	b)	$V_{\text{Truck,Car}} = V_{\text{Truck,Earth}} + V_{\text{Earth,Car}}$ $V_{\text{Truck,Car}} = 30 - 25 = +5 \text{ m s}^{-1}$ The observer in the car observes that the truck moves in the forward direction at a constant speed of 5 m s^{-1}.
	c)	$d_{\text{Car,Truck}} = V_{\text{Car,Truck}} t = (-5)(60) = -300 \text{ m}$
2	$V_{\text{spider,Earth}} = V_{\text{spider,passenger}} + V_{\text{passenger,train}} + V_{\text{train,Earth}}$ $V_{\text{spider,Earth}} = -0.5 + 1.2 + 3.1 = 3.8 \text{ m s}^{-1}$	
3	**a)**	Yes. In both Earth and reference frame M, the speeds of the carts are constant. Hence, their kinetic energies are constant.
	b)	No net external force acting on the system. Hence, no change in momentum and energy. Therefore, the isolated system containing only cart 1 is closed.
	c)	No net external force acting on the system. Hence, no change in momentum and energy. Therefore, the isolated system containing only cart 2 is closed.
4	**a)**	Merry-go-round: Its velocity changes, there is an acceleration towards the centre of the circle. It is a non-inertial reference frame.
	b)	Airplane taking off: The velocity must increase. There is an acceleration associated with it. It is a non-inertial reference frame.
	c)	Train at constant speed: The velocity is constant and no acceleration. It is an inertial frame of reference.

5 a)



Head-on collision in the Earth frame,

By RSOA = RSOS,

$$u_2 - u_1 = v_1 - v_2$$
$$0.80 - 0 = v_1 - v_2 \rightarrow v_1 = 0.80 + v_2$$

By PCOLM,

$$(0.36)(0) + (0.12)(0.80) = (0.36)v_1 + (0.12)v_2$$
$$(0.12)(0.80) = (0.36)(0.80 + v_2) + (0.12)v_2$$
$$v_2 = -0.40 \text{ m s}^{-1} \text{ (moves rightwards)}$$
$$v_1 = 0.40 \text{ m s}^{-1} \text{ (moves leftwards)}$$

The relative velocity is 0.80 m s^{-1}.

$$\Delta V_1 = v_1 - u_1 = 0.40 - 0 = 0.40 \text{ m s}^{-1} \text{ (leftwards)}$$
$$\Delta V_2 = v_2 - u_2 = -0.40 - (+0.80) = -1.20 \text{ m s}^{-1} \text{ (rightwards)}$$

b)

$$\Delta p_1 = m_1 \Delta V_1 = (0.36)(0.40) = 0.144 \text{ kg m s}^{-1} \text{ (leftwards)}$$
$$\Delta p_2 = m_2 \Delta V_2 = (0.12)(-1.20) = -0.144 \text{ kg m s}^{-1} \text{ (rightwards)}$$

Please note that the **total change in momentum** in the cart 1 and 2 system is zero. This indicates that there is no net external force acting on the system.

c)

$$\Delta E_{k,1} = \frac{1}{2}(0.36)(0.40^2 - 0^2) = 0.0288 \text{ J}$$
$$\Delta E_{k,2} = \frac{1}{2}(0.12)((-0.40)^2 - 0.80^2) = -0.0288 \text{ J}$$

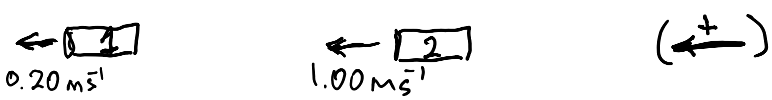
Please note that the **total change in the kinetic energy** of the system is zero. This indicates that there is no energy lost in this elastic collision.

d)

Reference frame M moves at a constant speed of 0.20 m s^{-1} towards right (towards cart 2)

$$u_{1,M} = u_{1,Earth} + u_{Earth,M} = 0 + 0.20 = 0.20 \text{ m s}^{-1} \text{ (leftwards)}$$
$$u_{2,M} = u_{2,Earth} + u_{Earth,M} = 0.80 + 0.20 = 1.00 \text{ m s}^{-1} \text{ (leftwards)}$$

In reference frame M,



Head-on collision in the Earth frame, by RSOA = RSOS,

		$u_{2,M} - u_{1,M} = v_{1,M} - v_{2,M}$ $1.00 - 0.20 = v_{1,M} - v_{2,M} \rightarrow v_{1,M} = 0.80 + v_{2,M}$ By PCOLM, $(0.36)(0.20) + (0.12)(1.00) = (0.36)v_{1,M} + (0.12)v_{2,M}$ $v_{2,M} = -0.20$ m s^{-1} (moves rightwards) $v_{1,M} = 0.60$ m s^{-1} (moves leftwards) Please check that the relative velocity is 0.80 m s^{-1} in this reference. It was the same in the Earth frame as well. This shows that the relative velocity is the same in all inertial frames of reference. Change in velocity in reference frame M, $\Delta V_{1,M} = v_{1,M} - u_{1,M} = 0.60 - 0.20 = 0.40$ m s^{-1} (leftwards) $\Delta V_{2,M} = v_{2,M} - u_{2,M} = -0.20 - (1.00) = -1.20$ m s^{-1} (rightwards) Change in momentum in reference frame M, $\Delta p_{1,M} = m_1 \Delta V_{1,M} = (0.36)(0.40) = 0.144$ kg m s^{-1} (leftwards) $\Delta p_{2,M} = m_2 \Delta V_{2,M} = (0.12)(-1.20) = -0.144$ kg m s^{-1} (rightwards) $\Delta E_{k,1} = \frac{1}{2}(0.36)(0.60^2 - 0.20^2) = 0.0576$ J $\Delta E_{k,2} = \frac{1}{2}(0.12)((-0.20)^2 - 1.00^2) = -0.0576$ J Please take note that the individual change in kinetic energy in cart 1 or cart 2 is different in this frame compared to the Earth frame. However, the **total change in kinetic energy is still zero**. This reinforces the fact that the total change in kinetic energy is the same in all inertial frames of reference regardless of the type of collision.
	e)	Now, the collision is elastic and the velocity of cart 1 is +0.30 m s^{-1} after the collision (moves towards left). In the Earth frame, by PCOLM, $(0.36)(0) + (0.12)(0.80) = (0.36)(0.30) + (0.12)v_2$ $v_2 = -0.10$ m s^{-1} (moves rightwards) $\Delta E_{k,1} = \frac{1}{2}(0.36)(0.30^2 - 0^2) = 0.0162$ J $\Delta E_{k,2} = \frac{1}{2}(0.12)((-0.10)^2 - 0.80^2) = -0.0378$ J $\sum \Delta E_k = 0.0162 + (-0.0378) = -0.0216$ J

		The loss in the kinetic energy is expected as this is an inelastic collision. The loss in kinetic energy goes to increase the internal energy of the system. Reference frame M, $u_{1,M} = u_{1,Earth} + u_{Earth,M} = 0 + 0.20 = 0.20$ m s^{-1} (leftwards) $u_{2,M} = u_{2,Earth} + u_{Earth,M} = 0.80 + 0.20 = 1.00$ m s^{-1} (leftwards) $v_{1,M} = v_{1,Earth} + v_{Earth,M} = 0.30 + 0.20 = 0.50$ m s^{-1} (leftwards) $v_{2,M} = v_{2,Earth} + v_{Earth,M} = -0.10 + 0.20 = 0.10$ m s^{-1} (leftwards) $\Delta E_{k,1} = \frac{1}{2}(0.36)(0.50^2 - 0.20^2) = 0.0378$ J $\Delta E_{k,2} = \frac{1}{2}(0.12)((-0.10)^2 - 1.00^2) = -0.0594$ J $\sum \Delta E_k = 0.0378 + (-0.0594) = -0.0216$ J Please take note that the energy loss in reference frame M is the same as that of in Earth frame. This indicates that the **gain in internal energy is the same** in all inertial frames of reference.
6	**a)**	$V_{CM} = \frac{(0.36)(0) + (0.12)(0.80)}{0.36 + 0.12} = 0.20$ m s^{-1} (leftwards)
	b)	$u_{1,CM} = u_{1,Earth} + u_{Earth,CM} = 0 - 0.20 = -0.20$ m s^{-1} (rightwards) $u_{2,CM} = u_{2,Earth} + u_{Earth,CM} = 0.80 - 0.20 = 0.60$ m s^{-1} (leftwards) $v_{1,CM} = 0.40 - 0.20 = 0.20$ m s^{-1} (leftwards) $v_{2,CM} = -0.40 - 0.20 = -0.60$ m s^{-1} (rightwards) Please take note that the velocities simply change sign after the elastic collision in this centre of mass frame (zero-momentum frame). This always happens for elastic collision in CM frames and it simplifies tedious calculations greatly. $\Delta V_{1,CM} = 0.20 - (-0.20) = 0.40$ m s^{-1} (leftwards) $\Delta V_{2,CM} = -0.60 - (+0.60) = -1.20$ m s^{-1} (rightwards) $\Delta p_{1,CM} = m_1 \Delta V_{1,CM} = (0.36)(0.40) = 0.144$ kg m s^{-1} $\Delta p_{2,CM} = m_2 \Delta V_{2,CM} = (0.12)(-1.20) = -0.144$ kg m s^{-1} $\Delta E_{k,1,CM} = \frac{1}{2}(0.36)(0.20^2 - (-0.20)^2) = 0$ $\Delta E_{k,2,CM} = \frac{1}{2}(0.12)((-0.60)^2 - 0.60^2) = 0$

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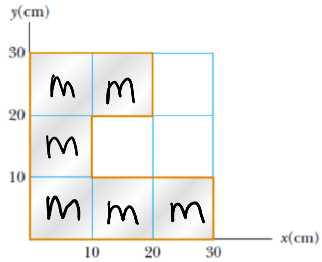


Since it is a uniform piece of sheet metal, let the mass of each square be *m*.

$$x_{CM} = \frac{3m(15) + m(5) + 2m(10)}{3m + m + 2m} = 11.7 \text{ cm}$$

$$y_{CM} = \frac{3m(5) + m(15) + 2m(25)}{3m + m + 2m} = 13.3 \text{ cm}$$

The coordinate of the CM is (11.7 cm, 13.3 cm) which is outside of the sheet metal.

8 a) Since the rod is thin, we assume that it has negligible thickness.

The infinitesimal mass of the rod: $dm = \lambda dx = \frac{M}{L} dx$

The centre of mass along the x-direction:

$$x_{cm} = \frac{1}{M}\int x dm = \frac{1}{M}\int x(\lambda dx) = \frac{\lambda}{M}\int_0^L x dx = \frac{\lambda}{M}\frac{L^2}{2}$$

$$x_{cm} = \frac{(M/L)}{M}\frac{L^2}{2} = \frac{L}{2}$$

b) It is a non-uniform rod which has mass per unit length varying with *x*.

The infinitesimal mass of the rod: $dm = \lambda dx = \alpha x dx$

Integrating it to find the total mass of the non-uniform rod gives $M = \frac{\alpha L^2}{2}$

$$x_{cm} = \frac{1}{M}\int x dm = \frac{1}{M}\int x(\lambda dx) = \frac{1}{M}\int x(\alpha x dx) = \frac{\alpha}{M}\int x^2 dx$$

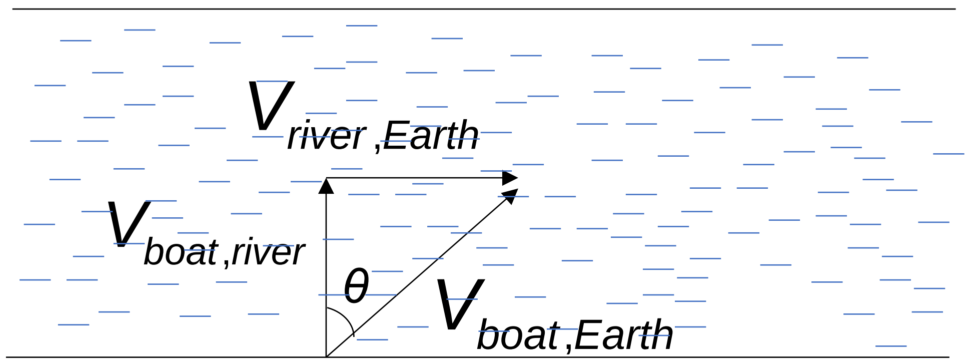
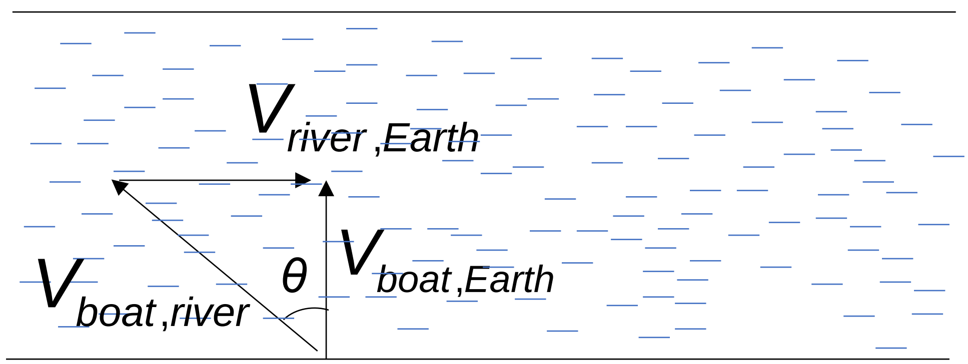
$$x_{cm} = \frac{\alpha}{M}\frac{L^3}{3} = \frac{\alpha L^3}{\left(\frac{\alpha L^2}{2}\right)3} = \frac{2L}{3}$$

9 a) Neglecting air resistance, the only external force acting on the projectile is the gravitational force. Thus, if the projectile did not explode, it would continue to move along the parabolic path indicated by the dashed line.

		Because the forces caused by the explosion are internal, they do not affect the motion of the center of mass of the system (fragments). Hence, after the explosion, the center of mass of the fragments follows the same parabolic path, the projectile would have followed if there had been no explosion.
	b)	The center of mass (CM) of the two-piece lands at a distance *R* from the launch point. One piece lands at a farther distance *R* from the landing point (2*R* from the launch point) of the CM. Both pieces have the same mass, the other piece must land at a distance R to the left of the landing point. This piece will be right back **at the launch point**.
10	**a)**	It is a head-on elastic collision. By RSOA = RSOS, $1.50-(-0.400)=v_2-v_1$ $1.90=v_2-v_1$ By PCOLM, $\sum p_{initial}=\sum p_{final}$ $(0.200)(1.50)+(0.300)(-0.400)=(0.200)v_1+(0.300)v_2$ $0.180=(0.200)v_1+(0.300)(1.90+v_1)$ $v_1=-0.780\text{ m s}^{-1}$ $v_2=1.12\text{ m s}^{-1}$
	b)	$v_{cm,before}=\dfrac{(0.200)(1.50)+(0.300)(-0.400)}{0.200+0.300}=0.360\text{ m s}^{-1}$ $v_{cm,after}=\dfrac{(0.200)(-0.780)+(0.300)(1.12)}{0.200+0.300}=0.360\text{ m s}^{-1}$ In the center of mass frame, $v_{cm,before}=v_{cm,after}$

	c) and d)	The velocities in the CM frame (zero-momentum frame) before the collision, $u_{1,CM} = u_{1,Earth} + V_{Earth,CM} = 1.50 - 0.360 = 1.14 \text{ m s}^{-1}$ $u_{2,CM} = u_{2,Earth} + V_{Earth,CM} = -0.400 - 0.360 = -0.760 \text{ m s}^{-1}$ $\sum p_i = (0.200)(1.14) + (0.300)(-0.760) = 0$ The velocities in the CM frame (zero-momentum frame) after the collision, $v_{1,CM} = v_{1,Earth} + V_{Earth,CM} = -0.780 - 0.360 = -1.14 \text{ m s}^{-1}$ $v_{2,CM} = v_{2,Earth} + V_{Earth,CM} = 1.12 - 0.360 = 0.760 \text{ m s}^{-1}$ $\sum p_f = (0.200)(-1.14) + (0.300)(0.760) = 0$ Please take note that the velocites change their signs after the elastic collision in the CM frame (zero-momentum frame). This fact greatly simplifies tedious calculations in **elastic** collisions.

Suggested Solutions for Tutorial Questions

1	**a)**	 $\vec{V}_{boat,Earth} = \vec{V}_{boat,river} + \vec{V}_{river,Earth}$ $\vec{V}_{boat,Earth} = \sqrt{5.00^2 + 10.0^2} = 11.2\text{ km h}^{-1}$ $\theta = \tan^{-1}\left(\frac{\vec{V}_{river,Earth}}{\vec{V}_{boat,river}}\right) = \tan^{-1}\left(\frac{5.00}{10.0}\right) = 26.6°$
	b)	 $\vec{V}_{boat,Earth} = \vec{V}_{boat,river} + \vec{V}_{river,Earth}$ $\vec{V}_{boat,Earth} = \sqrt{10.0^2 - 5.00^2} = 8.66\text{ km h}^{-1}$ $\theta = \tan^{-1}\left(\frac{\vec{V}_{river,Earth}}{\vec{V}_{boat,Earth}}\right) = \tan^{-1}\left(\frac{5.00}{8.66}\right) = 30.0°$

2	Total momentum of the system (Romeo, Juliet and boat) is zero. No net external force is exerted on the system as Juliet moves carefully moves to the rear of the boat, hence, the principle of conservation of momentum can be applied. $\sum p = 0$ $m_{juliet} v_{juliet} - \sum m\ v_{boat} = 0$ $m_{juliet} \dfrac{2.70}{t} = (77.0 + 55.0 + 80.0)\dfrac{d_{boat}}{t}$ $d_{boat} = \dfrac{(55.0)(2.70)}{(77.0+55.0+80.0)} = 0.700\text{ m}$	
3	**a)**	By PCOLM, $4mv = 4mv_{4m} + mv_m$ $v_m = 4v - 4v_{4m}$ By RSOA = RSOS, $v = v_m - v_{4m} \rightarrow v_m = v + v_{4m}$ $v + v_{4m} = 4v - 4v_{4m} \rightarrow 3v = 5v_{4m} \rightarrow v_{4m} = 3v/5$ $v_m = 8v/5$ % of kinetic energy transferred to *m*, $\dfrac{\frac{1}{2}m\left(\frac{8v}{5}\right)^2}{\frac{1}{2}4mv^2} = \dfrac{64}{4(25)}$ $= 0.64 \rightarrow 64\%$ **Method II: Using zero-momentum frame (center-of-mass frame)** $v_{cm} = \dfrac{4mv + m0}{5m} = \dfrac{4v}{5}$ The velocities of particle in the zero-momentum frame *before the collision*, $v_{4m,cm} = v_{4m,Earth} + v_{Earth,cm} = v - \dfrac{4v}{5} = \dfrac{v}{5}$ $v_{m,cm} = v_{m,Earth} + v_{Earth,cm} = 0 - \dfrac{4v}{5} = -\dfrac{4v}{5}$ After the collision, **the sign of velocities changes** in CM frame, hence, $v'_{4m,CM} = -\dfrac{v}{5}$ and $v'_{m,CM} = \dfrac{4v}{5}$ Rewriting the velocities *after the collision* in the Earth frame, $v'_{4m,Earth} = v'_{4m,cm} + v_{cm,Earth} = -\dfrac{v}{5} + \dfrac{4v}{5} = \dfrac{3v}{5}$ $v'_{m,Earth} = v'_{m,cm} + v_{cm,Earth} = \dfrac{4v}{5} + \dfrac{4v}{5} = \dfrac{8v}{5}$

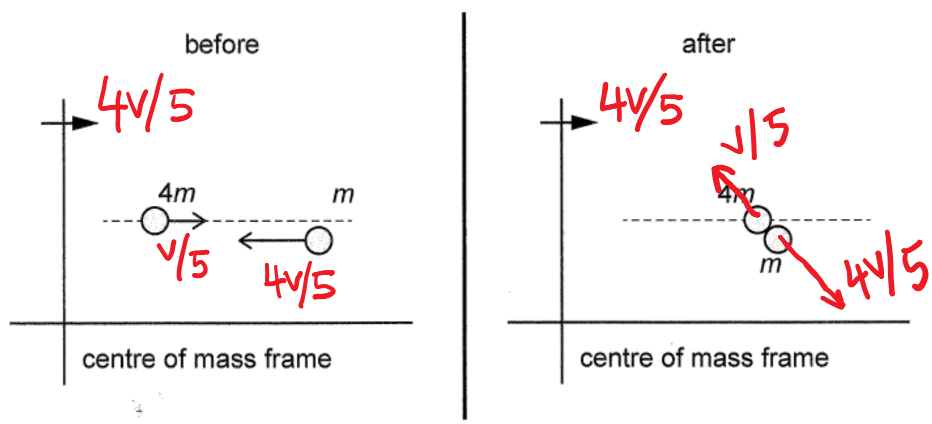
% of kinetic energy transferred to m,

$$\frac{\frac{1}{2}m\left(\frac{8v}{5}\right)^2}{\frac{1}{2}4mv^2} = \frac{64}{4(25)}$$

$$= 0.64 \rightarrow 64\%$$

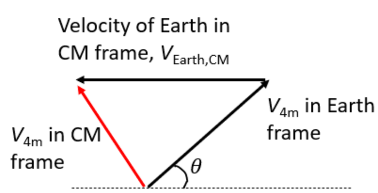
Note: There is a high chance that students might perform calculation error in solving simultaneous equations in the Earth frame (first method). The zero-momentum frame does not involve such equations.

b)



$$v_{cm} = \frac{4mv}{5m} = \frac{4}{5}v$$

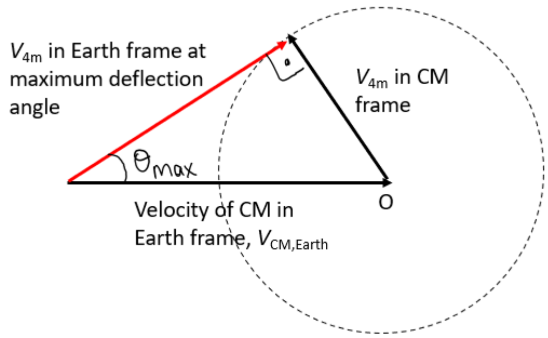
Using the vector diagram could help you find the direction of velocity of 4m in the center-of-mass frame readily.



$$V_{4m,CM} = V_{4m,Earth} + V_{Earth,CM}$$

Note: The diagram may help visualize the direction of velocities in earth and zero-momentum (CM) frames. The total momentum in the CM frame is zero.

c)



$$\sin\theta_{max} = \frac{V_{4m,CM}}{V_{CM,Earth}} = \frac{v/5}{4v/5} = \frac{1}{4}$$

$$\theta_{max} = 14.5°$$

4 a)

$1.0\ kg \rightarrow 4.0\ m\ s^{-1}$ 3.0 kg (at rest) $(\overset{+}{\rightarrow})$

After the inelastic collision,

$\leftarrow -0.50\ m\ s^{-1}$ $\rightarrow 1.5\ m\ s^{-1}$

$$\text{coefficient of restitution} = e = \begin{cases} 1 & \text{head-on elastic collision} \\ 0 < e < 1 & \text{inelastic collision} \\ 0 & \text{perfectly inelastic collision} \end{cases}$$

$$e = \frac{1.50 - (-0.50)}{4.0 - 0} = 0.5 \text{ inelastic collision}$$

b) Earth reference frame

$$\Delta KE_{\text{Earth frame}} = \frac{1}{2}(3.0)(1.5)^2 + \frac{1}{2}(1.0)(-0.5)^2 - \left(\frac{1}{2}(3.0)(0)^2 + \frac{1}{2}(1.0)(4.0)^2\right)$$

$$\Delta KE_{\text{Earth frame}} = -4.5 \text{ J}$$

Reference frame M moving at -1.0 m s^{-1} relative to Earth.

$$u_{1,M} = u_{1,E} + u_{E,M} = 4.0 + 1.0 = 5.0 \text{ m s}^{-1}$$

$$u_{3,M} = u_{3,E} + u_{E,M} = 0 + 1.0 = 1.0 \text{ m s}^{-1}$$

$$v_{1,M} = v_{1,E} + v_{E,M} = -0.5 + 1.0 = 0.5 \text{ m s}^{-1}$$

$$v_{3,M} = v_{3,E} + v_{E,M} = 1.5 + 1.0 = 2.5 \text{ m s}^{-1}$$

		$\Delta KE_{\text{M frame}} = \frac{1}{2}(3.0)(2.5)^2 + \frac{1}{2}(1.0)(0.5)^2 - \left(\frac{1}{2}(3.0)(1.0)^2 + \frac{1}{2}(1.0)(5.0)^2\right)$ $\Delta KE_{\text{M frame}} = -4.5\text{ J}$ Please note that ΔKE of the system regardless of inertial frames of reference. However, the change in kinetic energy of individual masses is different.
5	a)	→ 45.0 m s⁻¹ 0.100 kg 0.050 kg Earth frame $u_{CM} = \frac{0.100(45.0)+(0.050)(0)}{(0.100+0.050)} = 30.0\text{ m s}^{-1}$ $u_{0.100,CM} = u_{0.100,Earth} + u_{Earth,CM} = 45.0 - 30.0 = 15.0\text{ m s}^{-1}$ $u_{0.050,CM} = u_{0.050,Earth} + u_{Earth,CM} = 0 - 30.0 = -30.0\text{ m s}^{-1}$
	b)	$u_{rel} = u_{0.100,CM} - u_{0.050,CM} = 15.0-(-30.0) = 45.0\text{ m s}^{-1}$ Please note that the relative velocity in all inertial frames is the same.
	c)	Consider a perfectly inelastic collision. By the principle of conservation of linear momentum, $(0.100)(45.0) = (0.100+0.050)v_{common}$ $v_{common} = 30.0\text{ m s}^{-1}$ loss in KE + gain in internal energy of the system = 0 (by PCOE) Internal energy $= \frac{1}{2}(0.100)(45.0)^2 - \frac{1}{2}(0.100+0.050)(30.0)^2 = 33.8\text{ J}$
	d)	Kinetic energy after the collision would be $0.8\left(\frac{1}{2}(0.100)(45.0)^2\right) = 81\text{ J}$ By PCOLM, $(0.100)(45.0) = (0.100)v_1 + (0.050)v_2$ $v_1 = 45 - \frac{v_2}{2}$ By PCOE, $81\text{ J} = \frac{1}{2}(0.100)\left(45 - \frac{v_2}{2}\right)^2 + \frac{1}{2}(0.050)v_2^2$ $v_2 = 48.97\text{ m s}^{-1}$ $v_1 = 20.52\text{ m s}^{-1}$ Please take note that in this collision, only 20.25 J of energy is converted to internal energy.

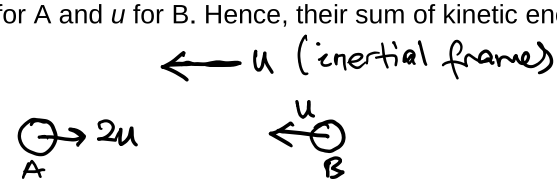
6

The kinetic energy of the carts in the Earth reference frame is

$$K = \frac{1}{2}mu^2 + \frac{1}{2}m(2u)^2 = \frac{1}{2}m5u^2$$

In another inertial frame of reference, the kinetic energy must be the same. This can be achieved by exchanging the speeds.

If we choose the inertial frame travelling towards A with a speed *u* then this would result in the speed of 2*u* for A and *u* for B. Hence, their sum of kinetic energy would be *K* again.



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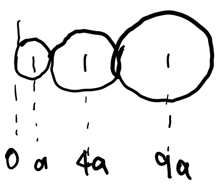


$$m_1 = \rho V_1 = \rho \frac{4}{3}\pi a^3 = m$$

$$m_2 = 2\rho V_2 = 2\rho \frac{4}{3}\pi (2a)^3 = 16m$$

$$m_3 = 3\rho V_3 = 3\rho \frac{4}{3}\pi (3a)^3 = 81m$$

$$x_{CM} = \frac{m_1 a + m_2 4a + m_3 9a}{m_1 + m_2 + m_3} = \frac{ma + 16m(4a) + 81m(9a)}{m + 16m + 81m} = 8.1a$$

8.1*a* away from the starting point O.

8

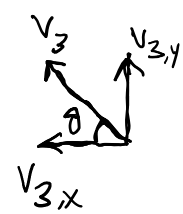
The center of mass velocity is the same before and after the collision.

$$V_{CM,y} = 300 \text{ m s}^{-1} = \frac{m(450) + m(0) + mV_{3,y}}{m + m + m} \rightarrow V_{3,y} = 450 \text{ m s}^{-1}$$

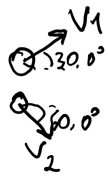
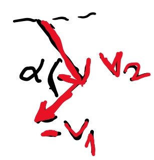
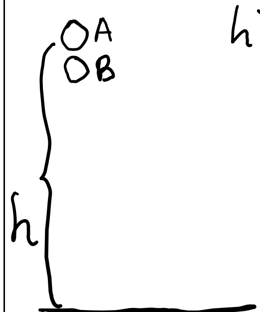
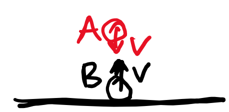
$$V_{CM,y} = 0 \text{ m s}^{-1} = \frac{m(0) + m(240) + mV_{3,x}}{m + m + m} \rightarrow V_{3,x} = -240 \text{ m s}^{-1}$$

$$V_3 = \sqrt{V_{3,x}^2 + V_{3,y}^2} = \sqrt{(-240)^2 + (450)^2} = 510 \text{ m s}^{-1}$$

$$\theta = \tan^{-1}\left(\frac{V_{3,y}}{V_{3,x}}\right) = \tan^{-1}\left(\frac{450}{240}\right) = 61.9^\circ \text{ above the horizontal}$$



<table>
<tr><td>9</td><td>a)</td><td>No net external force acting on the system, hence, the total momentum of the puck system is conserved in both vertical and horizontal directions.

$\sum p_x = (0.200)(3.0) = (0.200)v_1\cos(30.0°) + (0.200)v_2\cos(60.0°)$
$3.0 = v_1\cos(30.0°) + v_2\cos(60.0°)$
$\sum p_y = 0 = (0.200)v_1\sin(30.0°) - (0.200)v_2\sin(60.0°)$
$v_1\sin(30.0°) = v_2\sin(60.0°) \rightarrow v_1 = v_2\sqrt{3}$
$3.0 = v_2\sqrt{3}\cos(30.0°) + v_2\cos(60.0°) \rightarrow v_2 = 1.5\text{ m s}^{-1}$
$v_1 = 2.6\text{ m s}^{-1}$</td></tr>
<tr><td></td><td>b)</td><td>
$\vec{v}_2 - \vec{v}_1 = \sqrt{v_2^2 + v_1^2 - 2v_2v_1\cos(\alpha)}$ where $\alpha = 90.0°$
$\vec{v}_2 - \vec{v}_1 = 3.0\text{ m s}^{-1}$</td></tr>
<tr><td>10</td><td>a)</td><td>h ≫ diameter of balls
(↑+)

$V_{B,Earth} = V_{B,A} + V_{A,Earth}$
$V_{B,A} = V_{B,Earth} - V_{A,Earth}$
$V_{B,A} = v - (-v) = 2v$ (upwards)

Using PCOE for the ball A or B,

$mgh + 0 = 0 + \frac{1}{2}mv^2 \rightarrow v = \sqrt{2gh}$

$V_{B,A} = 2v = \sqrt{8gh}$</td></tr>
</table>

	b)	$V_{CM}=\dfrac{m_A(-v)+m_B(v)}{m_A+m_B}=\dfrac{m_A(-v)+nm_A(v)}{m_A+nm_A}=\left(\dfrac{n-1}{n+1}\right)v$ (upwards) Before the collision, their velocities in the CM frame are $u_{A,CM}=u_{A,Earth}+V_{Earth,CM}=-v-\left(\dfrac{n-1}{n+1}\right)v=-\dfrac{2n}{n+1}v$ $u_{B,CM}=u_{B,Earth}+V_{Earth,CM}=+v-\left(\dfrac{n-1}{n+1}\right)v=\dfrac{2}{n+1}v$ After the collision, their velocities in the CM frame would just change their signs $v_{A,CM}=\dfrac{2n}{n+1}v$ $v_{B,CM}=-\dfrac{2}{n+1}v$
	c)	Using PCOE, $\dfrac{1}{2}m_Av_A^2+0=mgH+0\rightarrow v_A=\sqrt{2gH}$ Their velocities in the Earth frame are $v_{A,Earth}=v_{A,CM}+V_{CM,Earth}=\dfrac{2n}{n+1}v+\left(\dfrac{n-1}{n+1}\right)v=\left(\dfrac{3n-1}{n+1}\right)v$ $v_A=\sqrt{2gH}=\left(\dfrac{3n-1}{n+1}\right)v=\left(\dfrac{3n-1}{n+1}\right)\sqrt{2gh}$ $\dfrac{H}{h}=\left(\dfrac{3n-1}{n+1}\right)^2$
	d)	$\dfrac{H}{h}=\left(\dfrac{3n-1}{n+1}\right)^2=\left(\dfrac{3n+3-4}{n+1}\right)^2=\left(\dfrac{3n+3}{n+1}-\dfrac{4}{n+1}\right)^2=\left(3-\dfrac{4}{n+1}\right)^2$ when $n\rightarrow\infty$, $\dfrac{4}{n+1}\rightarrow 0$ $\dfrac{H}{h}\rightarrow 9$

HCI H3 Physics 2024
A2 - Rotational Motion

Mr. Pang BiaoJin

1 General Information

1.1 Content: A2

- Kinematics of angular motion
- Dynamics of motion
- Rigid body rotation about an axis of fixed orientation

1.2 Learning Outcomes

Candidates should be able to:

1. show an understanding of and use the terms angular displacement, angular velocity, and angular acceleration of a rigid body with respect to a fixed axis.
2. solve problems using the equations of motion for uniform angular acceleration that are analogous to the equations of motion for uniform linear acceleration.
3. show an understanding of and use the terms angular momentum and moment of inertia of a rotating rigid body.
4. calculate the moment of inertia about an axis for simple objects by using calculus and the parallel-axis theorem or otherwise (knowledge of the perpendicular-axis theorem is not required).
5. show an understanding of torque produced by a force relative to a reference point, and apply the principle that torque is related to the rate of change of angular momentum to solve problems, such as those involving point masses, rigid bodies, or bodies with variable moment of inertia e.g. an ice-skater.
6. derive from the equations of motion, and apply the formula $K_{rot} = \frac{1}{2}I\omega^2$ for the rotational kinetic energy of a rigid body.
7. recall and apply the result that the motion of a rigid body can be regarded as translational motion of its centre of mass with rotational motion about an axis through the centre of mass to solve related problems, including situations where the frictional force between surfaces heuristically takes a limiting value governed by a coefficient of friction and the normal contact force (no distinction is made between the coefficient of static and kinetic friction).

2 Kinematics of Rotational Motion

In H2 Physics, we have learnt about some angular quantities for a point object in circular motion (e.g. angular velocity). In H3 Physics, we will extend our discussion to analyzing the rotation of an extended object.

We will constrain our analysis to the motion of a rigid body (a perfectly definite body in which the relative locations of all particles of which the object is composed remain constant).

2.1 Angular Quantities

Consider a point on a rigid body.

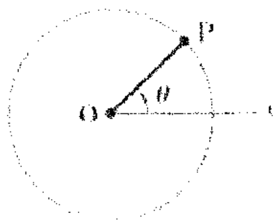


Figure 1: Angular Position of an arbitrary point P on a rotating body.

Body is rigid.

We define the *angular position* θ of this point.
This defines the angular position of the body. **(Qn: Why can we do this?)**

The point travels from point A to position B as shown in Figure 2 in a time Δt.

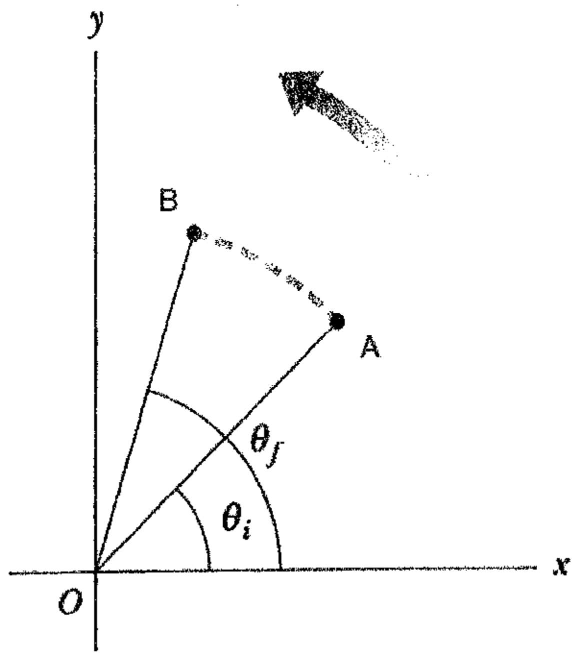


Figure 2: Motion of a particle on a rotating rigid object.

All points in rigid body rotate with same ω. $\vec{r}$ is fixed.

The *angular displacement* $\Delta\theta$ of the rigid body is then given by:

$$\Delta\theta = \theta_f - \theta_i \tag{1}$$

$\Rightarrow \Delta\theta$ is the same for all points in some time Δt

The *average angular velocity* of the body is given by

$$\omega_{ave} = \frac{\theta_f - \theta_i}{\Delta t} = \frac{\Delta\theta}{\Delta t} \tag{2}$$

The *instantaneous angular velocity* of the body is given by

$$\omega = \frac{d\theta}{dt} \tag{3}$$

If the instantaneous angular speed of an object changes from ω_i to ω_f over a time interval Δt, the object has an angular acceleration.

We can define the *average angular acceleration* of the body as

$$\alpha_{ave} = \frac{\omega_f - \omega_i}{\Delta t} = \frac{\Delta\omega}{\Delta t} \tag{4}$$

Finally, we can define the *instantaneous angular acceleration* as

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} \tag{5}$$

$\vec{V} = \vec{V}_{tan} + \cancel{\vec{V}_{normal}}$ 0 ∵ rigid body.

The velocity of the point is given by its tangential velocity (**Why so?**), which we recall from H2 Physics as:

$$v = r\omega \qquad (\vec{v} = \vec{\omega} \times \vec{r}) \tag{6}$$

Further, we can write expressions for the tangential and radial components of the acceleration as follows:

$$a_t = r\alpha \tag{7}$$

$$a_r = r\omega^2 \tag{8}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(\vec{\omega} \times \vec{r})$$

$$= \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \frac{d\vec{r}}{dt}$$

$$= \underbrace{\vec{\alpha} \times \vec{r}}_{\text{tangential}} + \underbrace{\vec{\omega} \times (\vec{\omega} \times \vec{r})}_{\text{normal}}$$

FYI: Directions of these vectors?

$$\vec{a}_n = \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

$$= -r\omega^2 \hat{r}.$$

$$\vec{a}_t = \alpha r \hat{\theta}.$$

ω follows right handed convention.

rotation.

$\uparrow \omega$.

$\alpha = \frac{d\omega}{dt}$.

If $\omega \uparrow$, $\vec{\alpha} \parallel \vec{\omega}$

If $\omega \downarrow$, $\vec{\alpha} \parallel -\vec{\omega}$.

Lecture Example 1 (Uni Phy by YnF, Q9.7)

The angle θ through which a disk turns is given by $\theta = a + bt - ct^3$, where

- a, b and c are constants,
- t is in seconds,
- θ is in radians.

When $t = 0s$, $\theta = \pi/4$ and the angular velocity is 2.00 rad s^{-1}.
When $t = 1.5s$, the angular acceleration is 1.25 rad s^{-1}.

(a) Find the values of a, b and c

(b) What is the angular acceleration when $\theta = \pi/4$?

a) $\frac{\pi}{4} = a + b(0) - c(0)^3 \Rightarrow a = \frac{\pi}{4}$.

$\omega = \frac{d\theta}{dt} = b - 3ct^2 \Rightarrow 2.00 = b - 3c(0)^2 \Rightarrow b = 2.00$.

$\alpha = \frac{d\theta^2}{dt^2} = -6ct \Rightarrow 1.25 = -6c(1.5) \Rightarrow c = -0.139$.

b). $\alpha = -6(-0.139)t$.

When $\theta = \frac{\pi}{4}$, $t = 0$.

$\alpha = -6(-0.139)(0)$

$= 0 \text{ rad } s^{-2}$.

$a = \frac{dv}{dt}$.

$dv = a\,dt$

2.2 Rotational EOM with Constant Angular Acceleration

Noting the similarity between the expressions for linear motion ($a = \frac{dv}{dt}$) and rotational motion ($\alpha = \frac{d\omega}{dt}$), we can write down analogous expressions for the Equations of Motion for rotational motion with *constant angular acceleration.*

$\int dv = \int a\,dt$.

$v = u + at$.

$d\omega = \alpha\,dt$.

$\int d\omega = \int \alpha\,dt$

$\omega = \omega_0 + \alpha t$.

etc.

$$\omega = \omega_0 + \alpha t \tag{9}$$

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2 \tag{10}$$

$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0) \tag{11}$$

$$\theta = \theta_0 + \frac{1}{2}(\omega + \omega_0)t \tag{12}$$

Lecture Example 2 (Uni Phy by YnF, Q9.19)

At $t = 0s$, a grinding wheel has an angular velocity of 24.0 $rads^{-1}$. It experiences a constant angular acceleration of 30.0 $rads^{-2}$ until a circuit breaker trips at $t = 2.00s$. From this moment onwards, it turns through 432 rad as it coasts to a stop at a constant angular acceleration.

(a) Through what total angle did the wheel turn between $t = 0s$ and the time it stopped?

(b) At what time did it stop?

(c) What was its acceleration as it slowed down?

$\omega = 24.0\ rads^{-1}$ at $t = 0$.

Constant $\alpha = 30.0\ rad\,s^{-2}$

a) $\theta = \theta_{(0\text{ to }2.00s)} + \theta_{(2.00s\text{ till stop})}$.

$= (\omega_i t + \frac{1}{2}\alpha t^2) + 432$.

$= (24.0)(2.00) + \frac{1}{2}(30.0)(2.00)^2 + 432$.

$= 540\text{ rad}$.

432.

b. $\omega_{2.00s} = \omega_{t=0} + \alpha t$

$= 24.0 + (30.0)(2.00)$

$= 84.0\ rads^{-1}$.

~~θ~~ ~~θ~~ $432 = 0 + \frac{1}{2}(0 + 84.0)(t - 2.00)$

$t = 12.3s$.

Tutorial: D1, **D2*** (starred questions are compulsory)

c. Using $\omega = \omega_0 + \alpha t$,

$0 = 84.0 + \alpha(12.3 - 2)$

$\alpha = -8.17\ rads^{-2}$.

3 Kinetic Energy of a Rotating Body

A rotating rigid body has kinetic energy.
Let us analyze a rigid object rotating about some axis with some angular velocity ω.

Consider a particle of mass m_i on the object, at some distance r_i from the rotation axis, shown in Figure 3. This particle is moving with some velocity v_i in the COM frame.

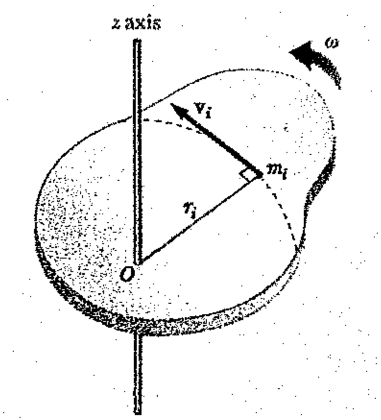


Figure 3: A rigid object rotating about the z axis with some angular speed ω.

Applying Equation 6, we can deduce that the kinetic energy of the point K_i is given by:

$$K_i = \frac{1}{2}m_i v_i^2 = \frac{1}{2}m_i r_i^2 \omega_i^2 \tag{13}$$

Recalling further that all particles share the same ω, and summing over all particles, we get an expression for the total kinetic energy K_{rot} of the rotating rigid object:

$$\begin{aligned} K_{rot} &= \sum_i (\frac{1}{2}m_i r_i^2 \omega_i^2) \\ &= \sum_i (\frac{1}{2}m_i r_i^2 \omega^2) \\ &= \frac{1}{2}(\sum_i m_i r_i^2)\omega^2 \\ &= \frac{1}{2}I\omega^2 \end{aligned} \tag{14}$$

Here, we define the quantity I as the *moment of inertia* of the body.

$$I = \sum_i m_i r_i^2 \tag{15}$$

Lecture Example 3 (Serway, Example 10.3)

Consider an oxygen molecule (O_2) rotating in the xy plane about the z axis. The rotation axis passes through the center of the molecule, perpendicular to its length.

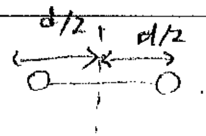
The mass of each oxygen atom is 2.66×10^{-26} kg, and at room temperature the average separation between the two atoms is $d = 1.21 \times 10^{-10}$ m.

You can treat the atoms as point particles.

(a) Calculate the moment of inertia of the molecule about the z axis.

(b) The angular speed of the molecule about the z axis is 4.60×10^{12} $rads^{-1}$.
Calculate its rotational kinetic energy.

a) $I = \sum_i m_i r_i^2 = m\left(\frac{d}{2}\right)^2 + m\left(\frac{d}{2}\right)^2$.



$= \frac{1}{2} m d^2$.

$= \frac{1}{2}(2.66 \times 10^{-26})(1.21 \times 10^{-10})^2$

$= 1.95 \times 10^{-46}$ kg m^2.

b) $K_{rot} = \frac{1}{2} I \omega^2$.

$= \frac{1}{2}(1.95 \times 10^{-46})(4.60 \times 10^{12})^2$.

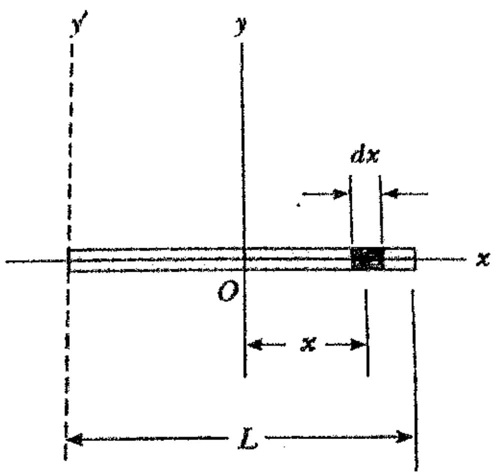
$= 2.06 \times 10^{-21}$ J.

3.2 Moment of Inertia

We now expand on the expression for moment of inertia defined in Equation 15. If we consider the rigid object to consist of infinitely many particle, each individual $m_i \to 0$. In this limit, the expression can be written as an integral over the object:

$$I = \lim_{m_i \to 0} \sum_i m_i r_i^2 = \int r^2 dm \tag{16}$$

Lecture Example 4 (Serway, Example 10.6)



$\lambda = \frac{M}{L}.$

$dm = \lambda dx.$
$= \frac{M}{L} dx.$

Figure 4: Uniform rigid rod of length L

Calculate the moment of inertia of a uniform rigid rod of length L and mass M about an axis perpendicular to the rod (the y axis) and passing through its center of mass (Figure 4).

$I = \int r^2 dm.$

$I = \int_{-L/2}^{L/2} x^2 \left(\frac{M}{L}\right) dx.$

$= \frac{M}{L} \int_{-L/2}^{L/2} \cancel{dx}\, x^2 dx.$

$= \frac{M}{L} \left[\frac{1}{3} x^3\right]_{-L/2}^{L/2}.$

$= \frac{1}{12} M L^2.$

Quick check: Do you expect the moment of inertia to be ~~be~~ greater, lower or the same if the axis were y' instead?

3.3 Applying the Parallel Axis Theorem to find Moment of Inertia

Suppose the moment of inertia of a body of mass M about an axis passing through the center of mass of the body is given by I_{CM}.

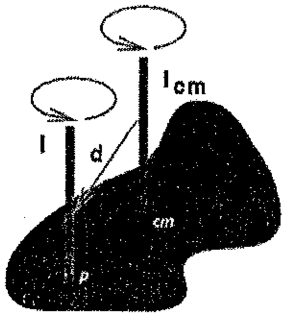


Figure 5: Parallel Axis Theorem

The parallel-axis theorem then states that the moment of inertia I about any axis **parallel to and a distance D from this axis** is given by:

$$I = I_{CM} + MD^2 \tag{17}$$

Lecture Example 5 (Serway, Example 10.6(modified))

Refer back to Figure 4. Calculate the moment of inertia of a uniform rigid rod of length L and mass M about an axis passing through one end of the rod (the y' axis)

(a) through direct integration, **without** using your answer in Example 4

(b) by using your answer in Example 4 and the parallel-axis theorem

a.
$$\begin{aligned} I &= \int r^2\,dm \\ &= \int_0^L x^2\left(\frac{M}{L}\right)dx \\ &= \frac{M}{L}\int_0^L x^2\,dx \\ &= \frac{M}{L}\left[\frac{1}{3}x^3\right]_0^L \\ &= \frac{1}{3}ML^2 \end{aligned}$$

b. From Eg 4,
$$I_{CM} = \frac{1}{12}ML^2$$
$$\begin{aligned} I &= I_{CM} + M\left(\frac{L}{2}\right)^2 \\ &= \frac{1}{12}ML^2 + \frac{1}{4}ML^2 \\ &= \frac{1}{3}ML^2 \end{aligned}$$

Tutorial: **D3***, **D4***, D5, D6 (starred questions are compulsory)

4 Torque

4.1 Fundamentals and Recap of H2 Content

We recall from H2 Physics that a force exerted on a rigid extended object pivoted about an axis tends to cause the object to rotate about that axis.

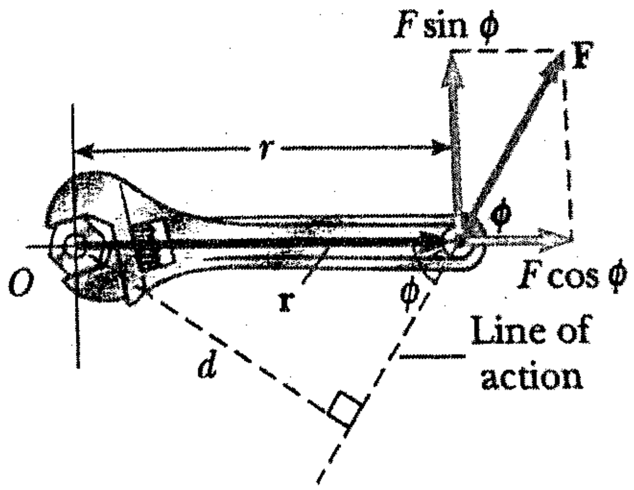


Figure 6: A force F applied on a wrench. $d = rsin\theta$ is known as the moment arm.

This tendency of the object to rotate is quantified by *torque* τ, of which magnitude is given by the product of the magnitude of the force exerted on it F and its moment arm d (See Figure 6 above for an example).

$$\tau = Fd \tag{18}$$

Note that torque is a **vector**.

Being slightly pedantic and considering the rotation direction, we notice that an object rotating about a fixed axis has two possible directions of rotation.

Typically, we define torque to be **positive** if it tends to produce **counter-clockwise** rotation and **negative** if it tends to produce **clockwise** rotation.

We thus have our final equation:

$$\tau = \pm Fd \tag{19}$$

This will suffice for the H3 Physics course.

4.2 Relating Torque to Angular Acceleration

We know from Newton's Second Law that a net force acting on an object will cause it to accelerate. Similarly, what happens when there is a net torque on an object?

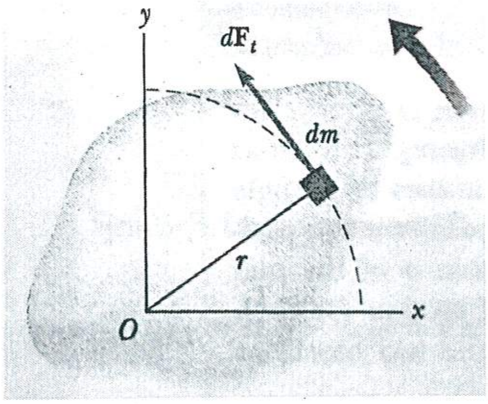


Figure 7: A rigid object rotating about an axis through O.

Consider a rigid object rotating about a fixed axis, such as in Figure 7 above.
Breaking the object down into infinitesimal mass elements dm, each has a tangential acceleration $\mathbf{a}_t$ caused by an external tangential force $d\mathbf{F}_t$ acting on it.

By Newton's Second's Law,

$$dF_t = (dm)a_t \tag{20}$$

The torque $d\tau$ associated with the force $d\mathbf{F}_t$ can then be expressed by:

$$d\tau = r(dF_t) = r(dm)a_t \tag{21}$$

Invoking our expression in Equation 7 for a_t, we thus have an expression for $d\tau$:

$$d\tau = \alpha r^2 dm \tag{22}$$

rigid body.

Note that every dm here has the same angular acceleration α. This allows us to obtain an expression for the net torque through integration.

$$\sum \tau = \int \alpha r^2 dm = \alpha \int r^2 dm = I\alpha \tag{23}$$

Finally, we simplify the expression by invoking the expression for the Moment of Inertia defined in Equation 16.

$$\sum \tau = I\alpha \tag{24}$$

Lecture Example 6

A light cable is wrapped several times around a uniform solid cylinder that can rotate freely about its axis. The cylinder has a diameter of 0.120 m and mass of 50 kg.

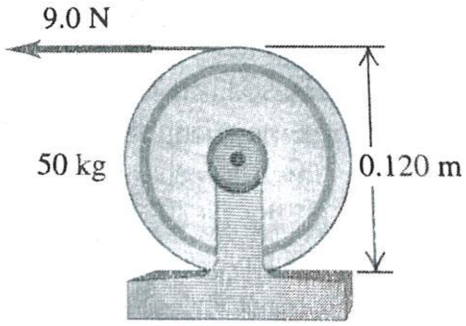


Figure 8: Cylinder and cable

The cable is now pulled with a force of 9.0 N, as shown in the Figure 8.

Assuming that the cable unwinds without stretching or slipping, what is its acceleration?

Torque due to the applied force $= Fr$.

Moment of inertia of cylinder $= \frac{1}{2}Mr^2$.

Angular acceleration is thus

$$\alpha = \frac{\tau}{I} = \frac{Fr}{\frac{1}{2}Mr^2} = \frac{2F}{Mr}.$$

$$= \frac{2(9.0)}{(50)(0.060)}$$

$$= 6.0 \text{ rad s}^{-1}.$$

Acceleration of cable is thus $a = R\alpha$

$$= (0.060)(6.0)$$

$$= 0.36 \text{ m s}^{-2}.$$

Lecture Example 7 (Serway, Example 10.10)

A uniform rod of length L and mass M is attached at one end to a frictionless pivot and is free to rotate about the pivot in the vertical plane, as shown in Figure 9.

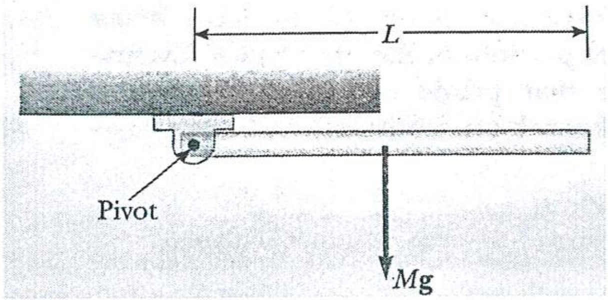


Figure 9: Rod rotating freely about a pivot

The rod is then released from rest in the horizontal position.

In terms of the gravitational acceleration g, what is the initial linear acceleration of the right end of the rod?

$\tau = Mg\left(\frac{L}{2}\right)$.

$\alpha = \frac{\tau}{I} = \frac{Mg\left(\frac{L}{2}\right)}{\frac{1}{3}ML^2} = \frac{3g}{2L}$.

using then $a_t = r\alpha$,

$a_t = L\alpha = L\left(\frac{3g}{2L}\right) = \frac{3g}{2}$.

Tutorial: D7, D8, **D9***, D10

5 Angular Momentum

5.1 Fundamental Definition

We first establish the rotational counterpart to linear momentum: *angular momentum L*.
Similar to how linear momentum is given by the product of mass and velocity, the form of angular momentum should not surprise you:

$$L = I\omega \tag{25}$$

5.2 Conservation of Angular Momentum

Revisiting what we know about conservation of linear momentum in H2 Physics, we recall:

$$\sum F = \frac{dp}{dt} \tag{26}$$

This then suggests that if $\sum F = 0$, $p_i = pf$.

Similarly, we note that:

$$\sum \tau = \frac{dL}{dt} \tag{27}$$

We then observe that if $\sum \tau = 0$, $L_i = L_f$.
(To be pedantic however, note that $\sum \tau$ and L have to be measured about the same origin.)

Lecture Example 8 (Uni Phy by YnF, Q10.37)

Find the magnitude of the angular momentum of the second hand on a clock about an axis through the centre of the face of the clock. The clock hand has a length of 15.0 cm and a mass of 6.00 g. Assume the second hand is a slender rod rotating with a constant angular velocity about one end.

$L = I\omega.$

$= (\frac{1}{3}ML^2)\omega.$

$= \frac{1}{3}(0.006)(0.15)^2\left(\frac{2\pi}{60}\right)$

$= 4.71 \times 10^{-6}$ kg m^{2}s^{-1}.

Lecture Example 9 (Uni Phy by YnF, Q10.42)

A diver comes off a board with arms straight up and legs straight down, giving her a moment of inertia of 18 kgm^2 about her axis of rotation. She tucks into a small ball, decreasing her moment of inertia to 3.6 kgm^2. While tucked, she makes two complete revolutions in 1.0 s from board to water. If she has not tucked at all, how many revolutions would she have made in 1.5 s from board to water?

Angular momentum of the isolated diver system is conserved.

Hence

$$I_i \omega_i = I_f \omega_f.$$

$$(18)\, \omega_i = 3.6 \times \left(\frac{2 \times 2\pi}{1.0}\right).$$

$$\omega_i = \frac{3.6}{18}(4\pi).$$

Revolutions in 1.5s.

$$= \omega_i \times \frac{1.5}{2\pi}$$

$$= 0.6 \text{ revolutions.}$$

Tutorial: **D11***, D12

6 Rolling Motion of a Rigid Object

To conclude our discussion, we extend our discussion to the rolling of a rigid object along a flat surface.

Typically, such analysis is not easy.
We can (and will, in this case), simplify our discussion by

1. focussing on the **center of mass** of the object
2. considering the case of *pure rolling*, where the object rolls without slipping.

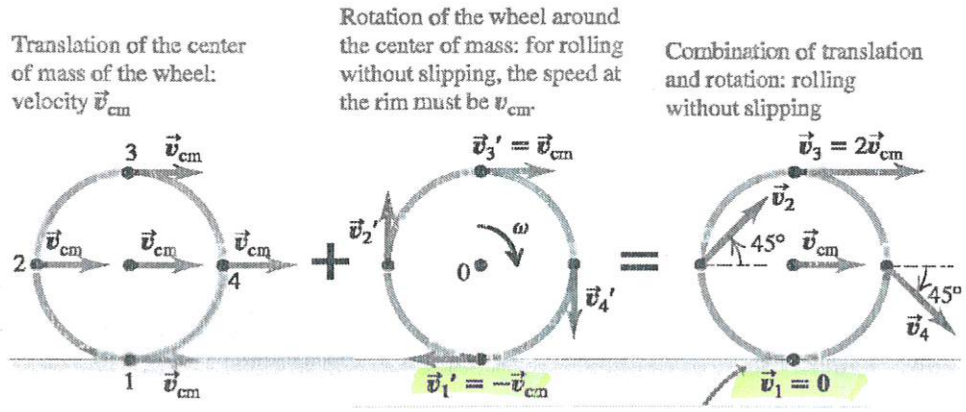


Figure 10: Pure Rolling Motion

We can break down pure rolling motion into a combination of translational motion and rotational motion (Figure 10 above).

For pure translational motion, the cylinder does not rotate, so each point moves to the right with speed v_{CM}.

For pure rotational motion, the rotation axis through the center of mass is stationary, with each point having the same angular speed ω. (COM frame of reference).

The <u>combination</u> of both these two motions represents pure rolling motion.

In the next page, we will now derive expressions for the velocity and acceleration, as well as the kinetic energy of such a rotating system.

Consider a uniform cylinder of radius R rolling without slipping (Figure 11).

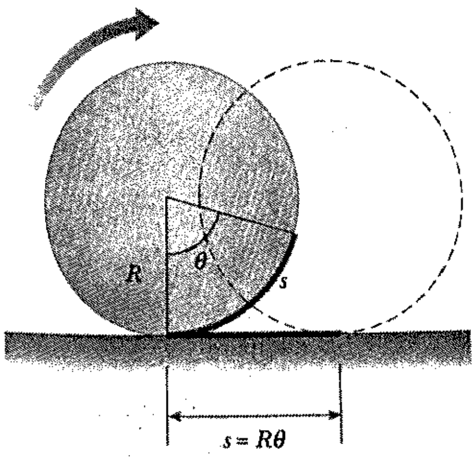


Figure 11: Pure Rolling Motion of a cylinder

Its center of mass moves a distance of $s = R\theta$ as it rotates through an angle θ. The linear speed of the center of mass v_{CM} is thus given by:

$$v_{CM} = \frac{ds}{dt} = R\frac{d\theta}{dt} = R\omega \tag{28}$$

Consequently, the linear acceleration of the center of mass a_{CM} is thus given by:

$$a_{CM} = \frac{dv_{CM}}{dt} = R\frac{d\omega}{dt} = R\alpha \tag{29}$$

Finally, by invoking the approach shown in Figure 10, we can write down the expression for the total kinetic energy K of the object:

$$K = \frac{1}{2}I_{CM}\omega^2 + \frac{1}{2}Mv_{CM}^2 \tag{30}$$

Tutorial: **D13***, D14

$\therefore$ $V = r\omega$.

a. $2.00 \times 10^{-3} = r\left(\frac{7.5(2\pi)}{60}\right)$.

$\therefore$

Diameter $= 2r = 0.0509m$.

b. $\alpha = \frac{a_t}{r}$

$= \frac{0.400}{0.02546}$

$= 15.7 rads^{-2}$.

12

$a_t = r\alpha$. $a_r = r\omega^2$.

a.

$a_t = r\alpha$.

$3.00 = 60.0\alpha$

$\alpha = 0.050 \text{ rad s}^{-2}$.

b. Using $\omega = \omega_0 + \alpha t$,

$\omega = 0 + 0.050(6.00)$.

$= 0.30 \text{ rad s}^{-1}$.

c. Using $\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$,

$\theta = (0) + \frac{1}{2}(0.050)(6.00)^2$

$= 0.90 \text{ rad}$.

d. $a_r = r\omega^2$

$= (60.0)(0.30)^2$.

$= 5.4 \text{ m s}^{-2}$.

~~$a_t = r\alpha$.~~

e. $a_{resultant} = \sqrt{a_r^2 + a_t^2}$

$= \sqrt{(5.4)^2 + [\cancel{(60.0)(0.05)}]^2}$ 3.00.

$= 6.17$

$= 6.2 \text{ m s}^{-2}$.

/3

$F_{fric} = \mu N$.
$= \mu m_A g$.

Extension from H2: Rotational KE of pulley.

Loss in GPE of B = Gain in t. KE of A & B
+ Gain in r. KE of pulley.
+ Energy loss through friction.

$$m_B g d = \frac{1}{2}(m_A + m_B)v^2 + \frac{1}{2}I\omega^2 + \mu m_A g d.$$

$$= \frac{1}{2}(m_A + m_B)v^2 + \frac{1}{2}\left(\frac{1}{2}mr^2\right)\left(\frac{v}{r}\right)^2 + \mu m_A g d.$$

$$= \frac{1}{2}(m_A + m_B)v^2 + \frac{1}{4}mv^2 + \mu m_A g d.$$

$$v^2\left(\frac{1}{2}m_A + \frac{1}{2}m_B + \frac{1}{4}m\right) = m_B g d - \mu m_A g d.$$

$$v^2 = \frac{4(m_B g d - \mu m_A g d)}{2m_A + 2m_B + m}.$$

4. Tips for dealing with dm :

$dm = \lambda \, dl.$

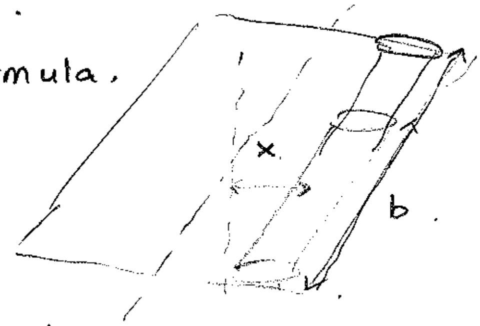
$dm = \sigma \, dA$

$dm = \rho \, dV.$

Idea: Split plate into many rods.

But not all rods have axis passing through.

So we need to modify the formula.



$$I = \int dI$$

$$= \int \left(\frac{1}{12}b^2 + x^2\right)\sigma b \, dx.$$

$$= \int \left(\frac{1}{12}b^3\sigma + \sigma b x^2\right) dx$$

$$= \sigma\left[\frac{1}{12}b^3 x + \frac{b}{3}x^3\right]_{-\frac{a}{2}}^{\frac{a}{2}}.$$

$$= 2\sigma\left(\frac{1}{12}b^3\left(\frac{a}{2}\right) + \frac{b}{3}\left(\frac{a}{2}\right)^3\right).$$

$$= 2\sigma\left(\frac{ab^3}{24} + \frac{a^3 b}{24}\right).$$

$$= 2\left(\frac{M}{ab}\right)\left(\frac{ab^3}{24} + \frac{a^3 b}{24}\right)$$

$$= \frac{M}{12}(a^2 + b^2).$$

// axis theorem $\left(I = \frac{1}{12}Mb^2 + Mx^2 \right)$

$dI = \frac{1}{12}b^2 dm + x^2 dm.$

$dm = \sigma \, dA.$

$= \sigma b \, dx.$

$\sigma = \frac{M}{ab}.$

2.

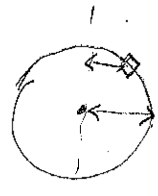
Idea: Take the solid sphere to be made out of many.

thin shells. of mass dm.

Then integrate over it.

Mass is distributed over the entire volume. $dm = \rho dV$.

$\rho = \dfrac{M}{\frac{4}{3}\pi R^3}$. (constant).



$dm = \rho dV$

$= \rho (4\pi r^2 dr)$. radius.

$I = \int r^2 dm$ distance from axis.

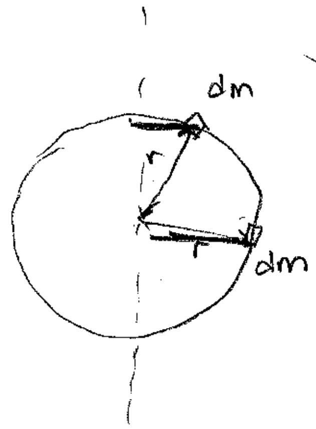
$= \int r^2 \rho (4\pi r^2) dr.$

$= 4\pi\rho \int r^4 dr.$

$= \dfrac{4\pi\rho}{5} \left[r^5 \right]_0^R.$

$= \dfrac{4\pi R^5}{5} \left(\dfrac{M}{\frac{4}{3}\pi R^3} \right).$

$= \dfrac{3}{5} MR^2$?



/5 (for real).

From earlier: $I_{shell} = \frac{2}{3} MR^2$.

$$\Downarrow$$

$$dI = \frac{2}{3}(dM) r^2 .$$

※ $\rho = \dfrac{M}{\frac{4}{3}\pi R^3}$.

$dm = \rho\, dV = \rho(4\pi r^2\, dr)$.

$$\Rightarrow dI = \frac{2}{3}(4\pi r^2 \rho)\, r^2\, dr .$$

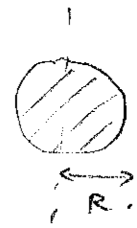
$$\begin{aligned}
I &= \int dI \\
&= \int_0^R \frac{2}{3}(4\pi r^2 \rho) r^2\, dr \\
&= \frac{8}{3}\pi\rho \int_0^R r^4\, dr \\
&= \frac{8}{3}\pi \left(\frac{M}{\frac{4}{3}\pi R^3}\right) \left[\frac{1}{5} r^5\right]_0^R . \\
&= 2 \frac{M}{R^3}\left(\frac{1}{5} R^5\right). \\
&= \frac{2}{5} M R^2 .
\end{aligned}$$

Challenge :

Moment of inertia of hollow cylinder through axis $= \frac{1}{2} M (r_1^2 + r_2^2)$

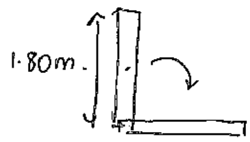
Hence derive the formula for moment of inertia for a solid sphere

$I = \frac{2}{5} MR^2$.



i.

$E_{initial} = E_{final}$ (Conservation of Energy).



$mgh = \frac{1}{2} I \omega^2$.

$I = \frac{1}{3} ML^2$.

$\omega = \frac{V_{end}}{l}$.

$mg\left(\frac{l}{2}\right) = \frac{1}{2}\left(\frac{1}{3} M l^2\right)\left(\frac{V_{end}}{l}\right)^2$

$V_{end} = \sqrt{3gl} = \sqrt{3(9.80)(1.80)}$

$= 7.27 ms^{-1}$.

D7. $(I = \frac{1}{2}MR^2.) \rightarrow$ using I for cylinder.

a. Loss in GPE = Gain in KE.

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2.$$

$$= \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{1}{2}M_1R_1^2 + \frac{1}{2}M_2R_2^2\right)\omega^2.$$

Using $V = R_1\omega, \Rightarrow \omega = \frac{V}{R_1}$

$$mgh = \frac{1}{2}mv^2 + \frac{1}{4}(M_1R_1^2 + M_2R_2^2)\left(\frac{V}{R_1}\right)^2.$$

$$mgh = v^2\left(\frac{1}{2}m + \frac{1}{4}(M_1R_1^2 + M_2R_2^2)\left(\frac{1}{R_1^2}\right)\right).$$

$$(1.5)(9.81)(2.00) = v^2\left[\frac{1}{2}(1.5) + \frac{1}{4}[(0.80)(0.025)^2 + (0.60)(0.050)^2]\left[\frac{1}{0.025^2}\right]\right]$$

$V = 3.40\ ms^{-1}.$

b. using instead $v = R_2\omega,$

$V = 4.95\ ms^{-1}.$

Explanation: V of the block is related to ω by $v = R\omega$.

Energy of system = KE_{trans} of block + KE_{rot} of R_1 + KE_{rot} of R_2.

$= \frac{1}{2}mv^2 + \frac{1}{2}I_1\omega^2 + \frac{1}{2}I_2\omega^2.$

If wrap around 1,

$KE_{trans,1} = \frac{1}{2}mv^2 = \frac{1}{2}m(R_1\omega)^2$

Wrap around 2,

$KE_{trans,2} = \frac{1}{2}mv^2 = \frac{1}{2}m(R_2\omega)^2.$

Since $KE_{rot} = \frac{1}{2}I_1\omega^2 + \frac{1}{2}I_2\omega^2$ is constant for both cases

and Initial energy = mgh is constant.

hence for R_2, the KE of block is higher.

D8.

a. $\tau = rF_{friction}$

$= \frac{1}{2}(0.090)(1.5)$

$= 0.0675$

≈ 0.068 Nm. #

b. Using $\omega = \omega_0 + \alpha t$,

$t = \dfrac{\omega - \omega_0}{\alpha}$.

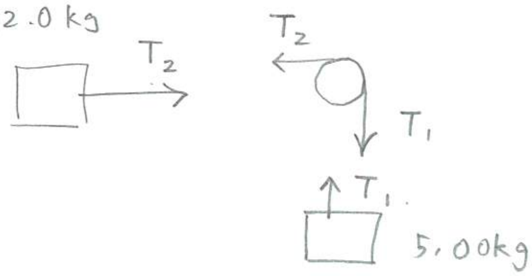
$= \dfrac{\omega - \omega_0}{\left(\frac{\tau}{I}\right)}$

$= \dfrac{(-1.6)(2\pi)}{\left(\frac{-0.0675}{0.11}\right)}$.

$= 16.4$ s.

$= 16$ s. #

D9.



Consider system of 5.00kg:

$\Sigma F = ma$.

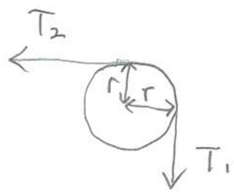
$(5.00)(9.81) - T_1 = (5.00)(a)$.

Common sense: which T would be larger? why?

Consider system of 12.0kg.

$T_2 = (12.0)(a)$.

Consider system of pulley.



Net torque.

$\tau = I\alpha$.

I_{disk}.

$(T_1 - T_2)(0.25) = \left[\frac{1}{2}(2.00)(0.250)^2\right]\alpha$.

$= \frac{1}{2}(2.00)(0.250)^2\left(\frac{a}{0.250}\right)$.

Recall:

$a = a_t = r\alpha$.

Solve for a:

$a = 2.73\,ms^{-2}$.

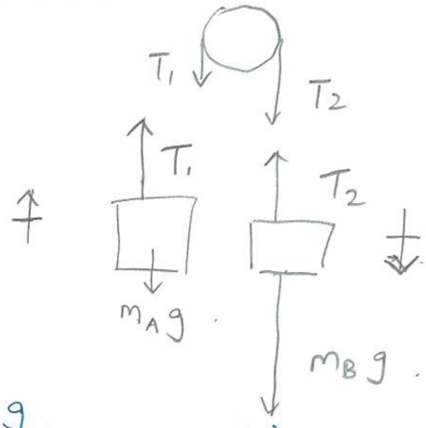
Solve for T_1 and T_2.

$T_1 = 35.4N$.

$T_2 = 32.7N$.

10.

Both objects have the same acceleration.



a.

For A,

$$T_1 - m_A g = m_A a. \Rightarrow T_1 = m_A a + m_A g. \quad \text{——— (1)}$$

For B,

$$m_B g - T_2 = m_B a. \Rightarrow T_2 = m_B a + m_B g. \quad \text{——— (2)}$$

For the pulley,

$$\Sigma \tau = (T_2 - T_1)(R) = I\alpha = I\left(\frac{a}{R}\right). \quad \text{——— (3)}$$

subst. (1) and (2) into (3) & simplify:

$$a = \frac{m_B - m_A}{\left(m_A + m_B + \frac{I}{R^2}\right)} g.$$

$$I = \frac{1}{2} m R^2.$$

$$= \frac{m_B - m_A}{m_A + m_B + \frac{\frac{1}{2} m R^2}{R^2}} g.$$

$$= \frac{75 - 65}{75 + 65 + \frac{1}{2}(6.0)}(9.81).$$

$$= 0.686014$$

$$= 0.69\,ms^{-2}.$$

If we naively did it the H2 way and neglect I_{pulley},

$$a = 0.700714\,ms^{-2}.$$

$$\%\ error = \frac{0.700714 - 0.686014}{0.686014}$$

$$\approx 2\%.$$

11.

a. Yes, because there is no external torque acting on the block and string system.

b. $L_i = L_f$.

$I_i w_i = I_f w_f$.

$M r_i^2 w_i = m r_f^2 w_f$.

$(0.300)^2 (1.75) = (0.150)^2 w_f$.

$w_f = 7.0 \text{ rads}^{-1}$.

c. By COE,

WD in pulling the cord = Change in KE of system

$= \frac{1}{2} I_f w_f^2 - \frac{1}{2} I_i w_i^2$.

$= \frac{1}{2}(0.025)[(0.150)^2(7.0)^2 - (0.300)^2(1.75)^2]$

$= 0.010 \text{J}$.

/12.

Angular momentum is conserved during the collision.

$L_i = L_f$.

$r_i P_i = I_f \omega_f$.

$r_i m_i v_i = \frac{1}{3} MR^2 \omega$.

$(0.500)(2.25)(0.500) = \frac{1}{3}(1.50)(0.75)^2 \omega$

$\omega = 2.0 \text{ rad s}^{-1}$.

By Conservation of Energy,

$\frac{1}{2} I \omega_i^2 + mgh = \frac{1}{2} I \omega_f^2$.

$\frac{1}{2}\left[\left(\frac{1}{3}\right)(1.50)(0.75)^2 \omega_i^2\right] + (1.50)(9.81)\left(\frac{0.75}{2}\right) = \frac{1}{2}\left[\left(\frac{1}{3}\right)(1.50)(0.75)^2 \omega_f^2\right]$

$\omega_f = 6.6 \text{ rad s}^{-1}$.

~~D13~~

a. $V_{cm} = \frac{s}{t} = \frac{18.29}{t} \approx \frac{18.29}{5.0} \approx 3.66 ms^{-1}$.

b. Then rotation rate ω is then given:

$V_{cm} = R\omega$.

$$\omega = \frac{V_{cm}}{R}$$

$$\approx \frac{3.66}{\left(\frac{21.6}{2} \times 10^{-2}\right)}$$

$$= 33.9 \text{ rad } s^{-1}$$

$$= 5.4 \text{ rev } s^{-1}.$$

c.

$$KE_{tot} = KE_{trans} + KE_{rot}$$

$$= \frac{1}{2} m V_{cm}^2 + \frac{1}{2} I \omega^2.$$

$$= \frac{1}{2} m V_{cm}^2 + \frac{1}{2}\left(\frac{2}{5} m R^2\right)\left(\frac{V_{cm}}{R}\right)^2.$$

$$= \frac{1}{2} m V_{cm}^2 + \frac{1}{10} m V_{cm}^2.$$

So $\frac{KE_{rot}}{KE_{tot}} = \frac{\frac{1}{10} m V_{cm}^2}{\frac{1}{10} m V_{cm}^2 + \frac{1}{2} m V_{cm}^2} = \frac{2}{7}$.

D14

Unrolling – assume with no slipping.

Angular speed of rotating hoop – in the COM frame.

a. Approach: Conservation of Energy.

Loss in GPE = Gain in KE

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

$$= \frac{1}{2}m(r\omega)^2 + \frac{1}{2}(mr^2)\omega^2$$

$$= mr^2\omega^2$$

$$gh = r^2\omega^2$$

$$\omega = \sqrt{\frac{gh}{r^2}}$$

$$= \sqrt{\frac{(9.81)(0.750)}{(0.080)^2}}$$

$$= 33.9 \text{ rad s}^{-1}$$

$$= 34 \text{ rad s}^{-1}$$

b. Speed of centre $= V_{cm}$

$$= r\omega$$

$$= (0.080)(33.9)$$

$$= 2.712$$

$$= 2.7 \text{ ms}^{-1}$$

Example 1 (IOAA 2007, theoretical round, questions 6 and 7)

a) A sun-orbiting periodic comet is farthest from the Sun at 31.5 AU and closest to the Sun at 0.5 AU. What is the orbital period of this comet?

b) For the comet in part (a) above, what is the area per unit time (in square AU per year) swept by the line joining the comet and the Sun?

Note: 1 AU = 1 astronomical unit, which is given by the average Sun-Earth distance, where 1 AU = 149.6 x 10^6 km. For part (b), you may use that the eccentricity of the orbit and the semi-minor axis b are related to the semi-major axis a by $b^2 = a^2 (1 – e^2)$, and that the area of an ellipse A_{ellipse} is given by $A_{\text{ellipse}} = \pi ab$. Refer to Figure 1 for the eccentricity of an orbit.

Solution

a) The major axis is 31.5 + 0.5 = 32.0 AU.
Hence, the semi-major axis is a = 32.0 / 2 = 16.0 AU.
We use Kepler's third law and compare the comet's orbit to that of the Earth, so that we can keep everything in "solar-system units" (a_{Earth} = 1.0 AU, the orbital period of the Earth is P_{Earth} = 1.0 year).

$$P^2 \propto a^3 \Rightarrow P = a^{3/2} = 16.0^{3/2} = 64.0 \text{ years.}$$

b) The area of an ellipse is given by $A_{\text{ellipse}} = \pi\, a\, b$.
The eccentricity is e = 15.5 / 16.0 = 0.969.
Since $b^2 = a^2 (1 – e^2)$, the semi-minor axis b = 3.97 AU.
Hence, the area of the ellipse is $\pi\, a\, b = \pi$ (16.0) (3.967) = 199 AU^2.
Thus, the area swept by the line joining the comet and the Sun is 199 / 64.0 = 3.12 AU^2 yr^{-1}.

Example 2

The distance between Earth's surface and an object of mass m is changed by an amount Δx.

Show that when $x \approx R_E$ and $\Delta x << R_E$, the gravitational potential energy of the system reduces to the expression $\Delta U = mg\Delta x$.

Solution

$$\begin{aligned}
\Delta U &= -\frac{GM_E m}{(R_E + \Delta x)} - \left(-\frac{GM_E m}{R_E}\right) \\
&= -\frac{GM_E m R_E}{R_E(R_E+\Delta x)} + \frac{GM_E m R_E}{R_E^2} \\
&= -\frac{GM_E m R_E}{R_E^2\left(1+\frac{\Delta x}{R_E}\right)} + gmR_E \\
&\approx -gmR_E\left(1-\frac{\Delta x}{R_E}\right) + gmR_E \\
&\approx -gmR_E + gm\Delta x + gmR_E \\
&\approx mg\Delta x
\end{aligned}$$

Example 3 (University Physics by Young and Freedman, 12.40)

A thin uniform rod has length *L* and mass *M*. A small uniform sphere of mass *m* is placed a distance *x* from one end of the rod, along the axi of the rod, as shown in the figure below.

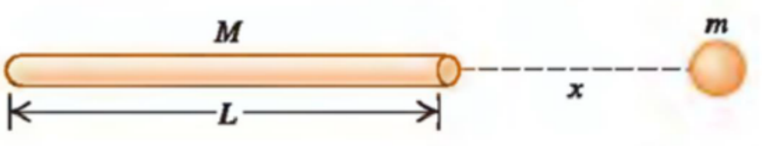


Figure 4: Thin uniform rod and small spherical mass

a) Calculate the gravitational potential energy of the rod-sphere system.

b) Show that your answer reduces to the expected result when $x >> L$.

Solution

a) Taking δM to be the mass of a small section of the rod, and $\lambda = M/L$ the mass per unit length of the rod, so that $\delta M = \lambda \delta r$ is the mass contained within a section of thickness δr. we have for the gravitational potential energy due to m and a section δM at a distance r from m,

$$\delta U = -\frac{G(\delta M)m}{r},$$

$$\delta U = -\frac{G\lambda(\delta r)m}{r} = -\frac{GMm}{L}\frac{1}{r}\delta r.$$

For the gravitational potential due to the whole rod, we have to integrate r from x to $x + L$,

$$U = -\int_x^{x+L} \frac{GMm}{L}\frac{1}{r}dr = -\frac{GMm}{L}\left[\ln r\right]_x^{x+L} = -\frac{GMm}{L}\left[\ln(x+L) - \ln(x)\right] = -\frac{GMm}{L}\left(\ln\frac{x+L}{x}\right).$$

b) Using that $L << x$, so that $L/x << 1$, we can do a Maclaurin expansion, so that

$$\ln\frac{x+L}{x} = \ln\left(1+\frac{L}{x}\right) \approx \frac{L}{x},$$

from which,

$$U = -\frac{GMm}{L}\left(\ln\frac{x+L}{x}\right) \approx -\frac{GMm}{L}\frac{L}{x} = -\frac{GMm}{x},$$

which is indeed what we expect if the length of the rod is much smaller than the distance between the rod and the mass m: from a very large distance, any object will look (or behave) like a point object.

Example 4 (IOAA 2010, theoretical round, question 2)

If the escape velocity from a solar-mass object's surface exceeds the speed of light, what would be its radius?

You may use the following values:

c = 299 792 458 m s^{-1},
G = 6.6726 x 10^{-11} N m^2 kg^{-2},
M_{Sun} = 1.9891 x 10^{30} kg.

Solution

At the edge of a black hole, the escape velocity is equal to the speed of light, c.
If an object is moving at the escape velocity, its initial speed is just enough to reach infinity with no speed (or kinetic energy) left. At infinity, the gravitational potential energy is also zero. Hence, the total energy is zero.

kinetic energy = - gravitational potential energy

$$\frac{1}{2}mc^2 = G\frac{Mm}{r}$$

$$\frac{1}{2}c^2 = G\frac{M}{r}$$

$$r = \frac{2GM}{c^2} = \frac{2(6.6726\times10^{-11})(1.9891\times10^{30})}{(299\ 792\ 458)^2} = 2\ 953.5\text{ m.}$$

Hence, if the escape velocity exceeds the speed of light, the radius must be smaller than 2,953.5 m. Note that the radius of a (non-rotating, un-charged) black hole as calculated above is called the Schwarzschild radius, r_S. (Since all numbers are given to at least five s.f., the final answer can be given to five s.f. as well.)

Example 5 (IOAA 2011, theoretical round, question 3)

On 9 March 2011, the Voyager probe was 116.406 AU from the Sun and moving at 17.062 km s^{-1}. Determine the type of orbit the probe is on: (a) elliptical, (b) parabolic, or (c) hyperbolic.

You may use the following values:

1 AU = 1.4960 x 10^{11} m,
G = 6.6726 x 10^{-11} N m^2 kg^{-2},
M_{Sun} = 1.9891 x 10^{30} kg.

Solution

We calculate the kinetic energy KE and gravitational potential energy GPE of the probe in SI units with respect to the Sun.

KE = ½ m v^2 = ½ m (17.062 x 10^3)^2 = 145,556,922 m (where m is the unknown mass of the probe).
GPE = -GMm/r = -7,621,575 m.

Since the total energy TE = KE + GPE > 0, the probe is on a hyperbolic orbit (c).

Example 6

a) For the Hohmann transfer orbit in Figure 10, write down the expressions for v_1 and v_3, the velocity of the object in circular orbits 1 and 3. Denote the large mass as M.

b) Write down expressions for v_a and v_p, the velocity of the object at apoapsis and periapsis of the elliptical orbits in terms of v_1, v_3, Δv and $\Delta v'$.

c) Using your understanding of elliptical orbits, write down an expression relating v_a and v_p. You may use any suitable lengths defined in the problem.

d) Applying energy considerations to the elliptical path, and using suitable results obtained in previous parts, show that Δv and $\Delta v'$ are given by Equations 30 and 31.

Solution

a)
$$F_g = F_c$$
$$G\frac{Mm}{r^2} = m\frac{v^2}{r}$$
$$v_1 = \sqrt{\frac{GM}{R}} \text{ and } v_3 = \sqrt{\frac{GM}{R'}}$$

b)
$$\mathrm{v_p} = \mathrm{v_1} + \Delta\mathrm{v}$$
$$v_a = v_3 - \Delta v'$$

c) By the principle of conservation of angular momentum,
$$v_p R = v_a R'$$

d) By the principle of conservation of energy,
$$\frac{1}{2}mv_p^2 - \frac{GMm}{R} = \frac{1}{2}mv_a^2 - \frac{GMm}{R'}$$
$$\frac{1}{2}mv_p^2 - \frac{1}{2}mv_p^2\left(\frac{R}{R'}\right)^2 = \frac{GMm}{R} - \frac{GMm}{R'}$$
$$\frac{1}{2}v_p^2\left(1 - \frac{R^2}{R'^2}\right) = GM\left(\frac{1}{R} - \frac{1}{R'}\right)$$
$$\frac{1}{2}v_p^2\left(\frac{R'^2 - R^2}{R'^2}\right) = GM\left(\frac{R' - R}{RR'}\right)$$
$$\frac{1}{2}v_p^2\left(\frac{R' - R}{R'}\right)\left(\frac{R' + R}{R'}\right) = GM\left(\frac{R' - R}{RR'}\right)$$
$$v_p^2 = 2GM\left(\frac{R' - R}{RR'}\right)\left(\frac{R}{R' - R}\right)\left(\frac{R'}{R' + R}\right)$$
$$v_p = \sqrt{\frac{2GM}{R}}\sqrt{\frac{R'}{R' + R}}$$
$$\Delta v = v_p - v_1 = \sqrt{\frac{GM}{R}}\sqrt{\frac{2R'}{R' + R}} - \sqrt{\frac{GM}{R}}$$
$$= \sqrt{\frac{GM}{R}}\left(\sqrt{\frac{2R'}{R' + R}} - 1\right)$$

Equivalently, we can derive
$$\Delta v' = v_3 - v_a = \sqrt{\frac{GM}{R'}}\left(1 - \sqrt{\frac{2R}{R + R'}}\right)$$

Tutorial A3: Planetary and Satellite Motion

Questions:

1 *University Physics by Young & Freedman / 12.39:*
A uniform solid 1000 kg sphere has a radius of 5.00 m.

(a) Find the gravitational force this sphere exerts on a point mass of 2.00 kg, placed at the following distances from the centre of the sphere:

(i) 5.01 m, and (ii) 2.5 m.

(b) Sketch a qualitative graph of the magnitude of the gravitational force F this sphere exerts on a point mass as a function of the distance r from the centre of the sphere.

2 *University Physics by Young & Freedman / 12.41*:

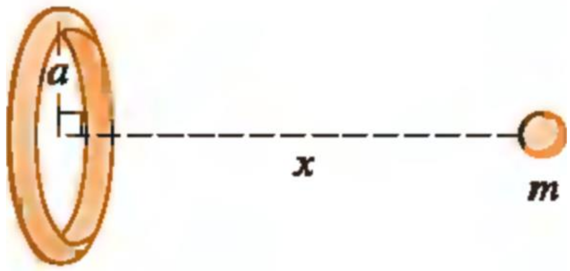
Consider the ring-shaped body of mass M as shown in the figure below. A particle with a mass m is placed a distance x from the centre of the ring, along the line through the centre of the ring and perpendicular to its plane.

(a) Calculate the gravitational potential energy U of this system.

(b) Show that your answer to part (a) reduces to the expected result when x is much larger than the radius of a of the ring.

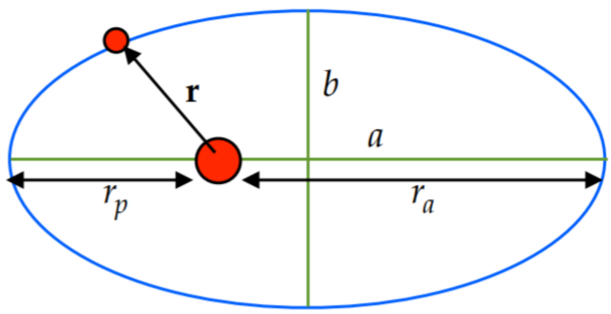
(c) Use $F = -\frac{dU}{dx}$ to find the magnitude and direction of the force on the particle.

(d) Show that your answer to part (c) reduces to the expected result when x is much larger than a.



3 *University Physics by Young & Freedman / 12.76*:
As Mars orbits the sun in its elliptical orbit, its distance of closest approach to the centre of the sun (at perihelion) is 2.067 x 10^{11} m, and its maximum distance from the centre of the sun (at aphelion) is 2.492 x 10^{11} m. Ignoring the influence of other planets, if the orbital speed of Mars at aphelion is 2.198 x 10^{4} m s^{-1}, what is its orbital speed at perihelion?

4 Shown on the right is a typical elliptical orbit.



The turning points r_p and r_a are the distances of closest approach and furthest recession. These points are usually denoted by the Greek prefixes *peri* (“near”) and *apo* (“away”). Thus, a planet’s point of closest approach to the Sun is called its perihelion, and its point of furthest recession is its aphelion (*helios* is sun in Greek).

(a) Show that, when the satellite is at either of the turning points r_p and r_a ,

$$r^2 + \frac{GMm}{E}r - \frac{L^2}{2mE} = 0,$$

where r is the distance between the satellite and the Earth,
E is the total energy of the satellite and Earth system,
and L is the angular momentum of the satellite.

(b) Since the equation above has two solutions, $r = r_p$ and $r = r_a$. it can be written as $(r - r_a)(r - r_p) = 0$. Using this, show that

$$r_a + r_p = -\frac{GMm}{E},$$

$$r_a r_p = -\frac{L^2}{2mE}.$$

(c) For an ellipse, we have that $r_a + r_p = 2a$ and $r_a r_p = b^2$, where a is the semi-major axis of the ellipse and b is the semi-minor axis. Using this, show that the energy and angular momentum of the orbit in terms of a and b are given as

$$E = -\frac{GMm}{2a},$$

$$L^2 = -2mEb^2.$$

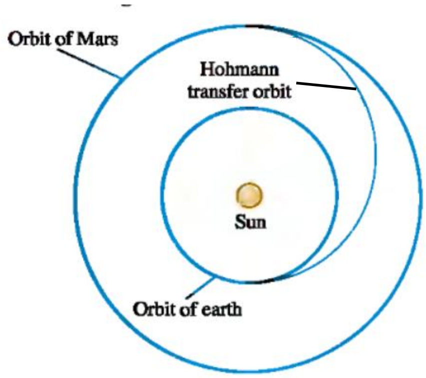
(d) Using that angular momentum is conserved and that the area of an ellipse is πab, show that

$$\frac{T^2}{4\pi^2} = \frac{a^3}{GM},$$

where T is the period of the orbit. (Hint: check the derivation of Kepler’s Second Law!)

5 *University Physics by Young & Freedman / 12.87 – modified*:

The most efficient way to send a spacecraft from the Earth to Mars is using a Hohmann transfer orbit. If the orbits of the Earth and Mars about the Sun are circular, the Hohmann transfer orbit is an elliptical orbit whose perihelion and aphelion are tangent to the orbits of the two planets. The spacecraft's rockets are fired briefly at Earth to put the spacecraft into the transfer orbit, after which the spacecraft coasts until it reaches Mars. The rockets are then fired again to put the spacecraft into the same orbit about the Sun as that of Mars.



Assume negligible gravitational forces acting on the spacecraft due to the Earth and Mars.

(a) For a flight from Earth to Mars, in what direction must the rockets be fired at the Earth?

(b) What is the time spent in the Hohmann transfer orbit during a one-way trip from the Earth to Mars?

In order to reach Mars from the Earth, the launch must be timed so that Mars will be at the right spot when the spacecraft reaches Mars' orbit around the sun.

(c) At launch, what must be the angle between the Sun-Mars line and the Sun-Earth line?

(d) What is the energy per unit mass supplied to or withdrawn from the spacecraft near Earth to place it into the transfer orbit? (Consider the force exerted by the rockets to be an external force.)

(e) What is the energy per unit mass supplied to or withdrawn from the spacecraft near Mars to align the spacecraft's orbit to that of Mars?

You may use any or all of the data provided below, as well as the equation for the energy of an orbit.

Mass of Sun = 1.99×10^{30} kg
Mass of Earth = 5.97×10^{24} kg
Mass of Mars = 6.42×10^{23} kg
Radius of Earth's orbit = 1.50×10^{11} m
Radius of Mars' orbit = 2.28×10^{11} m
Period of Earth's orbit = 365 days
Period of Mars's orbit = 687 days

6 *International Olympiad on Astronomy and Astrophysics 2008, theoretical round, question 4*:
Consider a potentially hazardous object (PHO) moving in a closed orbit under the influence of Earth's gravitational force. Let u be the inverse of the distance of the object from the Earth and p the magnitude of its linear momentum. As the object travels through points A and B, values of u and p are noted, as shown in the table below. Find the mass and the total energy of the object and sketch the shape of the u curve as a function of p from A to B.

	p (x 10^9 kg m s^{-1})	u (x 10^{-8} m^{-1})
A	0.052	5.15
B	1.94	194.17

7 *University Physics by Young & Freedman / Example 12.10 & 12.85*:

Suppose we drill a hole through the Earth (radius R_E, mass m_E) along a diameter and drop a mail pouch (mass m) down the hole.

(a) Derive an expression for the gravitational force on the pouch as a function of its distance r from the centre. Assume that the density of the Earth is uniform (not a very realistic model).

(b) Derive an expression for the gravitational potential energy $U(r)$ of the object-Earth system as a function of the object's distance from the centre of the Earth. Take the potential energy to be zero when the object is at the centre of the Earth.

(c) If an object is released in the shaft at the Earth's surface, what speed will it have when it reaches the centre of the Earth?

8 *University Physics by Young & Freedman / Example 12.89*:

Mass M is distributed uniformly over a disk of radius a. Find the gravitational force (magnitude and direction) between this disk-shaped mass and a particle with mass m located a distance x above the centre of the disk. Does your result reduce to the correct expression as x becomes very large?

9 *International Olympiad on Astronomy and Astrophysics 2013, theoretical round, question 17 – adapted*:

A spacecraft is orbiting the near-Earth asteroid Seneca (staying continuously very close to the asteroid), transmitting pulsed data to the Earth. Due to the relative motion of the two bodies (the asteroid and the Earth), the time it takes for a pulse to arrive at the ground station varies approximately between 2 and 39 minutes. Assuming that the Earth moves around the sun on a circular orbit and that the orbit of Seneca does not intersect the orbit of the Earth, calculate

(a) the semi-major axis a_S of Seneca's orbit around the Sun,

(b) the period of Seneca's orbit T_S.

Express your answers in terms of the Earth's orbit, that is, in astronomical units (AU) and years.

10 *International Olympiad on Astronomy and Astrophysics 2011, theoretical round, question 1 – fast food*:

Most single-appearance comets enter the inner Solar System directly from the Oort Cloud. Estimate how long it will take a comet to make this journey. Assume that in the Oort cloud, 35 000 AU from the Sun, the comet was at aphelion.

11 *International Olympiad on Astronomy and Astrophysics 2011, theoretical round, question 4 – adapted*:

Assume that Mars' moon Phobos moves around Mars in a perfectly circular orbit in the equatorial plane of the planet, find the length of time Phobos is above the horizon for a point on the Martian equator.
Mass of Mars M_{Mars} = 6.421 x 10^{23} kg; radius of Mars R_{Mars} = 3393 km; rotational period of Mars P_{Mars} = 24.623 hours; orbital radius of Phobos a_{Ph} = 9380 km.

Suggested solutions:

1(a)(i)	5.31×10^{-9} N	**(a)(ii)**	2.67×10^{-9} N				
2(a)	$-\dfrac{GMm}{\sqrt{x^2+a^2}}$	**(c)**	$-\dfrac{GMmx}{\left(x^2+a^2\right)^{3/2}}$				
3	2.65×10^{4} m s^{-1}						
5(b)	259 days	**(c)**	44°	**(d)**	9.13×10^{7} J kg^{-1}	**(e)**	6.00×10^{7} J kg^{-1}
6	5.00×10^{4} kg		-1.0×10^{12} J				
7(a)	$\dfrac{Gm_E m}{R_E{}^3} r$	**(b)**	$\dfrac{Gm_E m}{2R_E{}^3} r^2$	**(c)**	7.90×10^{3} m s^{-1}		
8	$\dfrac{2GMm}{a^2}\left[1-\dfrac{x}{\sqrt{a^2+x^2}}\right]$						
9(a)	2.46 AU	**(b)**	3.87 years				
10	1.2×10^{6} years						
11	4.25 hours						

Tutorial A3: Planetary and Satellite Motion

Suggested solutions:

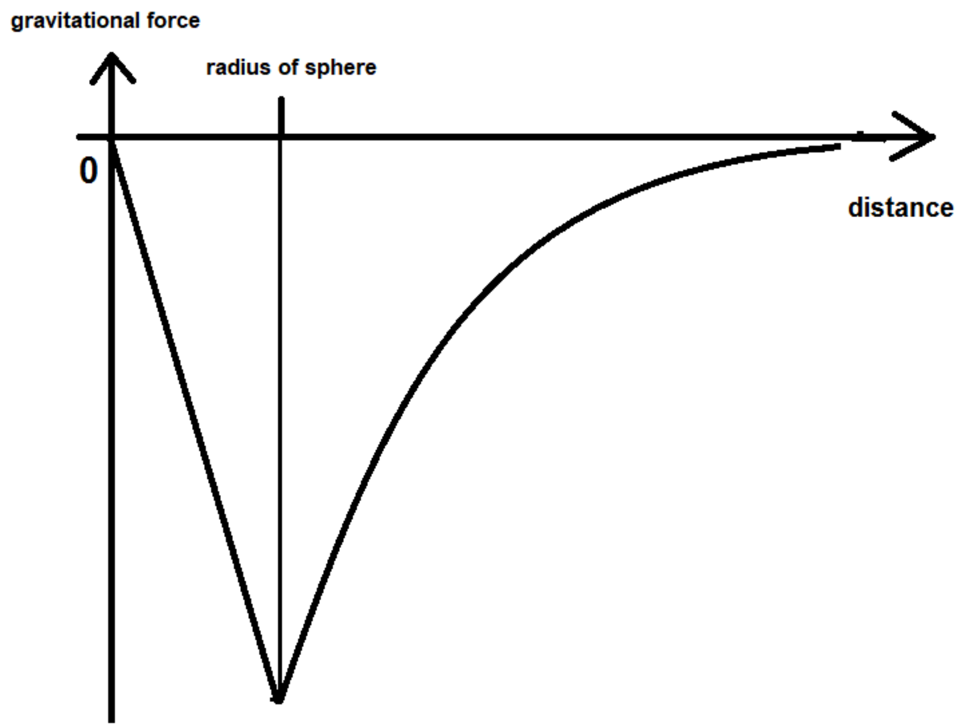
1 (a) (i) For a spherically symmetric distribution of mass, the force outside the distribution is equal to that of a point of the same mass located at the centre. Hence,

$$F_g = \frac{GMm}{r^2}$$
$$= \left(6.67\times10^{-11}\right)\frac{(1000)(2.00)}{(5.01)^2}$$
$$= 5.31\times10^{-9}\ \text{N}$$

(ii) For a spherically symmetric distribution of mass, the force at a point only depends on the mass "below" the point. Using the 1/8 of the sphere's mass is located within 2.5 m of the centre of the mass,

$$F_g = \frac{GMm}{r^2}$$
$$= \left(6.67\times10^{-11}\right)\frac{(125)(2.00)}{(2.5)^2}$$
$$= 2.67\times10^{-9}\ \text{N}$$

(b)



Inside the sphere, the force increases linearly with distance. Outside the sphere, the force falls of as $1/r^2$. The graph being negative indicates that the force is directed towards the centre of the sphere.

2 (a) Taking δM to be the mass of a small part of the ring, and $\lambda = M/(2\pi a)$ the mass per unit length of the ring, so that $\delta M = \lambda\delta r$ is the mass contained within a section of thickness δr. We then have for the gravitational potential energy due to m and a section δM at a distance r from m,

$$\delta U = -\frac{G(\delta M)m}{r} = -\frac{G(\lambda\delta r)m}{\sqrt{x^2+a^2}}.$$

We need to integrate over the whole ring. However, we find that the integration is independent of x and a, so we find for the total potential energy due to the ring

$$U = -\int_0^{2\pi a} \frac{G\lambda m}{\sqrt{x^2+a^2}}dr = -\frac{G\lambda m}{\sqrt{x^2+a^2}}[r]_0^{2\pi a} = -\frac{G\lambda m}{\sqrt{x^2+a^2}}2\pi a = -\frac{GMm}{\sqrt{x^2+a^2}},$$

where we used that $M = 2\pi a\lambda$ is the total mass of the ring.

(b) Using that $a << x$, we can do a Maclaurin expansion in a, so that

$$\left(x^2+a^2\right)^{-1/2} = f(a) \approx \left(x^2+0^2\right)^{-1/2} = x^{-1},$$

from which,

$$U = -\frac{GMm}{\sqrt{x^2+a^2}} = -GMm\left(x^2+a^2\right)^{-1/2} \approx -GMm\left(x^2+0^2\right)^{-1/2} = -\frac{GMm}{x},$$

which is indeed what we expect if the radius of the ring is much smaller than the distance between the rod and the mass m: from a very large distance, any object will look (or behave) like a point object.

(c)

$$U = -\frac{GMm}{\sqrt{x^2+a^2}} = -GMm\left(x^2+a^2\right)^{-1/2}$$

$$F_g = -\frac{dU}{dx} = GMm\left[-\frac{1}{2}\left(x^2+a^2\right)^{-3/2}2x\right] = -\frac{GMmx}{\left(x^2+a^2\right)^{3/2}}.$$

The minus sign indicates that the force is in the direction of decreasing x, which is decreasing distance between the ring and m. In other words, the minus sign tells us that the force between the ring and m is attractive.

(d) Using that $a << x$, we have

$$\left(x^2+a^2\right)^{-3/2} \approx \left(x^2+0^2\right)^{-3/2} = x^{-3},$$

from which,

$$U = -\frac{GMmx}{\left(x^2+a^2\right)^{3/2}} = -GMmx\left(x^2+a^2\right)^{-3/2} \approx -(GMmx)x^{-3} = -\frac{GMm}{x^2},$$

which is indeed what we expect if the radius of the ring is much smaller than the distance between the rod and the mass m: from a very large distance, any object will look (or behave) like a point object.

3 The angular momentum of Mars relative to the Sun is given by

$$L = mv_t r\ ,$$

where L is the angular momentum, m is the mass of Mars, v_t is the velocity of Mars perpendicular to the line Sun-Mars and r is the distance between Mars and the Sun. Using subscripts p for perihelion and a for aphelion, by the principle of conservation of angular momentum,

$$L = mv_{t,p}r_p = mv_{t,a}r_a$$

$$v_p = \frac{Lv_a r_a}{Lr_p} = \frac{v_a r_a}{r_p} = 2.650\times10^4 \text{ m s}^{-1},$$

where we use that the velocity at the aphelion and the perihelion is perpendicular to the line Sun-Mars, so $v_t = v$ there.

4 (a) This question is adapted from https://webhome.phy.duke.edu/~lee/P53/sat.pdf.

$$E = -\frac{GMm}{r} + \frac{1}{2}mv^2$$

Taking v_r the radial component of the velocity and v_t the tangential component,

$$E = -\frac{GMm}{r} + \frac{1}{2}mv_r^{\,2} + \frac{1}{2}mv_t^{\,2}$$

At the turning points, $v_r = 0$. Using that $L = m\, v_t\, r$, we get

$$E = -\frac{GMm}{r} + \frac{L^2}{2mr^2}$$

Multiplying by r^2/E and rearranging,

$$r^2 + \frac{GMm}{E}r - \frac{L^2}{2mE} = 0\ .$$

(b)

$$\left(r - r_a\right)\left(r - r_p\right) = 0\ ,$$

$$r^2 - rr_p - rr_a + r_a r_p = 0\ ,$$

$$r^2 - r\left(r_p + r_a\right) + r_a r_p = 0\ .$$

Comparing terms [see part (a)], we have

$$r_a + r_p = -\frac{GMm}{E} \quad \text{and} \quad r_a r_p = -\frac{L^2}{2mE}\ .$$

(c) With the given equations, this part becomes trivial:

$$2a = -\frac{GMm}{E} \quad \Rightarrow \quad E = -\frac{GMm}{2a}\ ,$$

$$b^2 = -\frac{L^2}{2mE} \quad \Rightarrow \quad L^2 = -2mEb^2\ .$$

(d) Recall from the derivation of Kepler's Second Law that

$$\frac{L}{2m}=\frac{\delta A}{\delta t}\,.$$

Note that the area swept out in one period T is equal to $A = \pi ab$, so that

$$\frac{L}{2m}=\frac{\pi ab}{T} \quad \Rightarrow \quad L=\frac{2m\pi ab}{T}$$

Squaring both sides,

$$L^2=\frac{4m^2\pi^2a^2b^2}{T^2}$$

Also, from part (c) we have

$$L^2=-2mEb^2=2m\frac{GMm}{2a}b^2=\frac{GMm^2b^2}{a}$$

Comparing the two equations for L^2,

$$\frac{4m^2\pi^2a^2b^2}{T^2}=\frac{GMm^2b^2}{a}$$

$$\frac{4\pi^2}{T^2}=\frac{GM}{a^3}.$$

5 (a) The spacecraft has to speed up to get into a higher orbit (larger 2*a*) about the Sun. By Newton's Third Law, the rockets must be fired opposite to the direction of motion of the spacecraft.

(b) The major axis of the Hohmann orbit will be equal to the sum of the radii of Mars' and Earth's orbits, $2a = (1.50 \times 10^{11}) + (2.28 \times 10^{11}) = 3.78 \times 10^{11}$ m, so $a = 1.89 \times 10^{11}$ m = 1.26 AU, where 1 AU = 1 astronomical unit is the radius of Earth's orbit. By Kepler's Third Law,

$$T_{\text{Hohmann}} = \left(\frac{a_{\text{Hohmann}}}{a_{\text{Earth}}}\right)^{3/2} T_{\text{Earth}} = 516 \text{ days.}$$

Hence, the spacecraft will spent 516 / 2 = 258 days in the Hohmann orbit.

(c) Referring to the figure in the question, we can assume that the Earth was "at the bottom" at the time beginning of the transfer, while Mars is "at the top" at the end of the transfer. In 258 days, Mars will have moved 258 / 365 x 360° = 135° in its orbit. Hence, at the beginning of the transfer, it was 135° "from the top." As the Earth was "at the bottom" at the that time, the angle Earth-Sun-Mars was 180 – 135 = 45°.

(d) As the spacecraft sped up, energy was <u>supplied</u> to it. Given that $E = -(GMm)/(2a)$ is the total energy of a system consisting of the Sun of mass M and a satellite (or spacecraft, or planet) of mass m in orbit about the Sun, the energy per unit mass supplied is

$$\frac{\Delta E}{m} = \left(-\frac{GM}{2a_{\text{Hohmann}}}\right) - \left(-\frac{GM}{2a_{\text{Earth}}}\right) = GM\left(\frac{1}{2a_{\text{Earth}}} - \frac{1}{a_{\text{Earth}} + a_{\text{Mars}}}\right) = 9.13 \times 10^{7} \text{ J kg}^{-1}.$$

(e) The spacecraft has to speed up again to get into a higher orbit, since the (semi)major axis of Mars' orbit is longer than that of the Hohmann transfer orbit. Hence, energy is again <u>supplied</u>. The energy per unit mass supplied is this time

$$\frac{\Delta E}{m} = \left(-\frac{GM}{2a_{\text{Mars}}}\right) - \left(-\frac{GM}{2a_{\text{Hohmann}}}\right) = GM\left(\frac{1}{a_{\text{Earth}} + a_{\text{Mars}}} - \frac{1}{2a_{\text{Mars}}}\right) = 6.00 \times 10^{7} \text{ J kg}^{-1}.$$

6 The total energy TE = KE + GPE = ½ $m\,v^2 - G\,M\,m\,/\,r = p^2\,/\,2m - G\,M\,m\,u$.
For point A we have TE = $(0.052 \times 10^9)^2 / 2m - G\,M\,m\,(5.15 \times 10^{-8})$
$= (1.352 \times 10^{15}) / m - (2.051 \times 10^7)\,m$ (in SI units) (1)
For point B we have TE = $(1.94 \times 10^9)^2 / 2m - G\,M\,m\,(194.17 \times 10^{-8})$
$= (1.881 \times 10^{18}) / m - (7.73 \times 10^8)\,m$ (in SI units) (2)
Note that we used $G = 6.67 \times 10^{-11}$ N kg^{-2} m^2 and $M = M_{earth} = 5.972 \times 10^{24}$ kg. We now have two equations with two unknowns, TE and the mass of the PHO m, for which we can solve. Subtracting (1) from (2), we get

$(1.88 \times 10^{18}) / m - (7.525 \times 10^{16})\,m$
$1.88 \times 10^{18} = (7.525 \times 10^8)\,m^2$
$m = 50{,}000$ kg.

Plugging the value into either of the two equations (1) or (2), we get

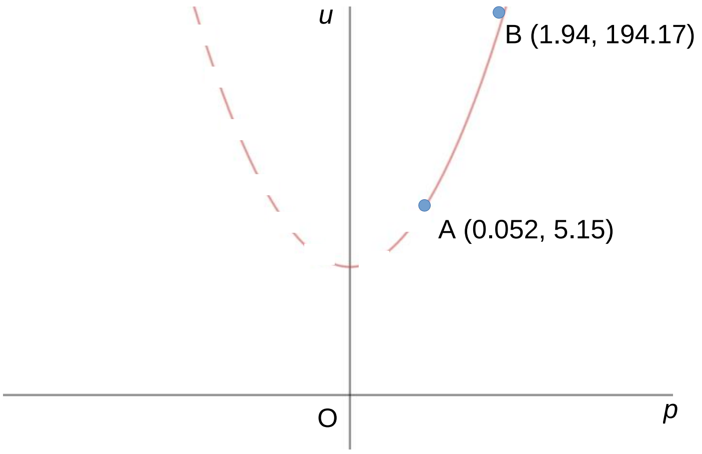
TE = $(1.352 \times 10^{15}) / 50{,}000 - (2.051 \times 10^7)(50{,}000) = -1.0 \times 10^{12}$ J.

Sanity check: the PHO is on a closed orbit, so the total energy must be negative.

For the second part of the question, we have

TE = $p^2 / 2m - G\,M\,m\,u$
$-1.0 \times 10^{12} = p^2 / 10^5 - (2.0 \times 10^{19})\,u$
$(2.0 \times 10^{19})\,u = p^2 / 10^5 + 1.0 \times 10^{12}$
$u = p^2\,(5.0 \times 10^{-15}) + (5.0 \times 10^{-8})$.

This is a parabolic function, shifted upwards from the horizontal axis, passing through points A and B. Note that, although the orbit of the PHO is elliptical (it is a closed orbit), the function u vs. p is parabolic. As the PHO orbits the Earth, it goes back and forth along part of the parabola. u is never zero – the PHO does not reach infinity – and p is never zero – the PHO never stops moving.



Sketch: definitely *not* to scale!

7 (a) Note that, for a spherically symmetric distribution of mass, only the sphere below the pouch matters.

$$F_g = G\frac{Mm}{r^2} = G\frac{\left(\frac{4}{3}\pi r^3 \rho_E\right)m}{r^2} = \frac{4}{3}\pi G r m \frac{m_E}{\frac{4}{3}\pi R_E{}^3} = \frac{G m_E m r}{R_E{}^3}$$

where we used M for the mass below the pouch, ρ_E the density of the Earth.

(b) Since the magnitude of the gravitational force is the gradient of the potential with respect to position, the gravitational potential is the integral of the gravitational force with respect to position. We have to integrate from the centre of the Earth to the position of the pouch, a distance *r* from the centre.

$$U = \int_0^r \frac{Gm_E mr'}{R_E^{\ 3}} dr' = \frac{Gm_E m}{R_E^{\ 3}} \left[\frac{1}{2} r'^2 \right]_0^r = \frac{Gm_E mr^2}{2R_E^{\ 3}}$$

As all the variables in the equation are positive, the gravitational potential *U* is positive. This is, because the gravitational potential is set to be zero at the centre of the Earth in this question.
Sanity check: if we drop a pouch into the hole, it will speed up towards the centre of the Earth, gaining kinetic energy and losing potential energy. Hence, the potential energy must be larger than zero everywhere, except at the centre of the Earth.

(c) At the Earth's surface,

$$U = \frac{Gm_E mR_E^{\ 2}}{2R_E^{\ 3}} = \frac{Gm_E m}{2R_E}$$

Loss in gravitational potential energy = gain in kinetic energy, from which

$$v_{final} = \sqrt{\frac{2Gm_E}{2R_E}} = 7900 \text{ m s}^{-1}$$

Note that we need the mass of the Earth m_E and the radius of the Earth R_E to solve this part of the question. Alternatively, we can use that g = 9.81 m s^{-2} at the surface of the Earth. However, we would still require the radius of the Earth. For homework exercises like these, you may look them up. In exams, any required values will be provided.

8 The most straightforward method, using Newton's law of gravity for *m* and a small mass *δM* and integrating does not work directly. This law gives only the magnitude of the force and not the direction. If this is not taken into account, the force is overestimated.

Let *δM* be the mass of a small part of the disk, and $\lambda = M/(\pi a^2)$ the mass per unit area of the disk, so that $\delta M = \lambda\ (r\ \mathrm{d}\theta)\ \mathrm{d}r$. We then have for the gravitational potential energy due to *m* and a section *δM* at a distance *s* from *m*,

$$\delta U = -\frac{G(\delta M)m}{s} = -\frac{G(\lambda\ r\ \mathrm{d}\theta\ \mathrm{d}r)m}{\sqrt{x^2+r^2}} ,$$

where *r* is the distance of *δM* from the centre of the disk. To get the gravitational potential energy due to a ring of radius *r*, we have to integrate *θ* from 0 to 2*π*, so that

$$U_{\text{ring}} = -2\pi\lambda\frac{G(r\ \mathrm{d}r)m}{\sqrt{x^2+r^2}} = -\frac{2\pi GM(r\ \mathrm{d}r)m}{\pi a^2\sqrt{x^2+r^2}} = -\frac{2GMm}{a^2}\frac{r}{\sqrt{x^2+r^2}}\mathrm{d}r ,$$

To get the gravitational potential energy due to the whole disk, we have to integrate *r* from 0 to the radius of the disk *a*,

$$U = -\frac{2GMm}{a^2}\int_0^a \frac{r}{\sqrt{x^2+r^2}}\mathrm{d}r .$$

Using the substitution $u = r^2 + u^2$, so that $\mathrm{d}u = 2r\ \mathrm{d}r$, $r\ \mathrm{d}r = ½\ \mathrm{d}u$,

$$U = -\frac{2GMm}{a^2}\int_0^a \frac{r}{\sqrt{x^2+r^2}}\mathrm{d}r = -\frac{2GMm}{a^2}\int_{x^2}^{a^2+x^2}\frac{1}{2\sqrt{u}}\mathrm{d}u = -\frac{2GMm}{2a^2}\left[2u^{1/2}\right]_{x^2}^{a^2+x^2} .$$
$$= -\frac{2GMm}{a^2}\left(\sqrt{a^2+x^2}-\sqrt{x^2}\right).$$

$$F_{\text{disk}} = -\frac{dU}{dx} = \frac{2GMm}{a^2}\left[\frac{1}{2}\left(x^2+a^2\right)^{-1/2}2x-1\right] = \frac{2GMm}{a^2}\left[\frac{x}{\sqrt{x^2+a^2}}-1\right].$$

Note that $F_{\text{disk}} < 1$, which means the force points in the direction of decreasing *x*, which is to say, decreasing distance between *m* and the disk: the force is attractive. Also, the *magnitude* of the force is

$$\left|F_{\text{disk}}\right| = \frac{2GMm}{a^2}\left[1-\frac{x}{\sqrt{x^2+a^2}}\right].$$

Using that $a << x$, we have (using a Maclaurin expansion)

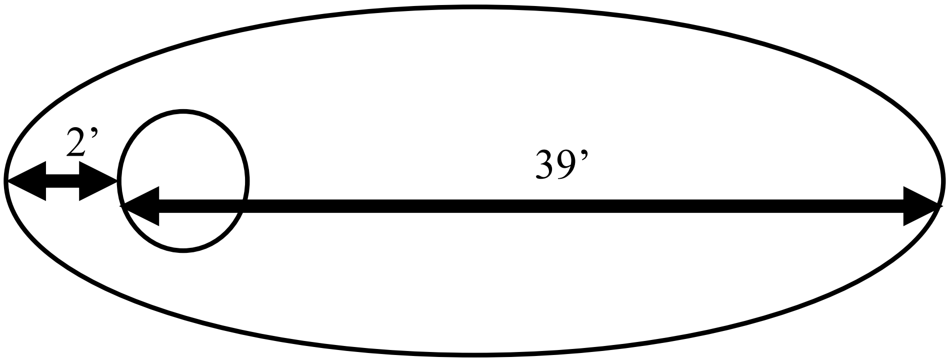
$$x\left(x^2+a^2\right)^{-1/2} \approx 1+0+-\frac{1}{2}\frac{a^2}{x^2} = 1-\frac{1}{2}\frac{a^2}{x^2} ,$$

from which,

$$\left|F_{\text{disk}}\right| = \frac{2GMm}{a^2}\left[1-\frac{x}{\sqrt{x^2+a^2}}\right] \approx \frac{2GMm}{a^2}\left[1-1+\frac{1}{2}\frac{a^2}{x^2}\right] = \frac{GMm}{x^2} .$$

This is indeed what we expect: at a large distance, the disk will look (or behave) like a point mass.

9 (a)



The signal from the satellite to the Earth travels at the speed of light. The time for it to travel will be 2’ when the Earth and Seneca are on the same side of the Sun and Seneca is at perihelion; the time will be 39’ when the Earth and Seneca are on opposite sides of the Sun and Seneca is at aphelion. Hence,

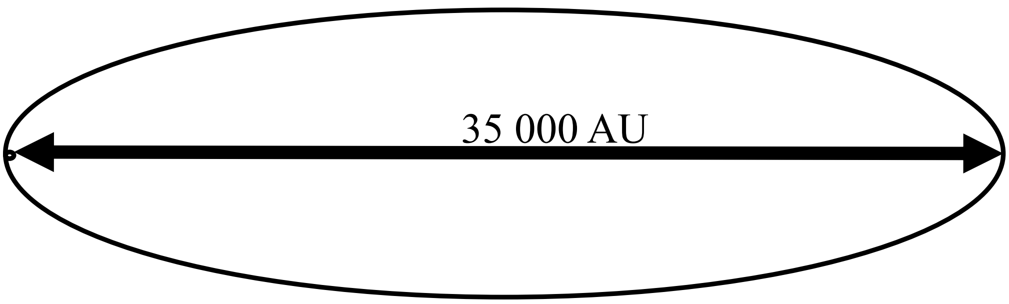
$$2\,a_S = (2 \text{ x } 60 \text{ s} + 39 \text{ x } 60 \text{ s}) \text{ x } (299\,792\,458 \text{ m s}^{-1}) = 7.38 \text{ x } 10^{11} \text{ m} = 4.93 \text{ AU},$$

$$a_2 = 4.93 / 2 = 2.47 \text{ AU}.$$

(b) By Kepler’s Third Law,

$$T_{\text{Seneca}} = \left(\frac{a_{\text{Seneca}}}{a_{\text{Earth}}}\right)^{3/2} T_{\text{Earth}} = 3.87 \text{ years.}$$

10 Note that a true single-appearance comet will be on a hyperbolic orbit. However, in this question, we are told that the comet is at aphelion when it is at 35 000 AU from the Sun. Thus, we have to assume the comet is on an elliptical orbit. To some extent, the comet is still a single-appearance comet: it has visited the inner solar system at most once since the emergence of homo sapiens a few 100 000 years ago.



As the comet enters the inner solar system, it will be at most a few astronomical units (AU) from the Sun. Hence, its perihelion will at most be a few AU, which is negligible to the distance to the aphelion.

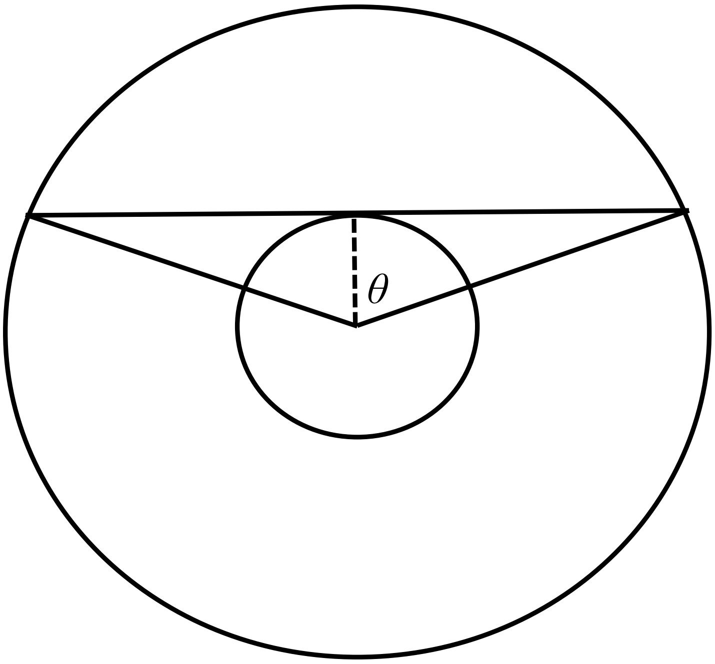
$$2\,a \approx 35\,000 \text{ AU},$$

$$a \approx 17\,500 \text{ AU},$$

$$T = a^{3/2} = 17500^{3/2} = 2.3 \text{ x } 10^6 \text{ years.}$$

Hence, a one-way trip will take about $(2.3 \text{ x } 10^6) / 2 = 1.2 \text{ x } 10^6$ years.

11



Let the smaller circle be the surface of Mars and the larger circle Phobos' orbit. If we are at the top of the smaller circle (at the end of the dashed line), we can only see Phobos when it is above the horizontal line. Otherwise, we would have to look through Mars itself (Phobos would be below the horizon).

Mars' angular speed about its own axis is ω_{Mars} = 360 / 24.623 = 14.6 degrees hr^{-1}.

We can use Kepler's Third Law to obtain Phobos' rotational period. However, we have to plug in $M = M_{Mars}$ for the mass of the central object (rather than the mass of the Sun) and $a = a_{Ph}$ (rather than the semi-major axis of a planet). With this, we find for the rotational period of Phobos T_{Ph} = 27 582 s = 7.6616 hr. From this, we find for Phobos' angular speed about the centre of Mars ω_{Ph} = 360 / 7.6616 = 47.0 degrees hr^{-1}.

So Phobos "overtakes" Mars at an angular speed of 47.0 – 14.6 = 32.4 degrees hr^{-1}.

From the figure above, we find cos $\theta = R_{Mars} / a_{Ph}$ = 3393 / 9380, from which θ = 68.8°. Throughout each orbit, Phobos will be visible for the amount of time it takes to cover an angle 2θ = 137.6 degrees, that is t = 137.6 / 32.4 = 4.25 hours.
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H3 Electric and Magnetic Fields Tutorial

There are four parts to this tutorial (A, B, C and D). Discussion questions are the sort of questions to expect in the A-levels or Prelim papers. Challenge questions will not be discussed unless time permits, and can be safely skipped.

Part A: Conductors in Electrostatic Equilibrium, Mathematical Preliminaries & Continuous Charge Distributions

Self-Review Questions

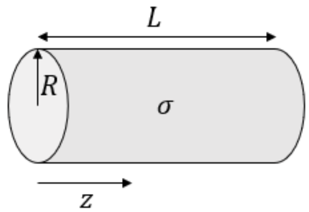
S1 A conductor is in electrostatic equilibrium. Explain:

(a) why the electric field strength inside the conductor is zero, and

(b) why the surface of the conductor is an equipotential surface.

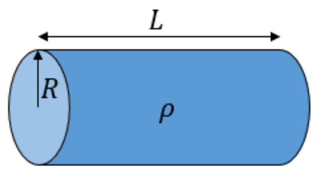
S2 **(a)** Find the total charge in a line of charge of length L, with linear charge density $\lambda = bx^2$, where b is a positive constant and x ranges from 0 to L.

(b)



Find the total charge in a thin cylindrical shell of charge of radius R and length L, with a surface charge density $\sigma = kz$ where z ranges from 0 to L.
(Note: "cylindrical shell" only refers to the curved surface. The circular base and top of a cylinder are not considered part of a cylindrical shell.)

(c)



Find the total charge in an insulating cylinder of radius R of length L with volume charge density $\rho = k(1 - r)$, where k is a positive constant.

(d) Find the total charge in an insulating sphere of radius R with volume charge density $\rho = a(b - r^2)$, where a and b are positive constants.

Discussion Questions

D1 **(2020 H3 Q6 – part)**

A uniformly charged thin disc of radius R lies in the *x*-*y* plane as shown in Fig. 6.1.

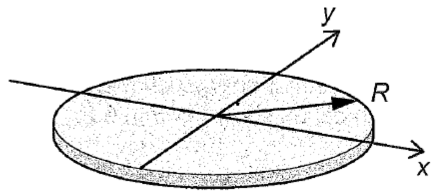


Fig. 6.1

The total amount of charge on the disc is Q.

(a)(i) State an expression for the surface charge density σ in terms of Q and r [1]

(b) The electric field strength can be determined by superimposing the point charge fields of infinitesimal charge elements. This can be done by summing the fields of charged rings of width dr, as shown in Fig. 6.2.

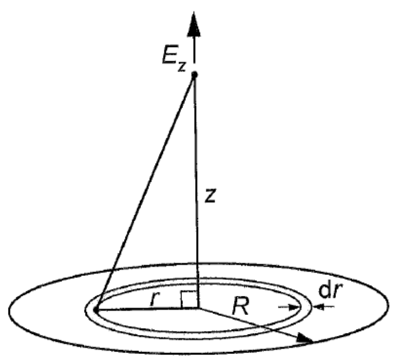


Fig. 6.2

(i) Show that the electric field at position $(0,0,z)$ is given by:

$$E_z = \frac{\sigma}{2\varepsilon_0}\left[1 - \frac{z}{\sqrt{z^2 + R^2}}\right]$$

[6]

(ii) Determine an expression for E_z when $z \ll R$. [2]

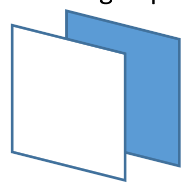
(iii) Determine an expression for E_z when $z \gg R$.
You may wish to use the approximation when x is small:

$$(1 + x)^n \approx 1 + nx$$

[2]

(iv) Comment on the form of the expressions in (b)(ii) and (b)(iii) [2]

D2 Two large plates with surface charge σ_1 and σ_2 are arranged parallel to each other, with separation d.



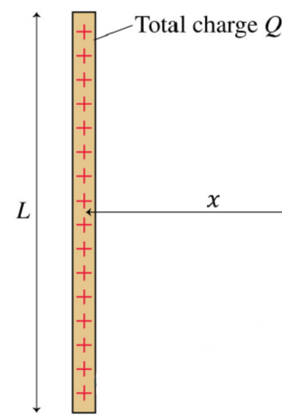
Determine the magnitude of the electric field strength between the plates, and to the left/right of the plates if:

(a) $\sigma_1 = -\sigma_2$

(b) $\sigma_1 = \sigma_2$

(c) Both plates are positively charged (but $\sigma_1 \neq \sigma_2$)

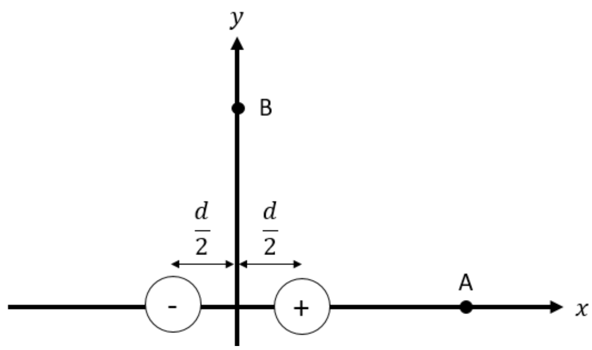
D3 A wire of length L has a charge Q. The charges are uniformly distributed across the rod with linear charge density λ.



(a) Calculate the electric potential V at point P at a distance x from the mid-point of the rod.

(b) Find the gradient of V, and hence show that you get the same results as Lecture Example 5.

D4 The figure below shows an electric dipole, with charges $-q$ and $+q$, centred on the origin.



(a) Find expressions for the magnitude of the electric field strength E at:

(i) a point A along the x-axis

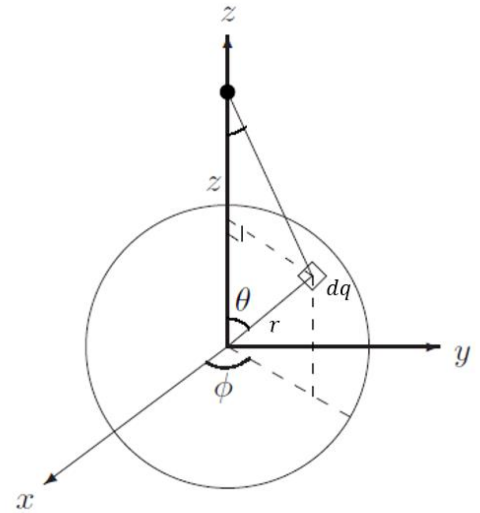
(ii) a point B along the y-axis

(b) By considering what happens to your expressions in **(a)** when x and y respectively become large, show that the electric field of a dipole is inversely proportional to the *cube* of the distance from the dipole.
You may wish to use the approximation when x is small: $(1+x)^n \approx 1 + nx$

(Optional) Challenging Questions

These questions will *not* come out in the A-levels, but are good for conceptual understanding. Use of computer algebra software (e.g. Integral Calculator https://www.integral-calculator.com/) is advised to deal with the tedious mathematics. You'll need to solve multiple integrals, so do have a look at Appendix 1 if you're stuck.

C1 A solid sphere of charge has volume charge density ρ_0 as shown below.

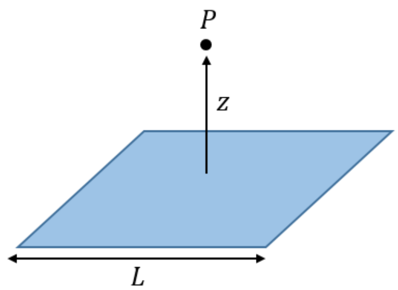


(a) Show that the electric potential V at a distance z from the centre of the sphere is $V = \frac{Q}{4\pi\varepsilon_0 z}$ (where Q is the total charge on the sphere) when $z > R$.
You may find the cosine rule useful: $c^2 = a^2 + b^2 - 2ab\cos\theta$

(b) Find the gradient of V, and hence verify that $E = \frac{Q}{4\pi\varepsilon_0 z^2}$ when $z > R$

C2 In question D1, we calculated the electric field strength due to a circular plane of charge, and then used that to find the electric field strength of an infinite plane.

Let's try the same thing, but using a square plane. The figure shows a square plane of positive charge, of side L. The charges are uniformly distributed with surface charge density σ.



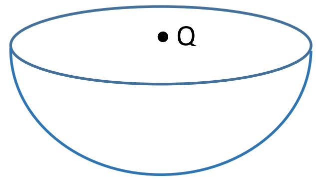
Set up the integral (but do not perform the integration manually!) to calculate the electric field strength at point P at a distance z from the centre of the plane. Why is this integral much more difficult to perform?

Part B: Gauss' Law

Self-Review questions

S1 The following charges are located at various places inside a submarine, $5.00\mu C$, $-9.00\mu C$, $27.0\mu C$ and $-84.0\mu C$. Calculate the net electric flux through the hull of the submarine.

S2 A point charge Q is located just above the center of the flat face of a hemisphere of radius R. What is the electric flux through the curved surface? What is the electric flux through the flat face?



Discussion Questions

D1 **(2020 H3 Q6a modified – continuation of AD1)**
A uniformly charged thin disc of radius R lies in the *x*-*y* plane as shown in Fig. 6.1.

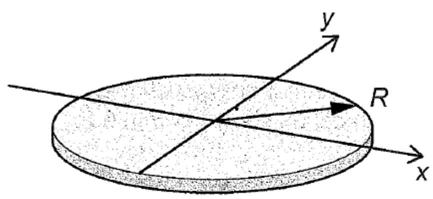


Fig. 6.1

The total amount of charge on the disc is Q.

(i) State an expression for the surface charge density σ in terms of Q and r [1]

(ii) Use your answer in **(i)** and apply Gauss's law with an appropriately chosen Gaussian surface to show that an approximation for the electric field at the position $(0,0,z)$, where $z \ll R$, is given by

$$E_z = 2\pi k\sigma$$

where k is a constant you will need to determine.

You may wish to draw a diagram to help your answer. [6]

(iii) Determine an expression for the electric potential at the point $(0,0,z)$ relative to the origin. [2]

(iv) Why is your answer in **(ii)** only an approximation? (Hint: The actual expression is found in Part A, D1)

D2 **(Griffiths, *Introduction to Electrodynamics*)**
A long cylinder of radius R carries a volume charge density that is proportional to the distance r from the axis: $\rho = kr$, for some constant k.

(a) Find the electric field strength inside and outside the cylinder.

(b) Hence, find the electric potential inside and outside the cylinder.

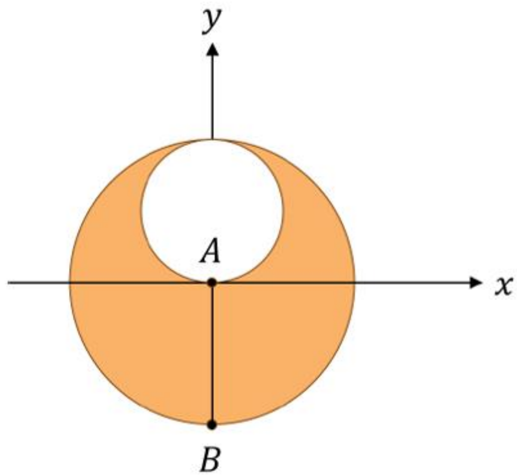
D3 **(a)** Using Gauss' Law, find expressions for the electric field strength E at a distance r from the centre of a sphere of charge of radius R with a constant volume charge density ρ_0

(b) Hence, calculate the electric potential V inside and outside the sphere.

(c) Plot a graph of E against r and V against r on the same axes

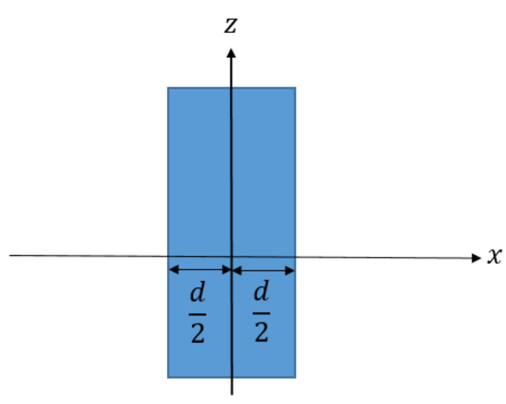
(d) *Optional, additional exercise:*
Repeat (a), (b), and (c) in the case where the volume charge density is instead given by $\rho = kr$, where k is a constant.

D4 An insulating sphere of radius $2R$ has a uniform charge density ρ. A spherical cavity of radius R is carved out as shown below:



Find the magnitude and direction of the electric field strength E at points A and B.

D5 A long, high, rectangular slab of insulating material of thickness *d* is placed at the origin. It has a uniform positive charge density ρ. The diagram below shows the side view:



(a) Find the magnitude of the electric field strength E inside the slab for points along the x-axis.

(b) Suppose an electron of charge -*e* and mass m_e is released along the x-axis somewhere inside the slab, and can move freely within the slab. Show that it exhibits simple harmonic motion with frequency $f = \frac{1}{2\pi}\sqrt{\frac{\rho e}{m_e \epsilon_0}}$

Part C: Ampere's Law

Self-Review Questions

S1 (2019 H3 Q2)

(a) State Gauss' Law for magnetic fields. [1]

(b) Explain why magnetic field lines always form closed loops. [2]

(c) Explain why the existence of magnetic monopoles would be inconsistent with Gauss' Law for magnetic fields. [2]

Discussion Questions

D1 (2019 H3 Q3)

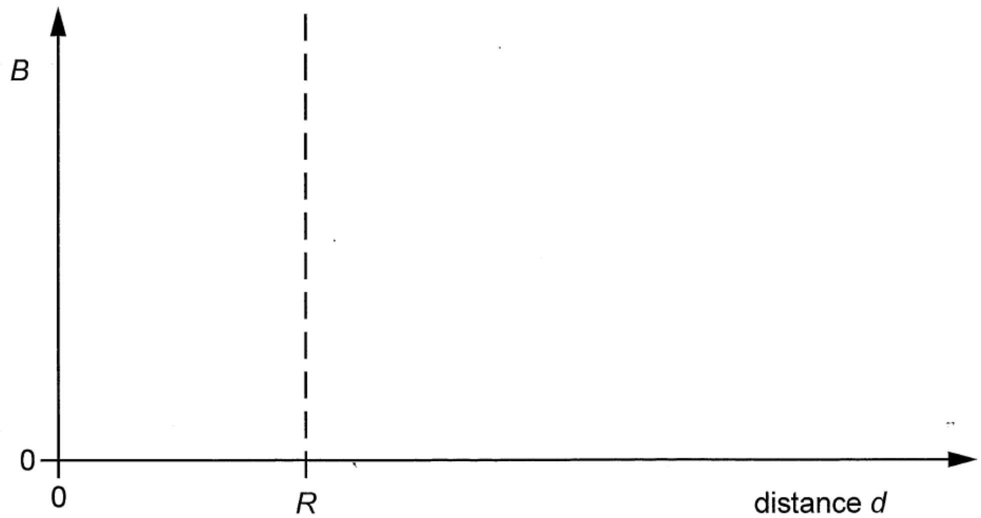
(a) State the line integral form of Ampere's Law. Define the symbols used. [3]

(b) A long straight copper wire of radius R carries a constant current I.

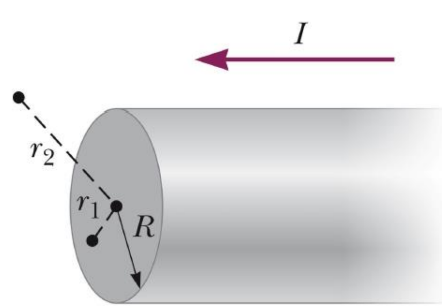
(i) Use Ampere's law to show that the magnetic flux density B at a distance d from the wire is: [2]

$$B = \frac{\mu_0 I}{2\pi d}$$

(ii) Sketch the magnetic flux density as a function of distance from the centre of the wire. [3]



D2 A long cylindrical copper wire of radius R carries a current I as shown. The current density J varies according to radial distance r from the centre of the wire, given by $J = br$ where b is a constant.



(a) Find the magnitude of the magnetic flux density B at a distance $r_1 < R$ and $r_2 > R$.

(b) Sketch a graph of B against r.

D3 **(HCI Prelim 2020 Q7a)**

Consider a cylindrical segment of a long straight wire carrying a current that is uniformly distributed across the cross-section of the wire of radius *R*.

(i) Derive expressions for the magnetic field at a distance *r* from the axis of the wire for $r \leq R$ and $r \geq R$, in terms of current density J, r and R. [4]

(ii) On Fig. 7.1, sketch the variation of the magnetic field *B* with the distance *r* from the center of the wire. [2]

Fig. 7.1

(iii) In a particular segment of the wire carrying a uniform current I, it is discovered that there is a cylindrical cavity of radius 0.1 *R* centered at a point that is 0.5 *R* away from the axis of the wire, as shown in Fig. 7.2

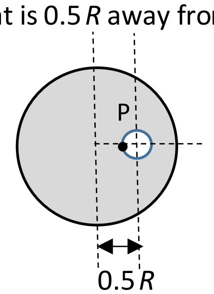


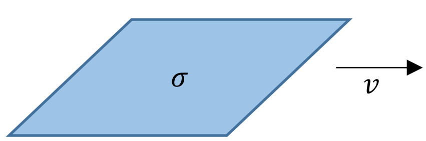
Fig 7.2

Show that the current density *J* is given by $J = 0.322\frac{I}{R^2}$ [2]

(iv) With reference to Fig. 7.2, derive an expression for the magnetic field at the point P in terms of μ_0, *I* and *R*.

Point P is on the edge of the cavity nearest to the axis of the wire. [7]

D4 **(a)** A wide, long insulating belt has a uniform positive charge per unit area σ on its upper surface. Rollers at each end move the belt to the right at a constant speed v.



Calculate the magnitude and direction of the magnetic field produced by the moving belt at points near its surface.

(b) A metal sheet oriented as the one above has a uniform surface current K flowing through it from left to right. Calculate the magnitude and direction of the magnetic field near the surface of the sheet.

D5 (HCI CT 2023 Q4b)

A coaxial cable is used for the transmission of high frequency electrical signals such as television signals.

Fig. 4.2 shows the typical construction of a coaxial cable and the radii of the different layers.

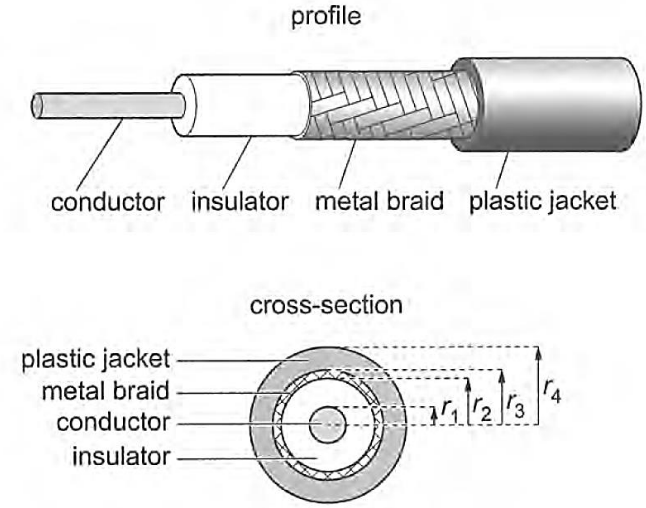


Fig 4.2

When there is a current I in the conductor in one direction, there is a current I in the metal braid in the opposite direction.

(a) State Ampere's Law in integral form, defining all terms. [2]

The current per unit cross-sectional area is a quantity known as the current density J. The current density in the conductor is J_1 and the current density in the metal braid is J_2. Assume that J_1 and J_2 are both constant.

(b) Show that:

$$J_2 = \frac{r_1^2}{r_3^2 - r_2^2}$$

[2]

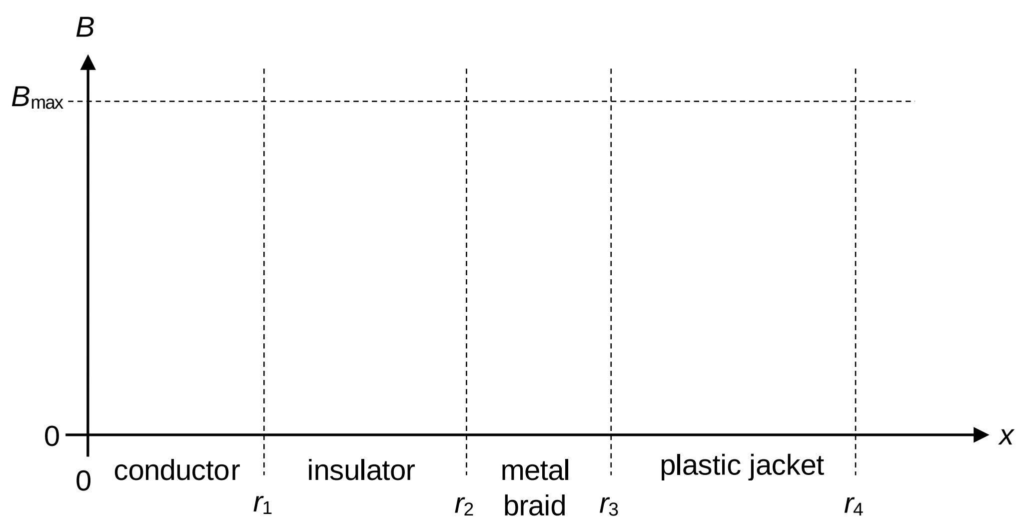
(c) Showing your working clearly, derive expressions for the magnetic flux density *B* in terms of J_1 and distance r from the centre of the coaxial cable for:

(i) $0 \leq r \leq r_1$, [1]

(ii) $r_1 \leq r \leq r_2$, [1]

(iii) and $r_3 \leq r \leq r_4$. [1]

(d) Use Fig. 4.3 to sketch how the magnetic flux density Bvaries between zero and a maximum value B_{max} with distance r from the centre of the coaxial cable, for $0 \leq x \leq r_4$.



[3]

(e) Standard transmission cables are made of two insulated wires, as shown in Fig. 4.4.

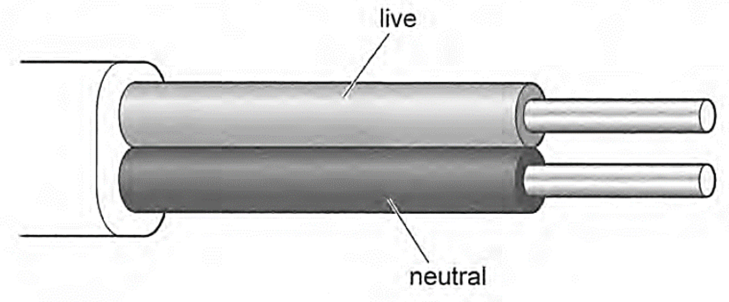


Fig. 4.4

Standard transmission cables are less expensive than coaxial cables.

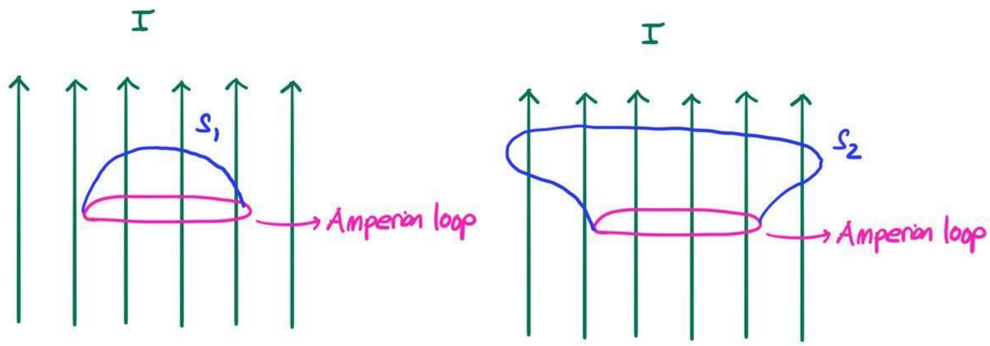
Suggest why standard transmission cables are **not** used for high frequency electrical signals. [1]

(Optional) Challenging Questions

These questions will *not* come out in the A-levels, but are good for conceptual understanding.

C1 (modified from Griffiths, *Introduction to Electrodynamics*)

In calculating the current enclosed by an Amperian loop, one must consider the total current I_{enc} flowing through a surface bounded by the Amperian loop. The trouble is, there are infinitely many surfaces that share the same boundary line – the figure below shows two of them. $\mathcal{S}_1$ is a hemisphere, and $\mathcal{S}_2$ is a "chef's hat" shape that bulges outwards near the top:



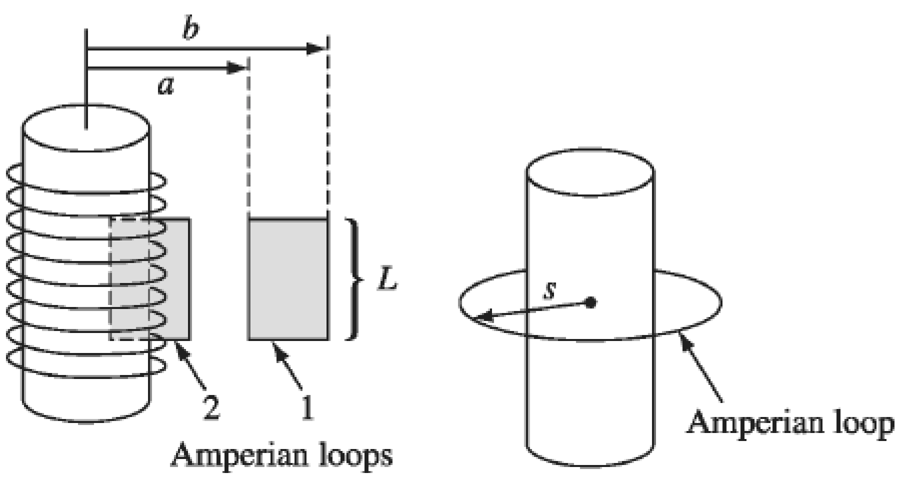
Of the infinite number of possible surfaces, which one(s) are we supposed to use? Explain.

C2 (Adapted from Griffiths, *Introduction to Electrodynamics*)

In Lecture Example 10, we derived the expression for the magnetic flux density B inside a long solenoid by assuming (correctly) that:

- $\vec{B} = 0$ everywhere outside the solenoid
- $\vec{B}$ inside the solenoid is a constant and points along the axis of the solenoid

Let's prove these assertions using Ampere's Law! Consider the following (long, tightly-wound) solenoid carrying current I. Draw two rectangular Amperian loops, 1 and 2; and a circular Amperian loop 3.



3

(a) Argue that $\vec{B}$ does not have a radial component. (Hint: use cylindrical symmetry and consider the effect of changing the direction of the current)

(b) Using Amperian loop 3, show that $\vec{B}$ does not have a "circumferential" component.

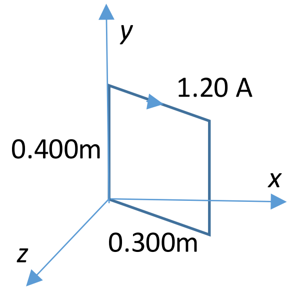
(c) Using Amperian loop 1 and the result of (a), show that $\vec{B} = 0$ everywhere outside the solenoid.

(d) Using Amperian loop 2 and the result of (c), show that everywhere inside the solenoid, $B = \mu_0 n I$ and that it points along the axis of the solenoid

Part D: Dipoles in Fields

Self-Review Questions

S1 A rectangular coil consists of 100 closely wrapped turns has a length of 0.400 m and width of 0.300m. The coil is hinged along the *y*-axis and its plane makes an angle of 30.0° with the *x*-axis.



What is the magnitude of a torque exerted on the coil by a uniform magnetic field of 0.800 T directed along the *x*-axis when the current is 1.20 A clockwise as shown in the diagram? What is the expected direction of rotation of the coil?

S2 **(a)** The SI unit for magnetic moments is ampere square metres ($\mathrm{A\ m^2}$), but it can also be expressed in joules per tesla ($\mathrm{J\ T^{-1}}$). Convert $1\ \mathrm{A\ m^2}$ into $\mathrm{J\ T^{-1}}$.

(b) The magnetic moment of Earth is approximately $8.00 \times 10^{22}\ \mathrm{A\ m^2}$.
The magnetic dipole moment of a single unpaired electron (also known as the Bohr magneton μB) is $\mu_B = 9.27 \times 10^{-24}\ \mathrm{J\ T^{-1}}$.

(i) If the magnetic moment of the Earth were caused by the complete magnetization of a huge iron deposit, how many unpaired electrons would this correspond to?

(ii) At two unpaired electrons per iron atom, how many kilograms of iron would this correspond to? (The density of iron is 7900 kg/m^3 so there are approximately 8.50×10^{28}iron atoms/m^3)

Discussion Questions

D1 **(H3 Specimen Paper Q7) – part (c) deals with dipoles**

(a) Fig 7.1 shows the cross section of a solid, egg-shaped object made out of a conducting material.

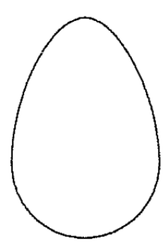


Fig. 7.1

The object is charged negatively.

(i) On Fig 7.1, sketch the electric field lines due to the conductor. [2]

(ii) State how the charge is distributed. [1]

(b) Fig 7.2 shows a uniformly positively-charged solid sphere of radius R made of an insulating material.

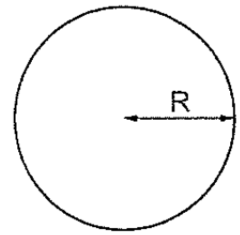


Fig. 7.2

The sphere has a constant charge density of ρ throughout its volume.
(The unit for charge density is C m^{-3}.)

(i) Using Gauss' Law, derive an expression for the electric field strength at a distance r from the centre of the sphere for the case where $r \geq R$. [4]

(ii) Derive an expression for the electric field strength as a function of r for the case where $r \leq R$. [2]

(iii) On Fig 7.3, sketch the electric field strength as a function of the distance r in the range $r = 0$ to $r = 3R$. [3]

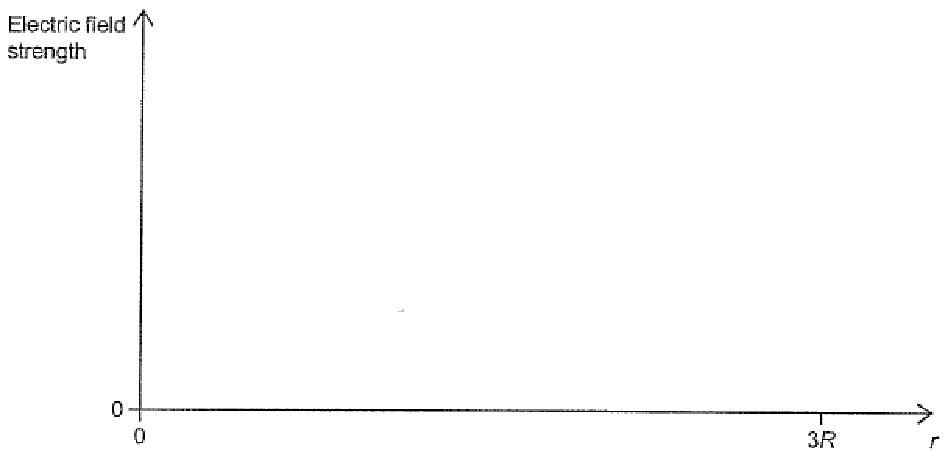


Fig. 7.3 [3]

(c) The positively-charged sphere in Fig 7.2 is attached, via an insulating rod of length L, to a sphere identical in material and dimensions but uniformly negatively charged with a charge density of $-\rho$ throughout its volume.

An electric dipole is thus formed.

(i) Find an expression for the magnitude of the electric dipole moment p. [2]

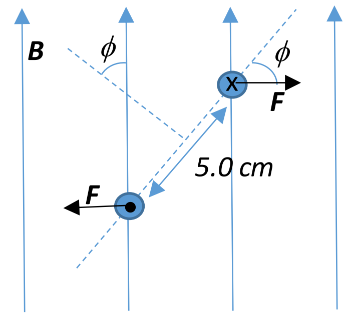
(ii) Suppose that the electric dipole is placed in an external uniform electric field E at an angle θ of $45°$ with respect to the lines of E.

Sketch this arrangement and include on your diagram arrows to show the force acting on each of the two charged spheres. [2]

(iii) State the torque τ on the system in terms of E, ρ and θ. [1]

(iv) There are two orientations of the dipole within the field where the dipole experiences zero torque. Describe what these orientations are and explain how an oscillating dipole is most likely to settle within the field. [3]

D2 A square loop of wire carrying current $I = 2.00$ A is in a uniform magnetic field $B = 0.830$ T. The normal to the plane of the loop makes an angle ϕ with the field, as shown below. The loop is free to rotate. Each side of the loop has length 5.00 cm and mass 20.0 g.



(a) The loop is released from rest when $\phi = 5°$.

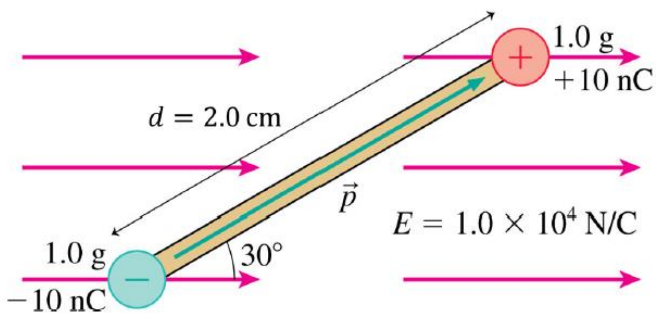
(i) Calculate the magnitude and direction of net torque on the loop.

(ii) Show that the oscillation is simple harmonic, and find the period of the oscillation.

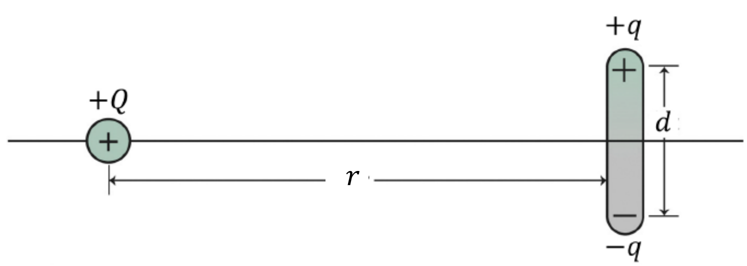
(b) Sketch a graph to show how magnetic potential energy U varies with ϕ as the loop is rotated through $\phi = 0°$ to $360°$.

(c) Find all equilibrium points, and discuss their stability.

D3 Two 1.0 g balls are connected by a 2.0 cm rod of negligible mass. One ball has a charge of +10 nC, the other has a charge of -10 nC. The rod is held in a 1.0×10^4 N C^{-1} uniform electric field at an angle of 30° with respect to the field, then released. Calculate its initial angular acceleration.



D4 A dipole with charges $\pm q$ and separation d is located at a distance r from a point charge Q, oriented as shown.



It is known that $r \gg d$.

(i) Without calculation, deduce the directions of the net torque and the net force on the dipole. Explain your reasoning.

The rest of this question is an optional challenge.

You may wish to use the approximation when x is small: $(1 + x)^n \approx 1 + nx$

(ii) Show that the magnitude of the torque τ on the dipole is $\tau = \frac{qQd}{4\pi\varepsilon_0 r^2}$ and determine its direction.

(iii) Show that the magnitude of the net force F on the dipole is $F = \frac{qQd}{4\pi\varepsilon_0 r^3}$ and determine its direction.

Answers to selected problems

A Continuous Charge Distributions

Self-Review Questions

S2	(a)	$$Q = \frac{bL^3}{3}$$
	(b)	$$Q = \pi kRL^2$$
	(c)	$$Q = 2\pi kL(R - R^2)$$
	(d)	$$Q = 4\pi a\left(\frac{bR^3}{3} - \frac{R^5}{5}\right)$$

Discussion Questions

D1	(a)(i)	$$\sigma = \frac{Q}{\pi R^2}$$
	(b)(ii)	$$E = \frac{\sigma}{2\varepsilon_0}$$
	(b)(iii)	$$E = \frac{\sigma R^2}{4\varepsilon_0 z^2}$$
D2	(a)	Between: $E = \frac{\sigma_1}{\varepsilon_0} = \frac{\sigma_2}{\varepsilon_0}$ Outside: $E = 0$
	(b)	Between: 0 Outside: $E = \frac{\sigma_1}{\varepsilon_0} = \frac{\sigma_2}{\varepsilon_0}$
	(c)	Between: $E = \frac{\lvert\sigma_1 - \sigma_2\rvert}{2\varepsilon_0}$ Outside: $E = \frac{\sigma_1 + \sigma_2}{2\varepsilon_0}$
D3	(a)	$$V = \frac{\lambda}{4\pi\varepsilon_0}\ln\frac{\sqrt{x^2 + L^2/4} + L/2}{\sqrt{x^2 + L^2/4} - L/2}$$
D4	(a)(i)	$$E_A = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{\left(x - \frac{d}{2}\right)^2} - \frac{1}{\left(x + \frac{d}{2}\right)^2}\right)$$
	(a)(ii)	$$E_B = \frac{qd}{4\pi\varepsilon_0\left(y^2 + \left(\frac{d}{2}\right)^2\right)^{3/2}}$$
	(b)	$$E_A \approx \frac{2qd}{4\pi\varepsilon_0 x^3}, \qquad E_B \approx \frac{qd}{4\pi\varepsilon_0 y^3}$$

(Optional) Challenging Questions

C2		$$E_z = \int_{x=-L/2}^{x=L/2}\int_{y=-L/2}^{y=L/2}\frac{1}{4\pi\varepsilon_0}\frac{\sigma\, z\, dx\, dy}{(x^2 + y^2 + z^2)^{3/2}}$$ OR $$E_z = \frac{\sigma z}{4\pi\varepsilon_0}\int_{y=-L/2}^{y=L/2}\frac{L}{(y^2 + z^2)\sqrt{y^2 + L^2/4 + z^2}}\, dy$$

B Gauss' Law

Self-Review Questions

S1		$\Phi_E = -6.89 \times 10^6\ \mathrm{N\ m^2\ C^{-1}}$
S2		$\Phi_{curved} = \dfrac{Q}{2\varepsilon_0}$ $\Phi_{flat} = -\dfrac{Q}{2\varepsilon_0}$

Discussion Questions

D1	(a)(i)	$\sigma = Q/\pi R^2$
	(a)(ii)	$k = 1/4\pi\varepsilon_0$
	(a)(iii)	$V_z = -Ez$
D2	(a)	$E = \begin{cases} \dfrac{1}{3\varepsilon_0} kr^2 & r \le R \\ \dfrac{kR^3}{3\varepsilon_0 r} & r \ge R \end{cases}$
	(b)	$\therefore V = \begin{cases} -\dfrac{kR^3}{3\varepsilon_0}\ln\dfrac{r}{a} & r \ge R \\ -\dfrac{kR^3}{3\varepsilon_0}\ln\dfrac{R}{a} - \dfrac{k}{9\varepsilon_0}(r^3 - R^3) & r \le R \end{cases}$ where $R < a < \infty$ is an arbitrary point of reference
D3	(a)	$E = \begin{cases} \dfrac{\rho r}{3\varepsilon_0} & r \le R \\ \dfrac{\rho R^3}{3\varepsilon_0 r^2} & r \ge R \end{cases}$
	(b)	$V = \begin{cases} \dfrac{\rho R^3}{3\varepsilon_0 r} & r \ge R \\ \dfrac{\rho(3R^2 - r^2)}{6\varepsilon_0} & r \le R \end{cases}$
	(d)	$\therefore E = \begin{cases} \dfrac{kr^2}{4\varepsilon_0} & r \le R \\ \dfrac{kR^4}{4\varepsilon_0 r^2} & r \ge R \end{cases}, \quad V = \begin{cases} \dfrac{kR^4}{4\varepsilon_0 r} & r \ge R \\ \dfrac{k(4R^3 - r^3)}{12\varepsilon_0} & r \le R \end{cases}$
D4		$E_A = \dfrac{\rho R}{3\varepsilon_0}$ upwards, $\quad E_B = \dfrac{17\rho R}{27\varepsilon_0}$ downwards
D5	(a)	$E = \dfrac{\rho x}{\varepsilon_0}$

C Ampere's Law

Discussion Questions

D2	(a)	$$B = \begin{cases} \dfrac{\mu_0 b r^2}{3} & r \le R \\ \dfrac{\mu_0 b R^3}{3r} & r \ge R \end{cases}$$
D3	(i)	$$B = \begin{cases} \dfrac{\mu_0 J r}{2} & r \le R \\ \dfrac{\mu_0 J R^2}{2r} & r \ge R \end{cases}$$
	(iv)	$$B = \frac{0.0805\mu_0 I}{R}$$
D4	(a)	$$B = \begin{cases} \dfrac{\mu_0 \sigma v}{2} \text{ out of the paper} & \text{(above the sheet)} \\ \dfrac{\mu_0 \sigma v}{2} \text{ into of the paper} & \text{(below the sheet)} \end{cases}$$
	(b)	$$B = \begin{cases} \dfrac{\mu_0 K}{2} \text{ out of the paper} & \text{(above the sheet)} \\ \dfrac{\mu_0 K}{2} \text{ into of the paper} & \text{(below the sheet)} \end{cases}$$
D5	(c)(i)	$$B = \frac{1}{2}\mu_0 J_1 r$$
	(c)(ii)	$$B = \frac{\mu_0 J_1 r_1^2}{2r}$$
	(c)(iii)	$B = 0$

D Dipoles in Fields

Self-Review Questions

S1		$\tau = 9.98$ N m
S2	(a)	$1\ \mathrm{Am}^2 = 1\ \mathrm{J\ T}^{-1}$
	(b)(i)	8.63×10^{45}
	(b)(ii)	4.01×10^{20} kg

Discussion Questions

D1	(b)(i)	$$E = \frac{R^3 \rho}{3\varepsilon_0 r^2}$$
	(b)(ii)	$$E = \frac{\rho r}{3\varepsilon_0}$$
	(c)(i)	$$p = \frac{4}{3}\pi R^3 \rho(2R + L)$$
	(c)(iii)	$\tau = Ep\sin\theta$
D2	(a)(i)	$\tau = 3.62 \times 10^{-4}$ N m clockwise
	(a)(ii)	$T = 0.563$ s
D3		$\alpha = 5.0$ rad s^{-2}

Tutorial solutions

Part A: Conductors in Electrostatic Equilibrium, Mathematical Preliminaries & Continuous Charge Distributions

Self-Review Questions

S1	
(a)	If the electric field strength inside the conductor is not zero, free charges will experience an electric force, and thus move. Then the conductor would not in electrostatic equilibrium.
(b)	The electric field at the surface of the conductor is normal to the surface. Therefore, there is no component of the electric field along the surface. (Otherwise, there would be a component of electric force acting on charges along the surface of the conductor, causing them to flow) The potential difference between any two points along the surface is therefore zero as $V = \int E\, d\ell$. Thus the surface is an equipotential surface.

S2	
(a)	$$Q = \int_0^L \lambda\, dx = \int_0^L bx^2\, dx = \left[\frac{bx^3}{3}\right]_0^L = \frac{bL^3}{3}$$
(b)	<u>Single integral method:</u> $$Q = \int \sigma\, dA$$ We need to replace dA with some variable to integrate over. Since σ only depends on z, slice the cylinder into thin strips of width dz. Then the area of each slice $dA = 2\pi R\, dz$. $$\therefore Q = \int_0^L kz\,(2\pi R\, dz) = 2\pi kR \int_0^L z\, dz = \pi kRL^2$$ <u>Double integral method:</u> $$Q = \int \sigma\, dA$$ Since area element $dA = R\, d\theta\, dz$, $$\begin{aligned} Q &= \int_{z=0}^{z=L} \int_{\theta=0}^{\theta=2\pi} kz\,(R\, d\theta\, dz) \\ &= \int_{z=0}^{z=L} 2\pi kzR\, dz \\ &= \pi kRL^2 \end{aligned}$$ (Note: Our solution is proportional to L^2, just like Lecture Example 1(b), so our answer is likely to be correct)
(c)	<u>Single integral method:</u> $$Q = \int \rho\, dV$$ We need to replace dV with some variable to integrate over. Since ρ only depends on r, referring to lecture example 3, slice the cylinder into thin cylindrical shells of radius r and thickness dr. Then $dV = \pi(r+dr)^2 L - \pi r^2 L = \pi L[(r+dr)^2 - r^2] = \pi L[r^2 + 2r\, dr + (dr)^2 - r^2] = 2\pi rL\, dr$ (we drop the $(dr)^2$ term as it is negligible). $$\therefore Q = \int_0^R k(1-r) \times (2\pi rL\, dr) = 2\pi kL \int_0^R (r - r^2)\, dr = 2\pi kL\left(\frac{R^2}{2} - \frac{R^3}{3}\right)$$ <u>Triple integral method:</u> $$Q = \int \rho\, dV$$

Since volume element $dV = r\, d\theta\, dr\, dz$,

$$Q = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=R} \int_{z=0}^{z=L} k(1-r)\, r\, d\theta\, dr\, dz$$
$$= \int_{r=0}^{r=R} \int_{z=0}^{z=L} 2\pi k(r-r^2)\, dr\, dz$$
$$= \int_{r=0}^{r=R} 2\pi kL(r-r^2)\, dr$$
$$= 2\pi kL\left(\frac{R^2}{2} - \frac{R^3}{3}\right)$$

(d)

Single integral method

$$Q = \int \rho\, dV, \qquad \rho = a(b-r^2)$$

We need to replace dV with some variable to integrate over. Since ρ only depends on r, referring to lecture example 3, cut the sphere into thin spherical shells of radius r and thickness dr.

Then $dV = \frac{4}{3}\pi(r+dr)^3 - \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(r^3 + 3r^2(dr) + 3r(dr)^2 + (dr)^3 - r^3) = 4\pi r^2\, dr$. (The $(dr)^2$ and $(dr)^3$ terms are negligible).

$$\therefore Q = \int_0^R a(b-r^2)\,(4\pi r^2) dr = 4\pi a \int_0^R br^2 - r^4\, dr = 4\pi a\left(\frac{bR^3}{3} - \frac{R^5}{5}\right)$$

Triple integral method

$$Q = \int \rho\, dV, \qquad \rho = a(b-r^2)$$

Since volume element $dV = (r\, d\theta)(r\sin\theta\ d\phi)(dr)$,

$$Q = \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=R} \int_{\phi=0}^{\phi=2\pi} a(b-r^2)\, r^2 \sin\theta\ d\theta\, dr\, d\phi$$
$$= \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=R} \int_{\phi=0}^{\phi=2\pi} a\sin\theta\,(br^2 - r^4)\, d\theta\, dr\, d\phi$$
$$= \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=R} 2\pi a \sin\theta\,(br^2 - r^4)\sin\theta\ d\theta\, dr$$
$$= \int_{r=0}^{r=R} 4\pi a(br^2 - r^4)\, dr$$
$$= 4\pi a\left(\frac{bR^3}{3} - \frac{R^5}{5}\right)$$

(Note: $\int_0^\pi \sin\theta\ d\theta = 2$)

B Gauss' Law

Self-Review questions

S1

The total charge inside the submarine, $Q = 5.00\ \mu\mathrm{C} - 9.00\ \mu\mathrm{C} + 27.0\ \mu\mathrm{C} - 84.0\ \mu\mathrm{C} = -61.0\ \mu\mathrm{C}$

Take the surface of the submarine as a Gaussian surface. Using Gauss' Law,

$$\oiint \vec{E}\cdot d\vec{A} = \frac{Q_{enc}}{\varepsilon_0}$$

(where $\oiint \vec{E}\cdot d\vec{A} = \Phi_E$, the net electric flux through the surface)

$$\therefore \Phi_E = \frac{-61.0\ \mu\mathrm{C}}{8.85\times 10^{-12}\ \mathrm{F\ m^{-1}}} = \frac{-61.0\times 10^{-6}}{8.85\times 10^{-12}} = -6.89\times 10^{6}\ \mathrm{N\ m^2\ C^{-1}}$$

S2	From Gauss' Law, the total flux through a surface that completely encloses the point charge is $\Phi = \frac{Q}{\varepsilon_0}$ Then, the total flux through the bottom hemisphere is half of that: $\Phi_{curved} = \frac{Q}{2\varepsilon_0}$ As the closed surface does not contain any charge, the net flux through it is zero. Therefore, the flux through the flat surface is $\Phi_{flat} = -\frac{Q}{2\varepsilon_0}$

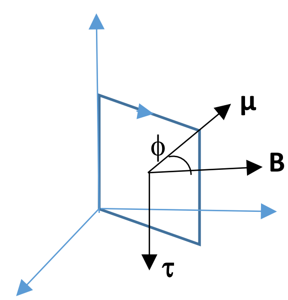
C Ampere's Law

Self-Review Questions

S1	
(a)	Gauss's Law for magnetic fields states that the total magnetic flux through a closed surface is zero
(b)	A magnetic field line that does not form an open loop will imply the presence of magnetic monopoles at the ends of the field line. As magnetic monopoles do not exist, magnetic field lines will always form a closed loop.
(c)	If a magnetic monopole exist, the magnetic flux calculated based on a closed surface that enclose the magnetic monopole will be non-zero which is inconsistent with Gauss's Law for magnetic fields.

D Dipoles in Fields

Self-Review Questions

S1	
	The magnetic dipole moment of all N turns is $\vec{\mu} = NI\vec{A}$ The torque is $\vec{\tau} = \vec{\mu} \times \vec{B}$, which is in the clockwise direction. The magnitude of the torque is $\tau = \mu B \sin\phi$ $\therefore \tau = NIAB \sin\phi$ $= (100)(1.20)(0.400 \times 0.300)(0.800) \sin 60°$ $= 9.98 \text{ N m}$ (the coil will rotate clockwise) 

S2	
(a)	Since $W = Fd$, we have $\text{J} = \text{N m}$ Since $F = BIL \Longrightarrow B = \frac{F}{IL}$, we have $\text{T} = \frac{\text{N}}{\text{A m}}$ $$\therefore 1\,\frac{\text{J}}{\text{T}} = 1\,\frac{\text{N m}}{\frac{\text{N}}{\text{A m}}} = 1 \text{ Am}^2$$
(b)(i)	The number of unpaired electrons $N = \frac{8.00\times10^{22}}{9.27\times10^{-24}} = 8.63 \times 10^{45}$
(b)(ii)	Number of iron atoms required is $\frac{N}{2} = 4.31 \times 10^{45}$ atoms The iron atoms occupy a volume of $V = \frac{4.31\times10^{45}}{8.50\times10^{28}} = 5.07 \times 10^{16} \text{ m}^3$ Mass of the iron is thus $M = V\rho = 4.01 \times 10^{20} \text{ kg}$

Tutorial solutions (Part A - Conductors in Electrostatic Equilibrium, Mathematical Preliminaries & Continuous Charge Distributions)

Discussion Questions

<table>
<tr><td>D1</td><td></td></tr>
<tr><td>(a)(i)</td><td>

Surface charge density σ = charge per unit area

Hence $Q = \sigma(\pi R^2)$

$$\therefore \quad \sigma = \frac{Q}{\pi R^2}$$

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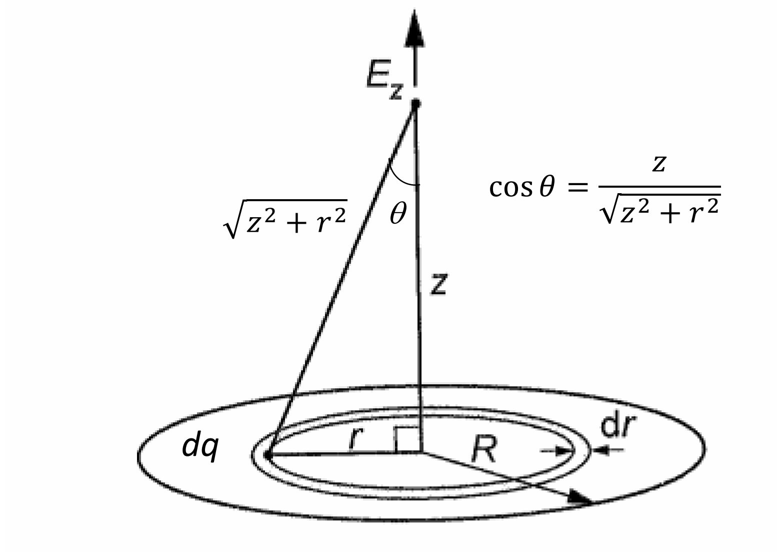
<tr><td>(b)(i)</td><td>



Consider infinitesimal charge element $dq = \sigma\, dA = \sigma(2\pi r\, dr)$. At (0, 0, z), the electric field E due to charge element dq is

$$dE = \frac{dq}{4\pi\varepsilon_0\left(\sqrt{z^2+r^2}\right)^2} = \frac{2\pi\sigma r\, dr}{4\pi\varepsilon_0(z^2+r^2)}$$

By symmetry, the radial components of E cancel out, so we only consider the z-component.

$$dE_z = dE\cos\theta = \frac{\sigma r\, dr}{2\varepsilon_0(z^2+r^2)}\frac{z}{\sqrt{z^2+r^2}}$$

$$E_z = \int_0^R \frac{\sigma r z}{2\varepsilon_0(z^2+r^2)^{\frac{3}{2}}}\, dr$$

$$= \frac{\sigma z}{2\varepsilon_0}\int_0^R \frac{r}{(z^2+r^2)^{\frac{3}{2}}}\, dr$$

$$= \frac{\sigma z}{2\varepsilon_0}\int_0^R \frac{1}{2}\frac{2r}{(z^2+r^2)^{\frac{3}{2}}}\, dr$$

$$= \frac{\sigma z}{2\varepsilon_0}\left[\frac{1}{2}\frac{1}{\left(-\frac{1}{2}\right)\sqrt{z^2+r^2}}\right]_0^R$$

$$= \frac{\sigma z}{2\varepsilon_0}\left(\frac{1}{z} - \frac{1}{\sqrt{z^2+R^2}}\right)$$

$$= \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{z}{\sqrt{z^2+R^2}}\right)$$

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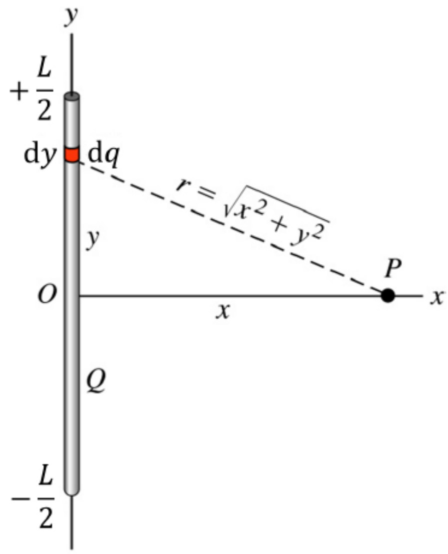
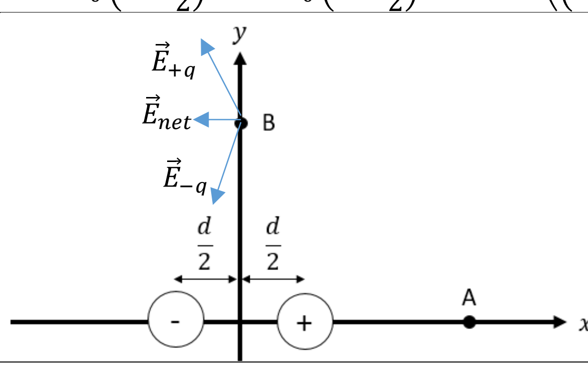
</table>

(b)(ii)	When $z \leq R,\ z^2 + R^2 \approx R^2$ $E_z = \frac{\sigma}{2\varepsilon_o}\left[1 - \frac{z}{\sqrt{z^2+R^2}}\right] = \frac{\sigma}{2\varepsilon_o}\left[1 - \frac{z}{\sqrt{R^2}}\right]$ $= \frac{\sigma}{2\varepsilon_o}\left[1 - \frac{z}{R}\right] \approx \frac{\sigma}{2\varepsilon_o}[1-0] = \frac{\sigma}{2\varepsilon_o}$ *Marker's Comment: The simplification of the expression and the approximation to be made must be clearly shown in the working.*
(b)(iii)	When $z \gg R, \frac{R}{z}$ is small $$E_z = \frac{\sigma}{2\varepsilon_0}\left[1 - \frac{z}{\sqrt{z^2+R^2}}\right] = \frac{\sigma}{2\varepsilon_0}\left[1 - \frac{z}{z\sqrt{1+\frac{R^2}{z^2}}}\right]$$ Since $\left(1+\frac{R^2}{z^2}\right)^{-1/2} \approx 1 - \frac{1}{2}\frac{R^2}{z^2}$ $$E_z \approx \frac{\sigma}{2\varepsilon_0}\left(1 - \left[1 - \frac{1}{2}\frac{R^2}{z^2}\right]\right) = \frac{\sigma R^2}{4\varepsilon_0 z^2}$$ *Marker's Comment: The binomial expansion must be used in the simplification of the expression.*
(b)(iv)	When $z \leq R,\quad E_z = \frac{\sigma}{2\varepsilon_o}$ The electric field at (0.0.z) is independent of z, equivalent to the case for an infinite charged sheet, the electric field lines are parallel lines emerging from the surface and constant everywhere (*R* is ∞). When $z \geq R,\ E_z \propto \frac{1}{z^2}$. The electric field at (0,0,*z*) of the disc is equivalent to the electric field of a point charge when *R* = 0. *Marker's Comment: Comments on the physical significance and recognition of the common scenarios that led to the expressions are expected.*

D2	
(a)	Between: the fields point in the same direction and have the same magnitude $$E = E_{+\sigma} + E_{-\sigma} = \frac{\sigma_1}{\varepsilon_0} = \frac{\sigma_2}{\varepsilon_0}$$ Outside: the fields point in opposite directions and have the same magnitude $$E = E_{+\sigma} - E_{-\sigma} = 0$$
(b)	Between: the fields point in opposite directions and have the same magnitude $$\therefore E = 0$$ Outside: the fields point in the same direction and have the same magnitude $$E = E_{+\sigma} + E_{-\sigma} = \frac{\sigma_1}{\varepsilon_0} = \frac{\sigma_2}{\varepsilon_0}$$
(c)	Between: the fields point in opposite directions but have different magnitudes $$E = \frac{\lvert\sigma_1 - \sigma_2\rvert}{2\varepsilon_0}$$ Outside: the fields point in the same direction $$E = \frac{\sigma_1 + \sigma_2}{2\varepsilon_0}$$

D3	
(a)	 The magnitude of the electric potential dV due to the small charge dq is given by: $dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{\sqrt{x^2+y^2}}$ Let the linear charge density be $\lambda = Q/L$. Since $dq = \lambda\, dy$, $dV = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\, dy}{\sqrt{x^2+y^2}}$ $\therefore V = \int_{y=-L/2}^{y=L/2}\frac{1}{4\pi\varepsilon_0}\frac{\lambda\, dy}{\sqrt{x^2+y^2}}$ $= \frac{\lambda}{4\pi\varepsilon_0}\ln\frac{\sqrt{x^2+L^2/4}+L/2}{\sqrt{x^2+L^2/4}-L/2}$
(b)	$\frac{dV}{dx} = \frac{\lambda}{4\pi\varepsilon_0}\left(-\frac{L}{x\sqrt{x^2+L^2/4}}\right) = -\frac{\lambda}{4\pi\varepsilon_0}\frac{2L}{x\sqrt{4x^2+L^2}}$ $-\frac{dV}{dx} = \frac{\lambda}{4\pi\varepsilon_0}\frac{2L}{x\sqrt{4x^2+L^2}} = E$ Which matches our answer in Lecture Example 5

D4	
(a)(i)	Take right as positive. At $A(x,0)$, $\vec{E}_{net,A} = \vec{E}_{+q} + \vec{E}_{-q} = \frac{q}{4\pi\varepsilon_0\left(x-\frac{d}{2}\right)^2} - \frac{q}{4\pi\varepsilon_0\left(x+\frac{d}{2}\right)^2} = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{\left(x-\frac{d}{2}\right)^2} - \frac{1}{\left(x+\frac{d}{2}\right)^2}\right)$
(a)(ii)	

	By symmetry, the vertical components $\vec{E}_{+q}$ and $\vec{E}_{-q}$ cancel out and the horizontal components sum, pointing to the left. $$E_{+q} = \frac{q}{4\pi\varepsilon_0\left(y^2+\left(\frac{d}{2}\right)^2\right)}$$ Take right as positive. The net electric field strength at $B(0,y)$ is twice the x-component of E due to the positive charge. Let θ be the acute angle with the y-axis: $$E_{net,B} = 2E_{+q,x} = 2\frac{q}{4\pi\varepsilon_0\left(y^2+\left(\frac{d}{2}\right)^2\right)}\sin\theta$$ By geometry, $\sin\theta = \frac{d/2}{\sqrt{y^2+\left(\frac{d}{2}\right)^2}}$ $$E_{net,B} = \frac{2q}{4\pi\varepsilon_0\left(y^2+\left(\frac{d}{2}\right)^2\right)}\frac{d/2}{\sqrt{y^2+\left(\frac{d}{2}\right)^2}} = \frac{qd}{4\pi\varepsilon_0\left(y^2+\left(\frac{d}{2}\right)^2\right)^{3/2}}$$
(b)	From (a)(i): $$E_A = \frac{q}{4\pi\varepsilon_0 x^2}\left(\frac{1}{\left(1-\frac{d}{2x}\right)^2}-\frac{1}{\left(1+\frac{d}{2x}\right)^2}\right)$$ As x becomes large, $\frac{d}{x}$ becomes small. $$\left(1-\frac{d}{2x}\right)^{-2} \approx 1-2\left(-\frac{d}{2x}\right) = 1+\frac{d}{x}$$ $$\left(1+\frac{d}{2x}\right)^{-2} \approx 1-2\left(\frac{d}{2x}\right) = 1-\frac{d}{x}$$ $$\therefore E_A \approx \frac{q}{4\pi\varepsilon_0 x^2}\left(1+\frac{d}{x}-\left(1-\frac{d}{x}\right)\right) = \frac{q}{4\pi\varepsilon_0 x^2}\frac{2d}{x} = \frac{2qd}{4\pi\varepsilon_0 x^3} \propto \frac{1}{x^3}$$ From (a)(ii): $$E_{net,B} = \frac{qd}{4\pi\varepsilon_0 y^3\left(1+\left(\frac{d}{2y}\right)^2\right)^{3/2}}$$ As y becomes large, $\frac{d}{y}$ becomes small, and $\left(\frac{d}{y}\right)^2$ becomes negligibly small. $$\left(1+\left(\frac{d}{2y}\right)^2\right)^{-3/2} \approx 1-\frac{3}{2}\left(\frac{d}{2y}\right)^2 \approx 1$$ $$\therefore E_{net,B} \approx \frac{qd}{4\pi\varepsilon_0 y^3} \propto \frac{1}{y^3}$$ Therefore, $E \propto 1/r^3$

(Optional) Challenging Questions

Note: these questions are mathematically tedious, and therefore will *not* come out in exams. However, the ideas are the same as whatever will come out. For details on multiple integrals, read Appendix 1.

C1	
	Using the cosine rule, the distance between dq in the sphere and a point along the z-axis is $\sqrt{z^2+r^2-2zr\cos\theta}$ The electric potential due to charge element dq at a distance $r > R$ is therefore:

$$dV = \frac{dq}{4\pi\varepsilon_0\sqrt{z^2 + r^2 - 2zr\cos\theta}}$$

The volume element in spherical coordinates is $dV = (r\ d\theta)(r\sin\theta\ \ d\phi)(dr)$. Since the volume charge density is uniform, $dq = \rho_0\ dV'$

$$\therefore V = \int_{r=0}^{r=R}\int_{\theta=0}^{\theta=\pi}\int_{\phi=0}^{\phi=2\pi}\frac{\rho_0\, r^2\sin\theta\ \ dr\ d\theta\ d\phi}{4\pi\varepsilon_0\sqrt{z^2 + r^2 - 2zr\cos\theta}}$$

This looks scary, but let's do one integral at a time. Notice that ϕ does not appear in the expression, so $\int_0^{2\pi} d\phi = 2\pi$. We factor that out in front:

$$V = \frac{2\pi\rho_0}{4\pi\varepsilon_0}\int_{r=0}^{r=R}\int_{\theta=0}^{\theta=\pi}\frac{r^2\sin\theta\ \ dr\ d\theta}{\sqrt{z^2 + r^2 - 2zr\cos\theta}}$$

Now let's try integrating to get θ. Let $u(\theta) = z^2 + r^2 - 2zr\cos\theta$, then $du = 2zr\sin\theta$. Then we force the numerator so that it looks like dy:

$$V = \frac{\rho_0}{2\varepsilon_0}\int_{r=0}^{r=R}\int_{\theta=0}^{\theta=\pi}\frac{r}{2z}\frac{2zr\sin\theta\ \ dr\ d\theta}{\sqrt{z^2 + r^2 - 2zr\cos\theta}}$$

Since $\int\frac{dy}{\sqrt{y}} = 2\sqrt{y}$,

$$V = \frac{\rho_0}{2\varepsilon_0}\int_{r=0}^{r=R}\left[\frac{r}{2z}\ 2\sqrt{z^2 + r^2 - 2zr\cos\theta}\right]_{\theta=0}^{\theta=\pi} dr = \frac{\rho_0}{2\varepsilon_0}\int_{r=0}^{r=R}\frac{r}{z}\left(\sqrt{z^2 + r^2 + 2zr} - \sqrt{z^2 + r^2 - 2zr}\right)dr$$

$$= \frac{\rho_0}{2\varepsilon_0}\int_{r=0}^{r=R}\frac{r}{z}\big(z + r - (z - r)\big)dr = \frac{\rho_0}{2\varepsilon_0}\int_{r=0}^{r=R}\frac{r}{z}2r\ dr = \frac{\rho_0}{\varepsilon_0 z}\int_{r=0}^{r=R} r^2\ dr$$

Phew! Now the last integral is easy.

$$V = \frac{\rho_0}{\varepsilon_0 z}\int_{r=0}^{r=R} r^2\ dr = \frac{\rho_0}{\varepsilon_0 z}\frac{R^3}{3} = \frac{\left(\frac{Q}{\frac{4}{3}\pi R^3}\right)R^3}{3\varepsilon_0 z} = \frac{Q}{4\pi\varepsilon_0 z}$$

Which is what we'd expect.

(b)

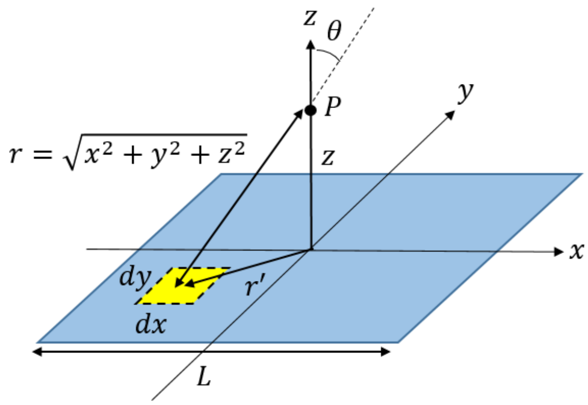
$$\frac{dV}{dz} = -\frac{Q}{4\pi\varepsilon_0 z^2}$$

$$E = -\frac{dV}{dz} = \frac{Q}{4\pi\varepsilon_0 z^2}$$

Again, as expected.

C2

Set the coordinate origin in the middle of the plane.



For an alternative method utilising the result of E due to a line charge, see below.

The magnitude of the electric field dE due to a small patch of charge dq at (x, y) is

$$dE = \frac{1}{4\pi\varepsilon_0}\frac{dq}{x^2+y^2+z^2}$$

Since P is directly above the middle of the plane, the x- and y- components of the electric field cancel out and the net field is only in the positive z-direction, so we only need to find dE_z due to dq.

$$dE_z = dE\cos\theta = \frac{1}{4\pi\varepsilon_0}\frac{dq}{x^2+y^2+z^2}\frac{z}{\sqrt{x^2+y^2+z^2}} = \frac{1}{4\pi\varepsilon_0}\frac{z\,dq}{(x^2+y^2+z^2)^{3/2}}$$

Since $dq = \sigma\,dA = \sigma\,dx\,dy$,

$$dE_z = \frac{1}{4\pi\varepsilon_0}\frac{\sigma\,z\,dx\,dy}{(x^2+y^2+z^2)^{3/2}}$$

$$\therefore E_z = \int_{x=-L/2}^{x=L/2}\int_{y=-L/2}^{y=L/2}\frac{1}{4\pi\varepsilon_0}\frac{\sigma\,z\,dx\,dy}{(x^2+y^2+z^2)^{3/2}}$$

This is a rather painful integral to evaluate, because **there isn't enough symmetry for us to get rid of more of the variables.**

However, if you're one of *those* people who insist on trying painful things, here's how it could be done. You solve a double integral by first integrating over one variable, then the other (the order doesn't matter – the second one will be extremely painful regardless):

$$E_z = \frac{1}{4\pi\varepsilon_0}\int_{x=-L/2}^{x=L/2}\int_{y=-L/2}^{y=L/2}\frac{\sigma\,z\,dx\,dy}{(x^2+y^2+z^2)^{3/2}}$$

$$= \frac{\sigma z}{4\pi\varepsilon_0}\int_{x=-L/2}^{x=L/2}\left[\frac{y}{(x^2+z^2)\sqrt{x^2+y^2+z^2}}\right]_{y=-L/2}^{y=L/2}dx$$

$$= \frac{\sigma z}{4\pi\varepsilon_0}\int_{x=-L/2}^{x=L/2}\frac{L}{(x^2+z^2)\sqrt{x^2+L^2/4+z^2}}\,dx \qquad \text{----------------------(1)}$$

$$= \frac{\sigma z}{4\pi\varepsilon_0}\left[\frac{2}{z}\tan^{-1}\left(\frac{Lx}{z\sqrt{4(x^2+z^2)+L^2}}\right)\right]_{x=-L/2}^{x=L/2}$$

$$= \frac{\sigma}{2\pi\varepsilon_0}\left(\tan^{-1}\left(\frac{L\cdot L/2}{z\sqrt{4(L^2/4+z^2)+L^2}}\right) - \tan^{-1}\left(\frac{L\cdot(-L/2)}{z\sqrt{4(L^2/4+z^2)+L^2}}\right)\right)$$

$$= \frac{\sigma}{2\pi\varepsilon_0}\left(\tan^{-1}\left(\frac{L^2}{2z\sqrt{5L^2+4z^2}}\right) - \tan^{-1}\left(-\frac{L^2}{2z\sqrt{5L^2+4z^2}}\right)\right)$$

Since $\tan^{-1}(-a) = -\tan^{-1}a$,

$$E_z = \frac{\sigma}{\pi\varepsilon_0}\tan^{-1}\left(\frac{L^2}{2z\sqrt{5L^2+4z^2}}\right)$$

Let's check that this is correct. As we're checking the field near the plane, $z \ll L, \frac{z}{L} \to 0$ and $\frac{L}{z} \to \infty$

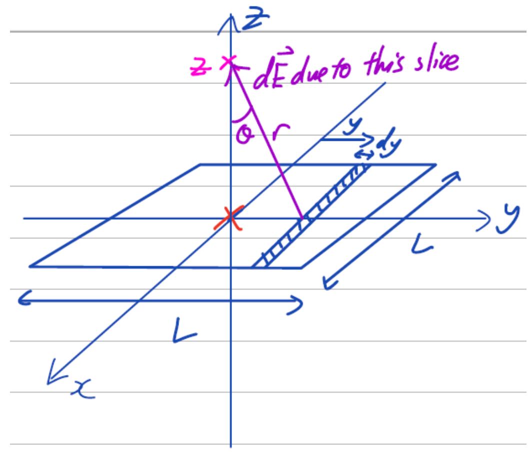
$$E_z = \frac{\sigma}{\pi\varepsilon_0} \tan^{-1}\left(\frac{L^2}{2zL\sqrt{5 + \frac{4z^2}{L^2}}}\right) = \frac{\sigma}{\pi\varepsilon_0} \tan^{-1}\left(\frac{L}{z}\frac{1}{2\sqrt{5 + \frac{4z^2}{L^2}}}\right) \approx \frac{\sigma}{\pi\varepsilon_0} \tan^{-1}\left(\frac{L}{z}\frac{1}{2\sqrt{5}}\right)$$

Since $\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}$

$$E_z \approx \frac{\sigma}{\pi\varepsilon_0}\frac{\pi}{2} = \frac{\sigma}{2\varepsilon_0}$$

Phew! That's the answer we wanted.

Alternatively, you could have sliced the plane of charge into many lines of charge:



In Lecture Example 5(a), we found that the electric field at a distance r due to one line of charge of length L is:

$$E_{line} = \frac{\lambda}{4\pi\varepsilon_0}\frac{2L}{r\sqrt{4r^2 + L^2}}$$

So, the electric field due to a rectangular strip of charge of length L of width dy is:

$$dE = E_{line}\, dy = \frac{\lambda}{4\pi\varepsilon_0}\frac{2L}{r\sqrt{4r^2 + L^2}}\, dy$$

Since $r = \sqrt{y^2 + z^2}$,

$$dE_z = dE\cos\theta = \frac{\sigma L\, dy}{2\pi\varepsilon_0\sqrt{y^2 + z^2}\sqrt{4x^2 + 4y^2 + L^2}}\frac{z}{\sqrt{y^2 + z^2}} = \frac{\sigma L z\, dy}{4\pi\varepsilon_0 (y^2 + z^2)\sqrt{y^2 + L^2/4 + z^2}}$$

$$\therefore E_z = \frac{\sigma z}{4\pi\varepsilon_0}\int_{y=-L/2}^{y=L/2} \frac{L}{(y^2 + z^2)\sqrt{y^2 + L^2/4 + z^2}}\, dy$$

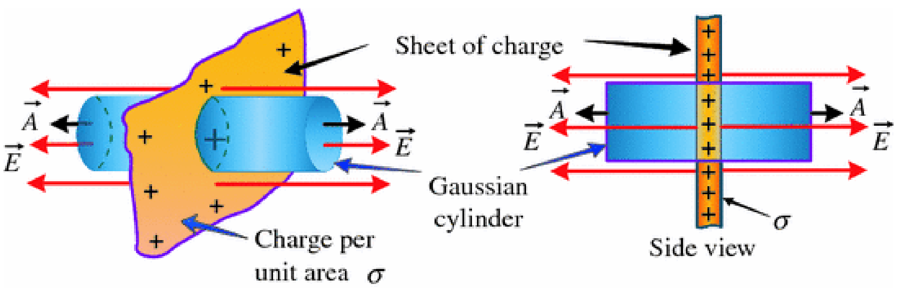
Which is the line marked (1) in the previous solution, swapping x for y. The rest of the solution works out the same.

Tutorial solutions (Part B - Gauss' Law)

Discussion Questions

<table>
<tr><td>D1</td><td></td></tr>
<tr><td>(a)(i)</td><td>
Surface charge density σ = charge per unit area
Hence $Q = \sigma(\pi R^2)$

$$\therefore \quad \sigma = \frac{Q}{\pi R^2}$$
</td></tr>
<tr><td>(a)(ii)</td><td>
Assume the circular thin disc to be an infinitely large uniformly charged circular sheet and by symmetry we say that the field at 'z' is a uniform outgoing field E_z and solve it by taking a gaussian cylinder with the top and bottom above and below the disc with $z \leq R$.



Applying Gauss's Law, let electric field at position (0,0,z) be E_z

$$\oiint E_z \cdot dA = \frac{Q}{\varepsilon_0}$$

$$E_z \oiint dA = \frac{\sigma A}{\varepsilon_0}$$

$$2AE_z = \frac{\sigma A}{\varepsilon_o}$$

$$E_z = \frac{\sigma}{2\varepsilon_o} = 2\pi\sigma\left(\frac{1}{4\pi\varepsilon_o}\right)$$

$$= 2\pi k\sigma$$

$$\text{where } k = \left(\frac{1}{4\pi\varepsilon_o}\right) = 9.0 \times 10^9 \text{ kg m}^3 \text{ C}^{-2}$$

Marker's Comment: All necessary steps must be shown since this is a "show" question. It is necessary to include a diagram with the Gaussian surface drawn with the E field lines drawn
</td></tr>
<tr><td>(iii)</td><td>
Let V_0 be the potential at the origin, and V_z be the potential at (0, 0, z).

$$V_z - V_0 = -\int_0^z E\, dr = -\int_0^z \frac{\sigma}{2\varepsilon_0}\, dr = -\left[\frac{\sigma r}{2\varepsilon_0}\right]_0^z = -\frac{\sigma z}{2\varepsilon_0} + 0$$

Comparing terms, we see that $V_0 = 0$ as we expect.

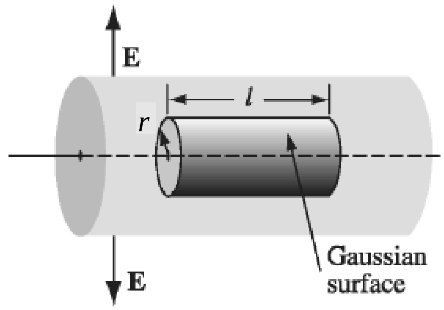
$$\therefore V = -\frac{\sigma z}{2\varepsilon_0}$$

Marker's Comment: The minus sign should not be omitted as it signifies the relationship between V and E.
</td></tr>
</table>

(iv)	In our calculation of Gauss' Law, we assumed that the electric flux is normal to the Gaussian surface that is parallel to the plane. This is only true near to the centre of the plane when d is small enough that edge effects can be ignored.

D2	(Griffiths pg 73-74)

(a) Draw a cylindrical Gaussian surface of radius r and length l:



For $r < R$, using Gauss' Law,

$$\oiint \vec{E}_{in} \cdot d\vec{A} = \frac{Q_{enc}}{\varepsilon_0}$$
$$= \frac{1}{\varepsilon_0}\int \rho \, dV$$
$$= \frac{1}{\varepsilon_0}\int_0^r (kr')(2\pi r' l \, dr')$$
$$= \frac{2\pi k l}{\varepsilon_0}\int_0^r r'^2 \, dr' = \frac{2\pi k l}{3\varepsilon_0} r^3$$
$$E(2\pi r l) = \frac{2\pi k l}{3\varepsilon_0} r^3$$
$$E = \frac{k r^2}{3\varepsilon_0}$$

The electric field points radially away from the axis. Thus the electric flux through the circular ends of the cylinder is zero, and we only need to consider the flux through the curved cylindrical area.

Mathematically, r cannot both be in the limits of the integral as well as be a variable to be integrated, so relabel the variable as r' and leave r in the limit.

For $r > R$, using Gauss' Law,

$$\oiint \vec{E}_{in} \cdot d\vec{A} = \frac{Q_{enc}}{\varepsilon_0} = \frac{1}{\varepsilon_0}\int \rho \, dV = \frac{1}{\varepsilon_0}\int_0^R (kr)(2\pi r l \, dr) = \frac{2\pi k l}{3\varepsilon_0} R^3$$
$$E(2\pi r l) = \frac{2\pi k l R^3}{3\varepsilon_0}$$
$$E = \frac{k R^3}{3\varepsilon_0 r}$$

$$E = \begin{cases} \dfrac{1}{3\varepsilon_0} k r^2 & r \le R \\ \dfrac{k R^3}{3\varepsilon_0 r} & r \ge R \end{cases}$$

Check: the two expressions should be equal at $r = R$, and from Appendix 3, outside the cylinder (wire) $E \propto 1/r$.

(b)

As the cylinder is infinitely long, $V \neq 0$ at infinity. So we choose a reference point $r = a > R$ as our reference (i.e. $V = 0$ when $r = a$).

If $r \geq R$,

$$V = -\int_a^r E\, dr' = -\int_a^r \frac{kR^3}{3\varepsilon_0 r'}\, dr' = -\frac{kR^3}{3\varepsilon_0}\ln\frac{r}{a}$$

(Note that if $a \to \infty, \ln\frac{r}{a} \to -\infty$)

If $r \leq R$,

$$\begin{aligned} V &= -\int_a^r E\, dr' \\ &= -\int_a^R \frac{kR^3}{3\varepsilon_0 r'}\, dr' - \int_R^r \frac{1}{3\varepsilon_0}kr^2\, dr' \\ &= -\frac{kR^3}{3\varepsilon_0}\ln\frac{R}{a} - \frac{k}{9\varepsilon_0}(r^3 - R^3) \end{aligned}$$

> As E is a piecewise function, we need to split the integral at $r' = R$ and perform the integrals separately.

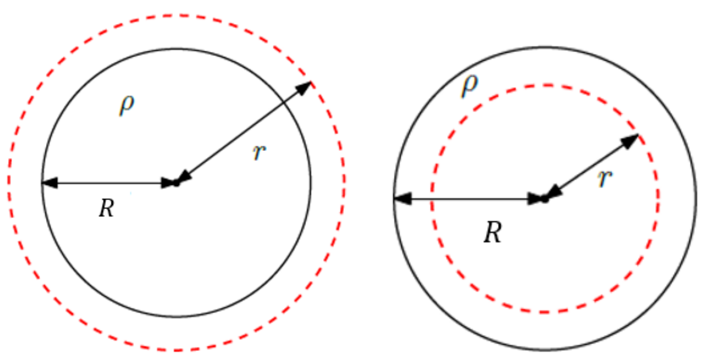
$$\therefore V = \begin{cases} -\dfrac{kR^3}{3\varepsilon_0}\ln\dfrac{r}{a} & r \geq R \\ -\dfrac{kR^3}{3\varepsilon_0}\ln\dfrac{R}{a} - \dfrac{k}{9\varepsilon_0}(r^3 - R^3) & r \leq R \end{cases}$$

If the a bothers you, try differentiating V – you'll see that it disappears, so that E is independent of a, as we'd want. This pesky extra term only crops up when you have an infinite amount of charge, because then the electric potential at infinite will no longer be zero.

D3

(a)

Draw a spherical Gaussian surface:



For $r > R$, using Gauss' Law,

$$\oiint \vec{E}_{out} \cdot d\vec{A} = \frac{Q_{enc}}{\varepsilon_0} = \frac{1}{\varepsilon_0}\,\rho V$$

$$E_{out}(4\pi r^2) = \frac{1}{\varepsilon_0}\,\rho\left(\frac{4}{3}\pi R^3\right)$$

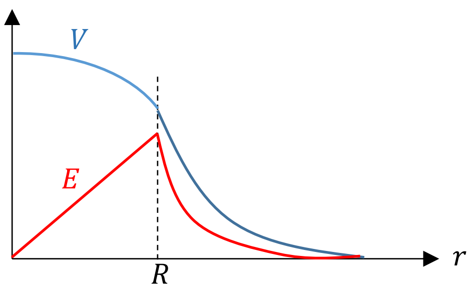
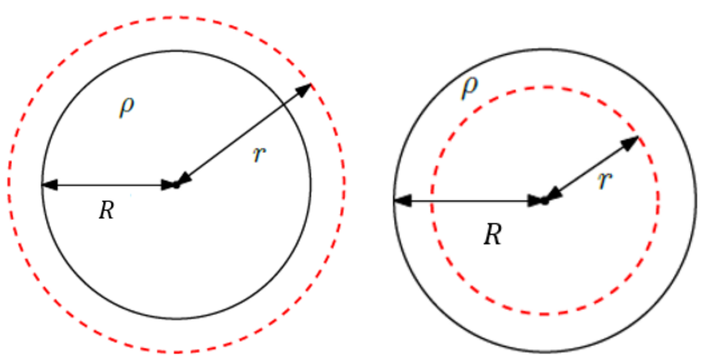
$$E_{out} = \frac{\rho R^3}{3\varepsilon_0 r^2}$$

For $r < R$, using Gauss' Law,

$$\oiint \vec{E}_{in} \cdot d\vec{A} = \frac{Q_{enc}}{\varepsilon_0} = \frac{1}{\varepsilon_0}\int \rho\, dV = \frac{1}{\varepsilon_0}\int 4\pi r^2\,\rho\, dr$$

$$E_{in}(4\pi r^2) = \frac{1}{\varepsilon_0}\,\rho\left(\frac{4}{3}\pi r^3\right)$$

$$E_{in} = \frac{\rho r}{3\varepsilon_0}$$

	$$\therefore E = \begin{cases} \dfrac{\rho r}{3\varepsilon_0} & r \le R \\ \dfrac{\rho R^3}{3\varepsilon_0 r^2} & r \ge R \end{cases}$$
(b)	Since there is a finite amount of charge, we can safely choose infinity as the reference point[1] because $V_\infty = 0$. If $r \ge R$, $$V = -\int_\infty^r E\,dr' = -\int_\infty^r \frac{\rho R^3}{3\varepsilon_0 r'^2}\,dr' = \frac{\rho R^3}{3\varepsilon_0 r}$$ If $r \le R$, $$V = -\int_\infty^r E\,dr' = -\int_\infty^R \frac{\rho R^3}{3\varepsilon_0 r'^2}\,dr' - \int_R^r \frac{\rho r'}{3\varepsilon_0}\,dr' = \frac{\rho R^3}{3\varepsilon_0 R} - \left[\frac{\rho r'^2}{6\varepsilon_0}\right]_R^r = \frac{\rho R^2}{3\varepsilon_0} - \frac{\rho r^2}{6\varepsilon_0} + \frac{\rho R^2}{6\varepsilon_0} = \frac{\rho(3R^2 - r^2)}{6\varepsilon_0}$$ $$\therefore V = \begin{cases} \dfrac{\rho R^3}{3\varepsilon_0 r} & r \ge R \\ \dfrac{\rho(3R^2 - r^2)}{6\varepsilon_0} & r \le R \end{cases}$$
(c)	
(d)	If $\rho = kr$, repeating the same calculations and the same Gaussian surfaces,  Using Gauss Law for $r > R$, $$\oiint \vec{E}_{out} \cdot d\vec{A} = \frac{Q_{enc}}{\varepsilon_0} = \frac{1}{\varepsilon_0}\int \rho\,dV = \frac{1}{\varepsilon_0}\int_0^R kr\,(4\pi r^2\,dr) = \frac{4\pi k}{\varepsilon_0}\int_0^R r^3\,dr = \frac{4\pi k}{\varepsilon_0}\frac{R^4}{4}$$ $$E_{out}(4\pi r^2) = \frac{\pi k R^4}{\varepsilon_0}$$ $$E_{out} = \frac{kR^4}{4\varepsilon_0 r^2}$$

[1] See solution to B2(b).

Using Gauss Law for $r < R$,

$$\oiint \vec{E}_{in} \cdot d\vec{A} = \frac{Q_{enc}}{\varepsilon_0} = \frac{1}{\varepsilon_0}\int \rho\, dV = \frac{1}{\varepsilon_0}\int_0^r kr' \left(4\pi r'^2\, dr'\right) = \frac{4\pi k}{\varepsilon_0}\frac{r^4}{4}$$

$$E_{in}(4\pi r^2) = \frac{\pi k r^4}{\varepsilon_0}$$

$$E_{in} = \frac{kr^2}{4\varepsilon_0}$$

$$\therefore E = \begin{cases} \dfrac{kr^2}{4\varepsilon_0} & r \leq R \\ \dfrac{kR^4}{4\varepsilon_0 r^2} & r \geq R \end{cases}$$

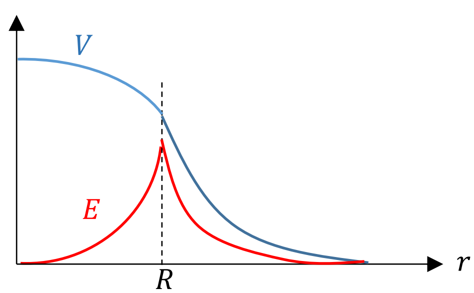
Since there is a finite amount of charge, we can safely choose infinity as the reference point because $V_\infty = 0$.
If $r \geq R$,

$$V = -\int_\infty^r E\, dr' = -\int_\infty^r \frac{kR^4}{4\varepsilon_0 r'^2} dr' = \frac{kR^4}{4\varepsilon_0 r}$$

If $r \leq R$,

$$V = -\int_\infty^r E\, dr' = -\int_\infty^R \frac{kR^4}{4\varepsilon_0 r'^2} dr' - \int_R^r \frac{kr^2}{4\varepsilon_0}\, dr' = \frac{kR^4}{4\varepsilon_0 R} - \left[\frac{kr'^3}{12\varepsilon_0}\right]_R^r = \frac{kR^3}{4\varepsilon_0 r} - \frac{kr^3}{12\varepsilon_0} + \frac{kR^3}{12\varepsilon_0} = \frac{k(4R^3 - r^3)}{12\varepsilon_0}$$

$$\therefore V = \begin{cases} \dfrac{kR^4}{4\varepsilon_0 r} & r \geq R \\ \dfrac{k(4R^3 - r^3)}{12\varepsilon_0} & r \leq R \end{cases}$$



D4

We can achieve the same effect of carving out the spherical hole by superposing a sphere of radius R with a uniform charge density $-\rho$ where the hole is. This corresponds to a charge of $-Q$.

Consider the large sphere of radius $2R$ before the cavity was carved. Repeating the calculations we did in D3, we get:

$$E_{2R} = \begin{cases} \dfrac{\rho(2R)^3}{3\varepsilon_0 r^2}, & r \geq 2R \\ \dfrac{\rho r}{3\varepsilon_0}, & r \leq 2R \end{cases}$$

For the small negatively-charged sphere of radius R we get:

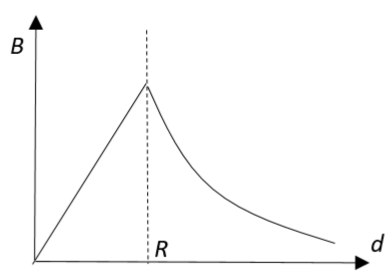
$$E_R = \begin{cases} \dfrac{\rho R^3}{3\varepsilon_0 r^2}, & r \geq R \\ \dfrac{\rho r}{3\varepsilon_0}, & r \leq R \end{cases}$$

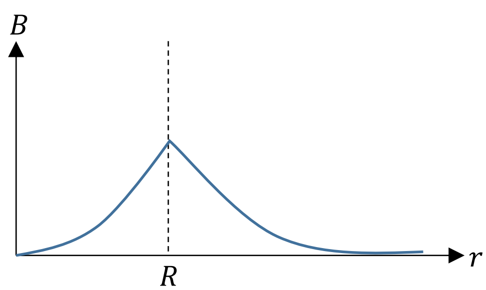
	At A, $E_{2R} = 0$ and $E_R = \frac{\rho R}{3\varepsilon_0}$, so: $E_A = \dfrac{\rho R}{3\varepsilon_0} \quad \text{upwards}$ At B, $E_{2R} = \frac{\rho(2R)}{3\varepsilon_0}$ downwards and $E_R = \frac{\rho R^3}{3\varepsilon_0(3R)^2} = \frac{\rho}{27\varepsilon_0 R}$ upwards, so: $E_B = \dfrac{2\rho R}{3\varepsilon_0} - \dfrac{\rho}{27\varepsilon_0 R} = \dfrac{17\rho R}{27\varepsilon_0} \quad \text{downwards}$

D5	
(a)	Consider a Gaussian cuboid centered at the origin, with a thickness $2x$. By Gauss's Law, $\oiint \vec{E} \cdot d\vec{A} = \dfrac{Q_{enc}}{\varepsilon_0}$ Since the electric field of the slab inside the Gaussian surface points to the left and right parallel to the x-axis, only the left and right surfaces with area A contribute to the flux, $\oiint \vec{E} \cdot d\vec{A} = 2EA$ Since $Q_{enc} = \rho V = \rho A(2x)$, $2EA = \dfrac{\rho A 2x}{\varepsilon_0}$ $\therefore E = \dfrac{\rho x}{\varepsilon_0}$ OR Slice the slab into many large planes of infinitesimal thickness. Each of these slices sets up an electric field with field strength: $dE = \dfrac{\sigma}{2\varepsilon_0} = \dfrac{\rho\, dx}{2\varepsilon_0}$ Take right as positive. At a point $-\frac{d}{2} < x < \frac{d}{2}$ the electric field will be: $E = \displaystyle\int_{-\frac{d}{2}}^{x} \frac{\rho\, dx'}{2\varepsilon_0} - \int_{x}^{\frac{d}{2}} \frac{\rho\, dx'}{2\varepsilon_0}\, dx' = \left[\frac{\rho x'}{2\varepsilon_0}\right]_{-\frac{d}{2}}^{x} - \left[\frac{\rho x'}{2\varepsilon_0}\right]_{x}^{\frac{d}{2}} = \frac{\rho x}{\varepsilon_0}$
(b)	The force that acts on the electron in the slab when it is at a distance x from the center is given by $m_e a = qE = -e\dfrac{\rho x}{\varepsilon_0}$ Hence comparing with characteristic equation of simple harmonic motion $a = -\omega^2 x$ We have $f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{\rho e}{m_e \varepsilon_0}}$

Tutorial solutions (Part C - Ampere's Law)

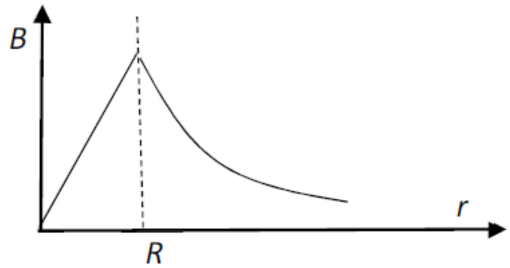
Discussion Questions

D1	
(a)	Ampere's Law states that $\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_{enc}$ where $\vec{B}$ is the magnetic flux density, $d\vec{\ell}$ is a line element along the integration path, μ_0 is the magnetic permeability of free space and I_{enc} is the current passing through the area enclosed by the integration path. OR where $\oint \vec{B} \cdot d\vec{\ell}$ is the net magnetic flux *along* the loop, μ_0 is the magnetic permeability of free space and I_{enc} is the enclosed current. *Note: $\oint \vec{B} \cdot d\vec{\ell}$ is the magnetic flux **along** the loop, not through or of the loop.* *Do not confuse **permeability** (μ) with **permittivity** (ε)!*
(b)(i)	Draw a circular Amperian loop of radius d centred on the wire. When $d > R$: $$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_{enc}$$ $$B(2\pi d) = \mu_0 I$$ $$B = \frac{\mu_0 I}{2\pi d}$$
(b)(ii)	Let the current density be J, and assuming that it is uniform across the cross-section of the wire: $$J = \frac{I}{\pi R^2} = \text{constant}$$ When $d < R$, $$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_{enc} = \mu_0 J(\pi d^2)$$ $$B(2\pi d) = \mu_0 \frac{I d^2}{\pi R^2}$$ $$B = \frac{\mu_0 I}{2\pi R^2} d$$ As $J =$ constant, no integration is needed: we can simply take $I_{enc} = J \times Area\ of\ loop$ i.e. $B \propto d$ when $d < R$. 
(b)(iii)	A magnetic field exists inside the wire as the current flows through all parts of the cross section of the wire, hence when we apply Ampere's Law over a circular path inside the wire, the path will enclose a fraction of the current that flows in the wire resulting in a non-zero magnetic field in the wire.

D2	
(a)	Draw a circular Amperian loop of radius *r* centred on the axis of the wire. The magnetic field is constant along this loop. Applying Ampere's Law, within the conductor, $r < R$, $$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_{enc} = \mu_0 \int_0^r J(2\pi r')dr'$$ $$B(2\pi r) = 2\pi\mu_0 \int_0^r (br')r'dr'$$ $$B = \frac{\mu_0 b}{r}\int_0^r r'^2 dr' = \frac{\mu_0 br^2}{3}$$ As J varies with r, integration is needed: $I_{enc} = \int J\, dA$ over the cross-sectional area of the loop. $\frac{dA}{dr'} = 2\pi r'$ so $dA = 2\pi r' dr'$ Outside the conductor, $r > R$, $$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_{enc} = \mu_0 \int_0^r J(2\pi r')dr'$$ $$B(2\pi r) = 2\pi\mu_0 \int_0^R (br)r\, dr$$ $$B = \frac{\mu_0 bR^3}{3r}$$ $$\therefore B = \begin{cases} \frac{\mu_0 br^2}{3} & r \le R \\ \frac{\mu_0 bR^3}{3r} & r \ge R \end{cases}$$
(b)	

D3	
(i)	Inside the wire, consider an Amperian circular loop of radius *r* centered around the axis of the wire. Due to symmetry, the magnetic field on this loop is constant. Using Ampere's Law, $$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_{enc}$$ $$B(2\pi r) = \mu_0 J\pi r^2$$ $$B = \frac{\mu_0 J\pi r^2}{2\pi r} = \frac{\mu_0 Jr}{2}$$ Outside the wire, using Ampere's Law, $$B(2\pi r) = \mu_0 J\pi R^2$$ $$B = \frac{\mu_0 JR^2}{2r}$$

(ii)



(iii)

As the wire carries a current I that is uniformly distributed throughout the wire (with the cavity), the current density J in the wire is given by

$$J = \frac{I}{(\pi R^2 - \pi(0.1R)^2)} = \frac{I}{0.99\pi R^2} = 0.322\frac{I}{R^2}$$

(iv)

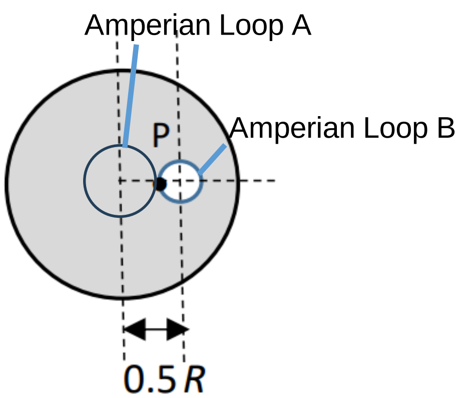
The magnetic field at P due to a uniform wire of radius R with no cavity is given by

$$B_{P0} = \frac{\mu_0 J r}{2} = \frac{\mu_0}{2}\left(0.322\frac{I}{R^2}\right)(0.4R) = 0.0644\mu_0 I/R$$

The magnetic field at P due to a uniform wire of radius 0.1R (centered at the center of cavity) is given by

$$B_{P1} = \frac{\mu_0 J R}{2} = \frac{\mu_0}{2}\left(0.322\frac{I}{R^2}\right)(0.1R) = 0.0161\mu_0 I/R$$

The net magnetic field at P is given by the superposition of the magnetic field at P due to a uniform current in the original wire and the magnetic field due to a uniform current in the ***opposite direction*** *flowing through the cavity.*



Assume the positive current to mean current flows into page and negative current to mean current flows out of page.

Using right hand grip rule, the positive current flowing within the Amperian loop A will cause a clockwise B-field along the Amperian loop. Hence direction of B_{P0} is vertically down.

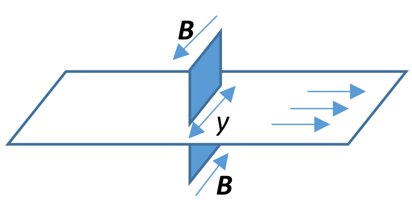
Using right hand rule, the negative current flowing within the Amperian loop B will cause an anti-clockwise B-field along the Amperian Loop. Hence direction of B_{P1} is also vertically down.

$$B_P = B_{P0} + B_{P1} = 0.0805\ \mu_0 I/R$$

D4

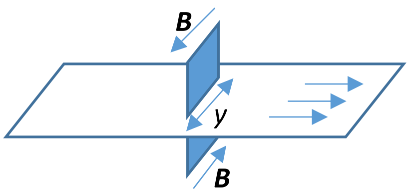
(a)

Consider a rectangular Amperian loop of width y across the belt as shown in the diagram,

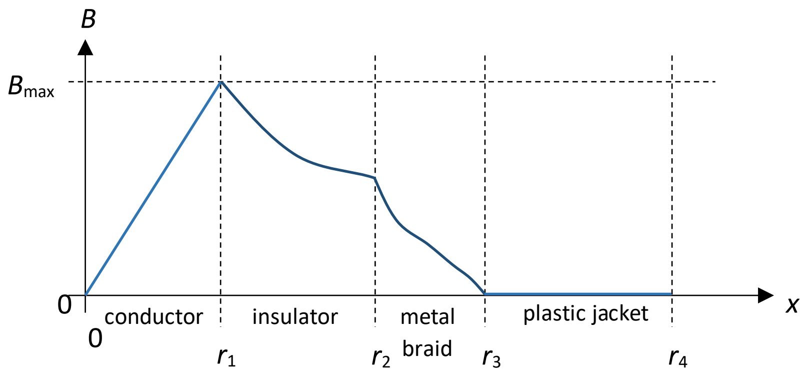


In time dt, dq of charge moves a distance dx. The enclosed current is thus given by:

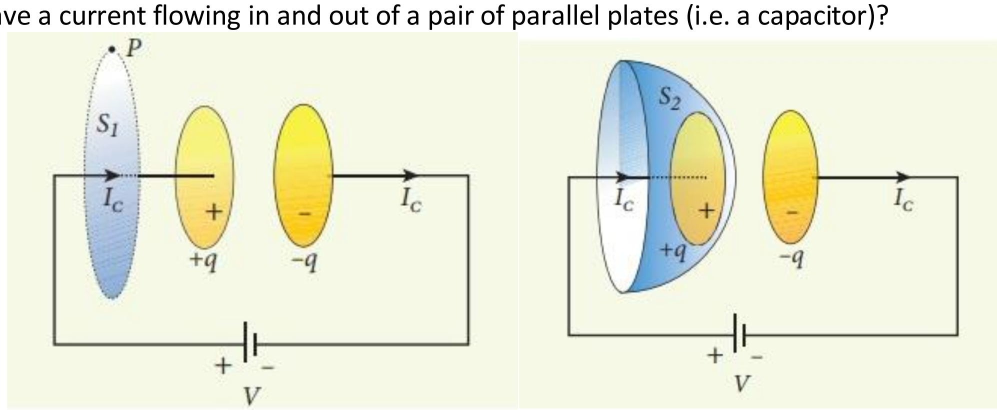
$$I_{enc} = \frac{dq}{dt} = \frac{\lambda\, dx}{dt} = \frac{\sigma y\, dx}{dt} = \sigma y v$$

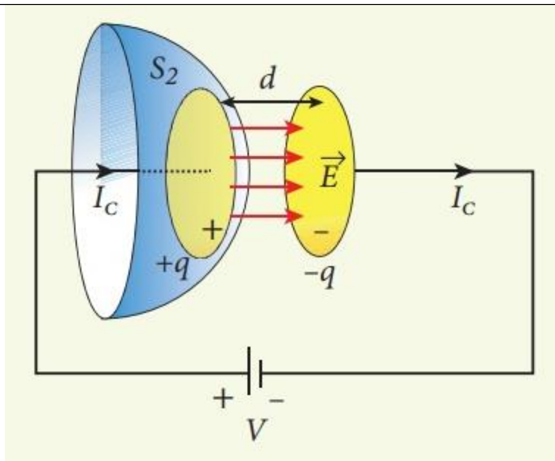
	Apply Ampere's Law to the loop, $\oint \vec{B}\cdot d\vec{\ell} = \mu_0 I_{enc}$ $2By = \mu_0 \sigma y v$ $\therefore B = \dfrac{\mu_0 \sigma v}{2}$ $B = \begin{cases} \dfrac{\mu_0 \sigma v}{2} & \text{out of the paper} \quad \text{(above the sheet)} \\ \dfrac{\mu_0 \sigma v}{2} & \text{into of the paper} \quad \text{(below the sheet)} \end{cases}$
(b)	 $I_{enc} = Ky$ Using Ampere's Law, $\oint \vec{B}\cdot d\vec{\ell} = \mu_0 I_{enc}$ $2By = \mu_0 K y$ $B = \begin{cases} \dfrac{\mu_0 K}{2} & \text{out of the paper} \quad \text{(above the sheet)} \\ \dfrac{\mu_0 K}{2} & \text{into of the paper} \quad \text{(below the sheet)} \end{cases}$

D5	
(a)	$\oint \vec{B}\cdot d\vec{\ell} = \mu_0 I_{enc}$ Where $\oint \vec{B}\cdot d\vec{\ell}$ is the net magnetic flux *along* the loop, μ_0 is the magnetic permeability of free space and I_{enc} is the enclosed current.
(b)	Since $I_1 = I_2$, $\int J_1\, dA = \int J_2\, dA$ $\int_0^{r_1} J_1\,(2\pi r\, dr) = \int_{r_2}^{r_3} J_2\,(2\pi r\, dr)$ $\pi r_1^2 J_1 = \pi(r_3^2 - r_2^2) J_2$ $J_2 = \dfrac{r_1^2}{r_3^2 - r_2^2} J_1 \quad \text{(shown)}$
(c)(i)	Draw a circular Amperian loop of radius r centred on the axis of the wire, where $0 \le r \le r_1$ $\oint \vec{B}\cdot d\vec{\ell} = \mu_0 I_{enc} = \mu_0 I_{1,enc} = \mu_0 \pi r^2 J_1$ $B(2\pi r) = \mu_0 \pi r^2 J_1$ $B = \dfrac{1}{2}\mu_0 J_1 r$
(c)(ii)	Draw a circular Amperian loop of radius r centred on the axis of the wire, where $r_1 \le r \le r_2$ $\oint \vec{B}\cdot d\vec{\ell} = \mu_0 I_{enc} = \mu_0 I_1 = \mu_0 \pi r_1^2 J_1$ $B(2\pi r) = \mu_0 \pi r_1^2 J_1$

	$$B=\frac{\mu_0 J_1 r_1^2}{2r}$$
(c)(iii)	Draw a circular Amperian loop of radius r centred on the axis of the wire, where $r_3 \leq r \leq r_4$ $$\oint \vec{B}\cdot d\vec{\ell}=\mu_0 I_{enc}=\mu_0(I_1-I_2)$$ Since $I_1=I_2=I$, $$B(2\pi r)=0$$ $$\therefore B=0$$
(d)	 *Marker's Comment: From r_2 to r_3, B takes the form $B=\frac{a}{r}-br$, where the first term initially dominantes, followed by the second term. Hence the first part of the graph is an inverse graph, followed by a linear graph. However, simply drawing a curve will suffice.*
(e)	The magnetic field outside a standard transmission cable, unlike that of a coaxial cable, is *not* zero when a current is flowing through it. A high frequency signal will mean that energy will be dissipated in metal components in the surroundings through electromagnetic induction.

(Optional) Challenging Questions

C1	They are all equivalent! Intuitively, all the current that flows through the flat circular area, also flows though any other open surface you can draw, so the current enclosed by the loop is the same. And any current that doesn't flow through the flat circular area, but happens to flow through one of the other surfaces (e.g. S2), also flows out again, so there's zero net contribution. There is one notable problem though (which is explicitly *not* in the H3 syllabus, so fret not): what happens when you have a current flowing in and out of a pair of parallel plates (i.e. a capacitor)?  Choosing the flat circular area (left), you get a current as expected. But choosing a surface that encompasses the plate (right), apparently no current flows through it. But Maxwell noticed that something *is* changing through the surface, even though there is no current: the electric field!

	 So he simply added a term to the RHS of Ampere's Law that accounts to the change in electric flux, and all was well. $\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_{enc} + \mu_0 I_d$ This is known as Maxwell's correction to Ampere's Law (the correction term is the "displacement current"), and **the H3 syllabus explicitly states that you don't need to know this. So this situation will *not* come out in the A-level exam, nor any school prelim papers.** But you took H3 to learn new physics, not just to get an "A", so now you are smarter! :)

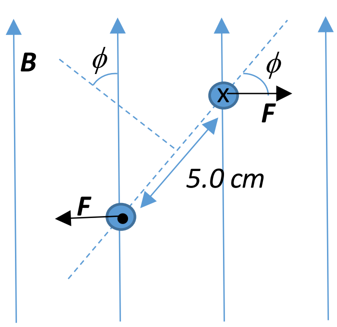
C2	(This can be found in Griffiths, page 236-237)
(a)	The system is cylindrically symmetric, so the magnetic field must also be cylindrically symmetric. Suppose $\vec{B}$ has a radial component, $\vec{B_r}$. When the current were flowing in one direction, suppose $\vec{B_r}$ is positive (e.g. radially away from the axis). If the current direction is reversed, the direction of $\vec{B_r}$ would be reversed too. Changing the direction of the current is physically equivalent to turning the solenoid upside down, but we know in real life that that does not affect the direction of the magnetic field. Therefore, $\vec{B_r} = 0$.
(b)	Let the circumferential component be $\vec{B}_\phi$. If it exists, it will be tangential to Amperian Loop 3 (which is a circle) Applying Ampere's Law to Amperian Loop 3, $\oint_{\mathcal{C}_3} \vec{B} \cdot d\vec{\ell} = B_\phi (2\pi s) = \mu_0 I_{enc}$ But $I_{enc} = 0$ because the loop encloses no current (the current is parallel to the plane of the loop) $\therefore B_\phi = 0$
(c)	In cylindrical coordinates, $\vec{B} = \vec{B_r} + \vec{B_\phi} + \vec{B_z}$ (just like how in Cartesian coordinates we can write $\vec{B} = \vec{B_x} + \vec{B_y} + \vec{B_z}$). But from (a) and (b), we've seen that $\vec{B_r} = 0$ and $\vec{B_\phi} = 0$. Thus $\vec{B}$ points only upwards or downwards. In general, we expect that value of B_z *may* depend on how far it is from the axis of the solenoid, i.e. $B_z = B_z(r)$. Furthermore, we expect that inside the solenoid, $\vec{B}$ points in one direction (e.g. upwards), and outside the solenoid it points in the other direction (e.g. downwards).

	Applying Ampere's Law to Amperian Loop 1, $$\oint_{C_1} \vec{B}\cdot d\vec{\ell} = B_z(a)L + 0 - B_z(b)L + 0 = B_z(a) - B_z(b) = \frac{\mu_0 I_{enc}}{L}$$ (Note that the two zeroes are because $B_z \perp d\vec{\ell}$, and the negative sign on the third term is because of the direction we are traversing the loop: on one side of the rectangle we are moving in the same direction as $\overrightarrow{B_z}$, at the other side we are moving in the opposite direction) But $I_{enc} = 0$ for any $b > a > R$. $\therefore \vec{B} = 0$ everywhere outside the solenoid
(d)	Applying Ampere's Law to Amperian Loop 1, $$\oint_{C_1} \vec{B}\cdot d\vec{\ell} = BL + 0 + 0 + 0 = \mu_0 I_{enc}$$ (where two of the zeroes are because $B_z \perp d\vec{\ell}$, and the third zero is because $\vec{B} = 0$ outside the solenoid.) If the loop encloses N turns of wire, then $I_{enc} = NI$. If $n = N/L$, then $I_{enc} = \mu_0 nL$ $$\therefore BL = \mu_0 nLI$$ $$\therefore B = \mu_0 nI$$

Tutorial solutions (Part D - Dipoles in Fields)

Discussion Questions

D1	
(a)(i)	Field lines normal to surface AND directed towards surface ; No lines inside the conductor ;
(a)(ii)	Charged particles are on the surface AND charge density is greater where the radius of curvature is smaller ;
(b)(i)	$\Phi = \iint E.dA = Q/\varepsilon_0$; $\iint E.dA = 4\pi r^2 E(r)$; $Q = 4/3\ \pi R^3\rho$; $E(r) = (4/3)\ \pi R^3\rho / \varepsilon_0 4\pi r^2 = R^3\rho / \varepsilon_0 3r^2$;
(b)(ii)	inside have $Q = 4/3\ \pi r^3\rho$; so $E(r) = r\rho/3\varepsilon_0$;
(b)(iii)	linear from origin (directly proportional) to R ; matches at $r = R$ (continuous but not smooth), R labelled ; $1/r^2$ for $r > R$;
(c)(i)	$p = Qd$; $= 4/3\ \pi R^3\rho\ (2R + L)$; dipole at 45 degrees to uniform field ;
(c)(ii)	dipole at 45 degrees to uniform field ; forces on charges along field lines in correct direction ;
(c)(iii)	$\tau = Ep\sin\theta$;
(c)(iv)	The dipole moment is parallel or anti-parallel to the field ; The dipole will settle in the orientation where an arrow drawn from the negative charge to the positive charge is in the same direction as the electric field lines ; Appropriate explanation of stable equilibrium, e.g. dipole experiences a restoring torque to this orientation if displaced slightly from it/electric potential energy of the dipole is lowest ($U = -\vec{p}.\vec{E}$) ;

D2	Sketch of the system: 
(a)(i)	$\mu = IA = (2.00)(0.05)^2 = 0.0050\ \text{A m}^2$ $\tau = \mu B \sin\phi = (0.0050)(0.830)\sin 5° = 0.000361696 \approx 3.62 \times 10^{-4}$ N m clockwise

(a)(ii)

To show that the oscillation is simple harmonic, we need to show that $\alpha \propto -\phi$.
Let the moment of inertia of the loop be i. By Newton's 2nd Law ($\tau = i\alpha$):

$$\tau = \mu B \sin\phi = i\alpha$$

$$\therefore \alpha = \frac{\mu B}{i}\sin\phi$$

Since $\phi = 5°$ is small, $\sin\phi \approx \phi$

$$\therefore \alpha \approx \frac{\mu B}{i}\phi \Longrightarrow \alpha \propto \phi$$

The torque acts in the opposite direction to the angular displacement. ∴ this is a simple harmonic oscillation.
Since $\alpha = -\omega^2\phi$,

$$\omega = \sqrt{\frac{\mu B}{i}}$$

$$\frac{2\pi}{T} = \sqrt{\frac{\mu B}{i}}$$

$$T = 2\pi\sqrt{\frac{i}{\mu B}}$$

(moment of inertia of rod about the centre: $\frac{1}{12}mL^2$. Moment of inertia of point mass at distance r is mr^2)
The moment of inertia of the square loop is:

$$i = 2\left(m\left(\frac{L}{2}\right)^2\right) + 2\left(\frac{1}{12}mL^2\right) = 2mL^2\left(\frac{1}{4}+\frac{1}{12}\right) = 2(0.020)(0.0500)^2\frac{1}{3} = 3.33\times10^{-5}\ \mathrm{kg\ m^2}$$

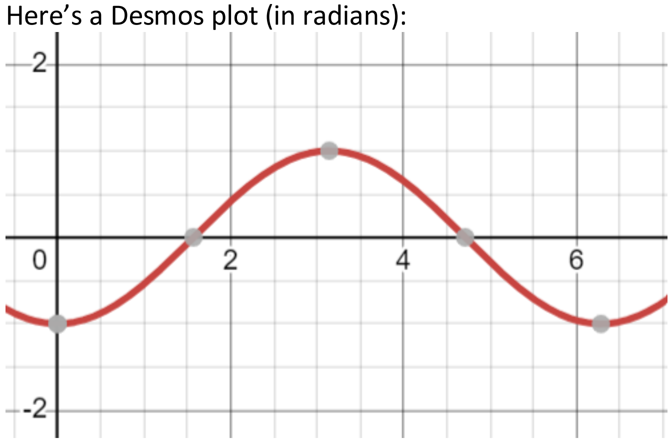
$$\therefore T = 2\pi\sqrt{\frac{3.33\times10^{-5}}{(0.0050)(0.830)}} = 0.563\ \mathrm{s}$$

(b)

$$\Delta U = \int_{\phi_1}^{\phi_2}\tau\,d\phi = \int_{\phi_1}^{\phi_2}\mu B\sin\phi\ d\phi = -\mu B\cos\phi_2 - (-\mu B\cos\phi_1)$$

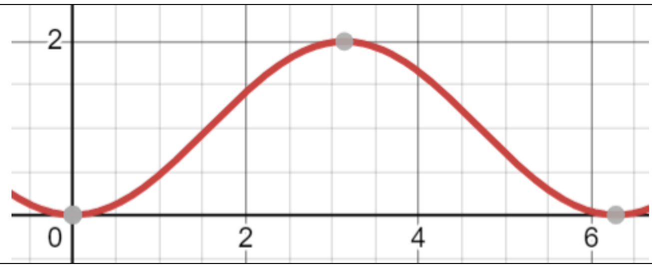
$$\therefore U = -\mu B\cos\phi$$

Plot a $y = -\cos x$ curve, with minima at $\phi = 0°$ and $360°$ and maxima at $\phi = 180°$
Here's a Desmos plot (in radians):



(note that the horizontal axis is in radians.)

Alternatively, if you decided to pick $\phi = 0°$ as the reference (zero) potential: (this is similar to choosing a different reference height to be $h = 0$ in the formula $GPE = mgh$)



(c)

There are equilibrium points at $\phi = 0°$ (or $360°$) and $180°$ because at those points, $|\tau| = \left|\frac{dU}{d\phi}\right| = 0$

There is a stable equilibrium point at $0°/360°$. A small displacement to the left or right will result in a restoring torque back to the equilibrium point.

There is an unstable equilibrium point at $180°$. A small displacement to the left or right will result in a torque pushing the loop further from the equilibrium point.

D3

Magnitude of electric dipole moment:

$$p = |q|d = 2.0 \times 10^{-10}\,\mathrm{C\,m}$$

Magnitude of torque exerted on the dipole with respect to the center of mass by the field:

$$\tau = pE\,|\sin\theta| = 1.0 \times 10^{-6}\,\mathrm{N\,m}$$

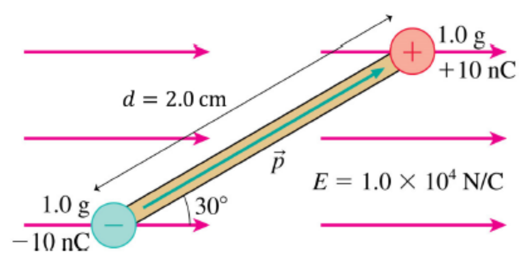
Moment of inertia with respect to the center of mass:

$$I = m_1 r_1^2 + m_2 r_2^2 = 2.0 \times 10^{-7}\,\mathrm{kg\,m^2}$$

Magnitude of angular acceleration: rod rotates about its center of mass

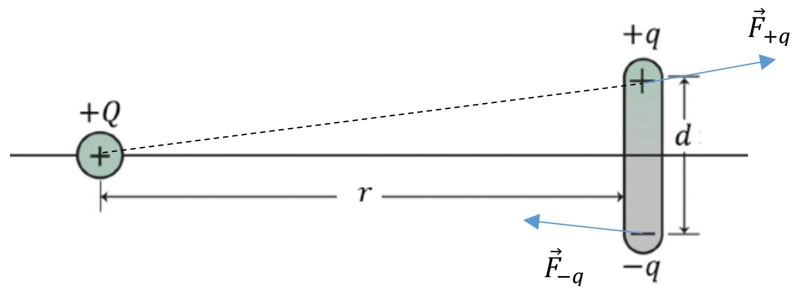
$$\tau = I\alpha \quad \Rightarrow \quad \alpha = \frac{\tau}{I} = 5.0\,\mathrm{rad/s^2}$$

The torque and the angular acceleration will decrease as the rod rotates toward alignment with $\vec{\mathbf{E}}$.



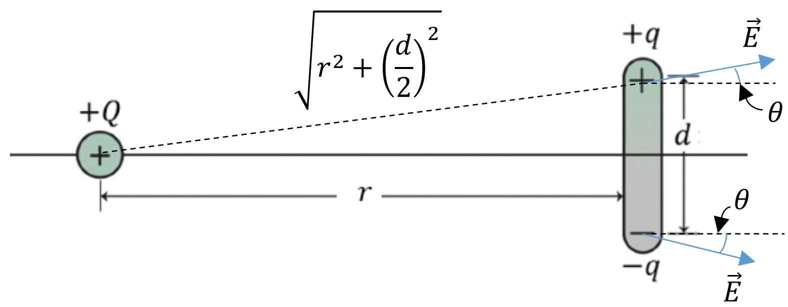
D4

(i)



Direction of net torque must be clockwise as $\vec{F}_{+q}$ and $\vec{F}_{-q}$ both give clockwise torques.
Direction of net force is upwards, because both $\vec{F}_{+q}$ and $\vec{F}_{-q}$ have a component in the upwards direction, and their left and right components cancel out.

(ii)



$$\vec{\tau} = \vec{p} \times \vec{E}$$
$$\vec{p} = q \cdot \vec{d}$$

At $+q$ and at $-q$, the magnitude of the electric field due to Q is:

$$E = \frac{Q}{4\pi\varepsilon_0} \frac{1}{r^2 + \left(\frac{d}{2}\right)^2} = \frac{Q}{4\pi\varepsilon_0} \frac{1}{r^2\left(1 + \left(\frac{d}{2r}\right)^2\right)}$$

Since $r \gg d, \frac{d}{r} \approx 0$ (we completely neglect even higher powers of d/r) and $\theta \approx 0$ (the direction of $\vec{E}$ is approximately parallel to the horizontal). So:

$$E \approx \frac{Q}{4\pi\varepsilon_0} \frac{1}{r^2}$$

Since $\vec{E} \perp \vec{p}$,

$$|\vec{\tau}| = \left|\vec{p} \times \vec{E}\right| = pE = (qd)\frac{Q}{4\pi\varepsilon_0 r^2}$$

From the right hand rule for cross products, $\vec{\tau}$ is directed into the page (i.e. clockwise torque)

$$\therefore \tau = \frac{qQd}{4\pi\varepsilon_0 r^2} \text{ (clockwise)}$$

(iii)

As drawn in (i), the net force on the dipole is $\vec{F}_{+q} + \vec{F}_{-q}$
By symmetry, the horizontal components cancel out. Both vertical components are identical and point upwards. Let θ be the angle with the horizontal. Take upwards as positive.

$$F_{net} = 2F_{+q,x} = 2\frac{Qq}{4\pi\varepsilon_0}\frac{1}{r^2\left(1+\left(\frac{d}{2r}\right)^2\right)}\sin\theta$$

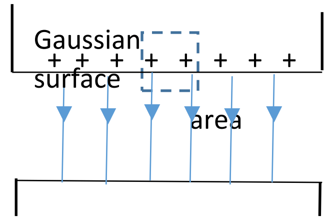
Since $\sin\theta = \frac{d/2}{\sqrt{r^2+\left(\frac{d}{2}\right)^2}}$

$$F_{net} = 2\frac{Qq}{4\pi\varepsilon_0}\frac{1}{r^2+\left(\frac{d}{2}\right)^2}\left(\frac{d/2}{\sqrt{r^2+\left(\frac{d}{2}\right)^2}}\right) = \frac{Qq}{4\pi\varepsilon_0}\frac{d}{\left(r^2+\left(\frac{d}{2}\right)^2\right)^{3/2}} = \frac{Qq}{4\pi\varepsilon_0}\frac{d}{r^3\left(1+\left(\frac{d}{2r}\right)^2\right)^{3/2}}$$

Since $r \gg d$, $\left(1+\left(\frac{d}{2r}\right)^2\right)^{-3/2} \approx 1-\frac{3}{2}\left(\frac{d}{2r}\right)^2 \approx 1$ (we neglect powers of d/r greater than 1 because d/r is small)

$$F_{net} \approx \frac{Qq}{4\pi\varepsilon_0}\frac{d}{r^3}(1) = \frac{qQd}{4\pi\varepsilon_0 r^3} \text{ upwards}$$

Case study: The parallel plate capacitor



From Gauss' Law, $E = \frac{\sigma}{\epsilon_0} = \frac{Q}{\epsilon_0 A}$

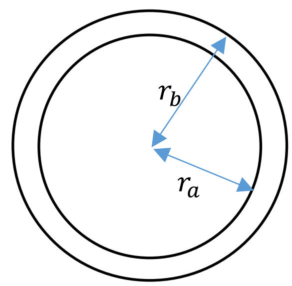
Where σ is surface charge density, A is the surface

Also, for parallel plates, $E = \frac{V}{d}$

Hence $V = \frac{d}{\epsilon_0 A} Q$

The capacitance of a parallel plate capacitor is thus $C = \frac{\epsilon_0 A}{d}$

Similarly, by considering the electric field between the two conductors of a capacitor, we can also find that the capacitance of a spherical capacitor comprising two concentric spherical conducting shells separated by a vacuum with inner radius r_a (positive charge) and outer radius r_b (negative charge) as $C = 4\pi\epsilon_0 \frac{r_a r_b}{r_b - r_a}$.



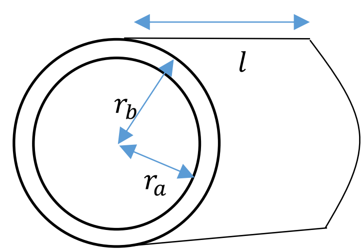
From Gauss' Law, $E = Q/(4\pi\epsilon_0 r^2)$ is 'emitted' from positively charged inner sphere,

$$-\frac{dV}{dr} = Q/(4\pi\epsilon_0\, r^2) \qquad , \qquad V = Q/(4\pi\epsilon_0 r)$$

By performing integration, $V_b - V_a$ = (Q / $4\pi\epsilon_0$)($\frac{1}{r_b}$ - $\frac{1}{r_a}$) = (Q / $4\pi\epsilon_0$)($r_a - r_b$)/ ($r_b\, r_a$)

Since $V_a > V_b$, $C = Q/(\ V_a - V_b\ )$ = $4\pi\epsilon_0 \frac{r_a r_b}{r_b - r_a}$.

Also, the capacitance of a cylindrical capacitor of inner radius r_a (positive charge) and outer radius r_b (negative charge) and length l is $C = \frac{2\pi\epsilon_0 l}{ln(r_b/r_a)}$



From Gauss' Law, $E = Q/(2\pi\epsilon_0 r l)$ is 'emitted' from positively charged inner tube,

$$-\frac{dV}{dr} = Q/(2\pi\epsilon_0 r l) \qquad , \qquad V = -Q\ (\ln r\ )/(2\pi\epsilon_0 l)$$

By performing integration, $V_b - V_a$ = (- Q ($\ln r_b - \ln r_a$) / $2\pi\epsilon_0 l$) = (Q ($\ln\frac{r_a}{r_b}$) / $2\pi\epsilon_0 l$)

Since $V_a > V_b$, $C = Q/(\ V_a - V_b\ )$ = ($2\pi\epsilon_0 l$)/ $\ln\left(\frac{r_b}{r_a}\right)$.

Solutions to Self Review Questions

1. Applying Gauss's Law over a Gaussian surface of radius *r*, that just covers the sphere of radius *R*,

$$E(4\pi r^2) = \frac{\sigma(4\pi R^2)}{\epsilon_0}$$

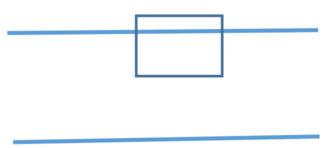
Hence the surface charge density is given by

$$\sigma = E\epsilon_0 \left(\frac{r}{R}\right)^2 = 1.33\ \mu C/m^2$$

Capacitance is given by

$$C = \frac{Q}{V} = 4\pi\epsilon_0 R = 13.3\ pF$$

2. Consider a Gaussian surface as shown in the diagram below,



Applying Gauss's Law yields

$$EA = \frac{\sigma A}{\epsilon_0} = \left(\frac{V}{d}\right) A$$

Hence the separation between the plates,

$$d = \frac{\epsilon_0 V}{\sigma} = 4.42\ \mu m$$

3. Combining the 15 µF and 3.0 µF capacitors in series gives an effective capacitance of 2.5 µF.

Combining the 2.5 µF and 6.0 µF capacitors in parallel gives an effective capacitance of 8.5 µF.

Combining the 8.5 µF and 20 µF capacitors in series gives an effective capacitance of 5.96 µF.

When 15 V is applied across the arrangement,

Total charge is

$$Q = CV = (5.96 \times 10^{-6})(15) = 89.4\ \mu C$$

This is also the charge on the 20.0 µF capacitor.

The potential difference across the 20.0 µF capacitor is thus $V = \frac{Q}{C} = \frac{89.4}{20} = 4.47$ V.

Hence the potential difference across the parallel arrangement is 10.53 V. This is also the potential difference across the 6.0 µF capacitor which gives it a charge of 63.2 µC.

That leaves only a charge of 26.2 µC for the other parallel branch and that is also the charge on each of the two capacitors in series in that branch.

4. Energy stored in the capacitor is given by

$$\xi = \frac{1}{2}CV^2 = \frac{1}{2}(450 \times 10^{-6})(295)^2 = 19.6\ J$$

5. The electric field due to charges on the surfaces of the dielectric is given by

$E_{dielectric} = E - E' = 0.70 \times 10^5$ V/m

Hence the charge density on the surfaces of the dielectric can be determined from

$$E_{dielectric} = \frac{\sigma_{dielectric}}{\epsilon_0}$$

$$\sigma_{dielectric} = (0.70 \times 10^5)(8.854 \times 10^{-12}) = 6.20 \times 10^{-7}\ C/m^2$$

Dielectric constant, $K = \frac{E}{E\prime} = 1.28$

6. The circuit can be described by the equation

$$\xi = iR + L\frac{di}{dt}$$

Solving yields

$$i = \frac{\xi}{R}\left(1 - e^{-(R/L)t}\right)$$

$$i = 1.2(1 - e^{-5t})$$

Final steady value of current is 1.2A and

Time needed for current to reach 50.0% ; i.e. 0.60 A is given by $t = 0.139$ s.

7. a) Mutual inductance is the ratio of the magnetic flux linkage in one coil due to current flowing in the other,

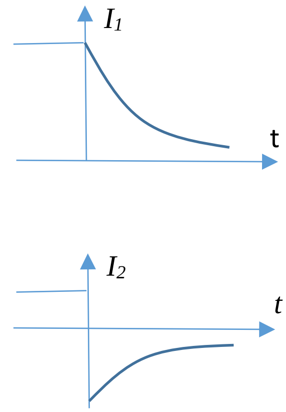
$$M_{21} = \frac{N_2 \Phi_2}{i_1} = \frac{400 \times 0.032}{6.52} = 1.96\ H$$

b) Similarly, for the other coil, since $M_{12} = M_{21}$,

$$M_{12} = \frac{N_1 \Phi_1}{i_2} = M_{21}$$

$$\Phi_1 = \frac{(2.54)(1.96)}{700} = 7.12 \times 10^{-3}\ Wb$$

(b)



8. For a solenoid of length *l*, the inductance is given by $L = \frac{\mu_0 N^2 A}{l}$

Thus the energy stored in the solenoid is given by

$$U = \frac{1}{2} L I^2 = \frac{\mu_0 N^2 A I^2}{2l} = 2.44 \times 10^{-6}\ J$$

9. Resonance frequency $f = \frac{1}{2\pi}\sqrt{\frac{1}{LC}} = 711$ Hz

10. At steady state, there is no p.d. across the inductor. Both resistors are connected in parallel to the cell and has the same p.d.

$I_1 = \frac{18}{6000} = 3$ mA $\quad I_2 = \frac{18}{2000} = 9$ mA

(a) At $t = 0$, the switch is opened and the cell is no longer part of the circuit. The current in the inductor is initially 9 mA (clockwise) and this will be the current in the outer loop through both resistors. The p.d. across both resistors is thus 72 V, hence the induced emf at the inductor is 72 V. Since the current is initially clockwise, the point *b* will be at a higher potential than *a*.

Solutions to Tutorial Questions

1. Consider a Gaussian cylindrical surface (radius *r*) in the region between the two conductors in the co-axial cable.

Applying Gauss's Law yields

$$E(2\pi rL) = \frac{Q_{inner}}{\epsilon_0}$$

The potential difference between the inner conductor surface (radius *a*) and the inner surface of the outer conductor (radius *b*) is given by

$$V = \int_a^b E dr = \frac{Q_{inner}}{2\pi L\epsilon_0}\int_a^b \frac{dr}{r} = \frac{Q_{inner}}{2\pi L\epsilon_0} ln\left(\frac{b}{a}\right)$$

Hence capacitance is given by

$$C = \frac{Q}{V} = \frac{2\pi L\epsilon_0}{ln(b/a)} = 2.68 \times 10^{-9}\, F$$

2. When fully charged, potential difference across capacitor C_1 is the same as the cell. The amount of charge on it is given by $Q = CV = 120\ \mu C$.

When switch S_1 is now open and S_2 closed, charges redistribute between the two capacitors until the potential difference across each are the same. Hence we can write

$$\frac{Q_1}{C_1} = \frac{Q - Q_1}{C_2}$$

Solving yields $Q_1 = 80.0\ \mu C$ and $Q_2 = 40.0\ \mu C$.

3. For a parallel plate capacitor, $E.A = \frac{Q}{\epsilon_0}$

Hence the energy stored in the capacitor is given by $\xi = \frac{QV}{2} = \frac{Q^2}{2\epsilon_0 A}x$

Change in energy stored is given by

$$d\xi = \frac{Q^2}{2\epsilon_0 A}(dx)$$

Hence the force between the plates is given by

$$F = \frac{d\xi}{dx} = \frac{Q^2}{2\epsilon_0 A}$$

4. a) The dielectric constant is given by

$$K = \frac{E_0}{E} = \frac{V_0}{V} = \frac{45}{11.5} = 3.91$$

b) We can model this as two capacitors connected in parallel with the charges split between them such that the same potential difference exist across the plates of both capacitors.

For the one-third part with dielectric,

$$Q_1 = C_{dielectric}V = K\frac{C_0}{3}V$$

For the two-third part with vacuum,

$$Q_2 = \frac{2C_0}{3}V$$

Hence

$$C_0V_0 = Q = Q_1 + Q_2 = \left(\frac{2}{3} + \frac{K}{3}\right)C_0V$$

$$V = \frac{3V_0}{2+K} = \frac{3 \times 45}{2 + 3.91} = 22.8\, V$$

5. The potential difference across each slab of dielectric is given by

$V_1 = \frac{Q}{C_1} = \frac{Q}{K_1 2C_0}$ and $V_2 = \frac{Q}{C_2} = \frac{Q}{K_2 2C_0}$

where $C_0 = \frac{\epsilon_0 A}{d}$ is the capacitance across the two parallel plates in vacuum of separation *d.*

Total potential difference,

$$V = \frac{Q}{2C_0}\left(\frac{K_1 + K_2}{K_1K_2}\right)$$

$$C = \frac{Q}{V} = (2C_0)\left(\frac{K_1K_2}{K_1 + K_2}\right) = \frac{2\epsilon_0 A}{d}\left(\frac{K_1K_2}{K_1 + K_2}\right)$$

6. The magnetic field at the center of the solenoid is given by

$$B = \mu_0 nI = (4\pi \times 10^{-7})\left(\frac{400}{0.25}\right)(80)$$
$$= 0.161\ T$$

The magnetic energy density is given by

$$u = \frac{B^2}{2\mu_0} = 1.03 \times 10^4\ J/m^3$$

Total energy stored is given by

$$\xi = u.Al = 0.129\ J$$

Since energy ξ stored in the magnetic field is given by

$$\xi = \frac{1}{2}LI^2 = 0.129$$

Inductance is thus given by

$$L = \frac{2(0.129)}{80^2} = 4.02 \times 10^{-5}\ H$$

7. (a) The circuit can be described by the equation

$$\xi = iR + L\frac{di}{dt}$$

Solving yields

$$i = \frac{\xi}{R}\left(1 - e^{-(R/L)t}\right)$$

$$0.22 = \frac{6.00}{4.90}\left(1 - e^{-(4.90/0.140)t}\right)$$

Hence $t = 5.66$ ms.

(b) Using the same equation,

$$i = \frac{6.00}{4.90}\left(1 - e^{-(4.90/0.140)10}\right) = 1.22\ A$$

(c) When current is decreasing, the equation is given by

$$-L\frac{di}{dt} = iR$$

Solving yields

$$i = \frac{\xi}{R}e^{-\frac{R}{L}t}$$

$$0.16 = 1.22e^{-\frac{4.90}{0.14}t}$$

Hence $t = 0.058$ s.

8. In an LC circuit, the charges oscillate with a frequency of $f = \frac{1}{2\pi}\omega = \frac{1}{2\pi}\sqrt{\frac{1}{LC}}$

Hence if the circuit oscillations are in tune with the radio signal of frequency $f_{signal} = 6.3 \times 10^{12}\ Hz$

The capacitance can be found by

$$C = \frac{1}{L}\left(\frac{1}{2\pi f}\right)^2 = 608 \times 10^{-12}\ F$$

9. For the *LC* circuit consisting of a fully charged capacitor and inductor connected in series,

$$\frac{q}{C} = -L\frac{d^2q}{dt^2}$$

We can see that it is of the same form as the characteristic equation of SHM where the resonant frequency can be determined by comparison, ie,

$\omega_0 = \frac{1}{\sqrt{LC}} = 4472$ rad.s^{-1}

With the addition of a resistor,

$$\frac{q}{C} = -L\frac{d^2q}{dt^2} - R\frac{dq}{dt}$$

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} = 0$$

Auxiliary equation is of the form

$$y^2 + \frac{R}{L}y + \frac{1}{LC} = 0$$

This has complex roots of

$$y_{1,2} = -\frac{R}{2L} \pm i\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

Hence solution is of the form

$$q = e^{-R/2L}\left(A\cos\left(t\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}\right) + B\sin\left(t\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}\right)\right)$$

With frequency

$\omega = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} = 4360$ rad.s^{-1}