Additional Practice Questions

1. **N92/II/15 (partial)**

The equation of a plane Π is $x-6y+2z=-5$, and the point A has coordinates $(3,-12,1)$. Write down

(i) a vector perpendicular to Π,

(ii) a vector equation for the straight line ℓ which passes through A and is perpendicular to Π.

Find the coordinates of the point of intersection of ℓ and Π, and hence show that the perpendicular distance from A to Π is $2\sqrt{41}$.

Π : $\underset{\sim}{r} \cdot \begin{pmatrix} 1 \\ -6 \\ 2 \end{pmatrix} = -5$

(i) $\begin{pmatrix} 1 \\ -6 \\ 2 \end{pmatrix}$ //

(ii) ℓ : $\underset{\sim}{r} = \begin{pmatrix} 3 \\ -12 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -6 \\ 2 \end{pmatrix}$ //, where $\lambda \in \mathbb{R}$

Let $\begin{pmatrix} 3+\lambda \\ -12-6\lambda \\ 1+2\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -6 \\ 2 \end{pmatrix} = -5$

$\Rightarrow 3+\lambda+72+36\lambda+2+4\lambda=-5 \Rightarrow \lambda=-2$

$\therefore$ coords. of pt. of intersection, B, are $(1,0,-3)$ //.

Hence, $\perp$ dist. fr. A to Π = AB

$= \left| \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} - \begin{pmatrix} 3 \\ -12 \\ 1 \end{pmatrix} \right|$

$= \left| \begin{pmatrix} -2 \\ 12 \\ -4 \end{pmatrix} \right|$

$= 2 \left| \begin{pmatrix} -1 \\ 6 \\ -2 \end{pmatrix} \right|$

$= 2\sqrt{1+36+4}$

$= 2\sqrt{41}$ // (shown)

2. **N90/II/15**

The point A has coordinates $(3,-1,5)$ and the line ℓ has equation $\mathbf{r}=\begin{pmatrix}8\\0\\-1\end{pmatrix}+t\begin{pmatrix}-6\\1\\4\end{pmatrix}$.

Find the coordinates of the point B on ℓ such that AB is perpendicular to ℓ.

The plane Π has equation $\mathbf{r}\cdot\begin{pmatrix}1\\-1\\3\end{pmatrix}=15$.

Find the coordinates of the point C where ℓ intersects Π.

Find a vector perpendicular to the plane ABC. Hence show that the acute angle between Π and the plane ABC is 68°, correct to the nearest degree.

Since B is on ℓ, let $\overrightarrow{OB}=\begin{pmatrix}8-6t\\t\\-1+4t\end{pmatrix}$. So, $\overrightarrow{AB}=\begin{pmatrix}5-6t\\t+1\\-6+4t\end{pmatrix}$

want $AB \perp \ell$, ie. $\overrightarrow{AB}\cdot\begin{pmatrix}-6\\1\\4\end{pmatrix}=0$

$\Rightarrow \begin{pmatrix}5-6t\\t+1\\-6+4t\end{pmatrix}\cdot\begin{pmatrix}-6\\1\\4\end{pmatrix}=0$

$\Rightarrow -30+36t+t+1-24+16t=0$

$\Rightarrow t=1$

$\therefore$ Coords of B are $(2,1,3)$ //.

To find intersection betⁿ ℓ & Π,

let $\begin{pmatrix}8-6t\\t\\-1+4t\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\3\end{pmatrix}=15 \Rightarrow 8-6t-t-3+12t=15$

$\Rightarrow t=2$

$\therefore$ Coords. of C are $(-4,2,7)$ //.

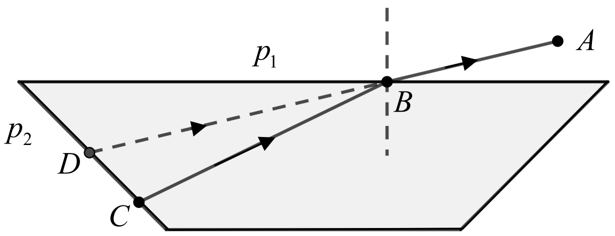
$\overrightarrow{AB}=\begin{pmatrix}-1\\2\\-2\end{pmatrix}$, $\overrightarrow{AC}=\begin{pmatrix}-7\\3\\2\end{pmatrix}$. $\therefore$ vector $\perp$ to plane ABC $=\overrightarrow{AB}\times\overrightarrow{AC}=\begin{pmatrix}10\\16\\11\end{pmatrix}$ //.

Hence, the acute $\angle$ betⁿ the planes

$=\cos^{-1}\left|\widehat{\begin{pmatrix}1\\-1\\3\end{pmatrix}}\cdot\widehat{\begin{pmatrix}10\\16\\11\end{pmatrix}}\right| = \cos^{-1}\left|\dfrac{10-16+33}{\sqrt{11}\sqrt{477}}\right| = 68^\circ$ //

(to nearest deg.)

3. **VJC/2018 MYE/Q6**



The diagram above shows the vertical cross-section of a slab of glass in the form of a trapezoidal prism, where the top surface is a plane p_1 and the left side of the glass is a plane p_2. C and D are points in p_2. The light from a particle placed at C travels in a straight line to B in the glass. The light is refracted at B and travels in a straight line to A $(3,1,2)$ in the air. To an observer at A, the particle at C appears to be at D $(0,7,-4)$.

The plane p_1 has equation $z=0$ and AB is parallel to $\mathbf{i}-2\mathbf{j}+2\mathbf{k}$.

(i) Find the coordinates of B. [3]

$\overrightarrow{BC}$ is in the direction of $-4\mathbf{i}+a\mathbf{j}+\mathbf{k}$, where a is a positive constant. The line BC makes an angle of $\cos^{-1}\left(\frac{1}{9}\right)$ with the normal to p_1 at B.

(ii) Find the value of a. [3]

(iii) Given that the distance BC is 18, show that the position vector of C is $-6\mathbf{i}+19\mathbf{j}+2\mathbf{k}$. [2]

The plane p_2 has equation $\mathbf{r}\cdot\mathbf{n}=34$.

(iv) Given that p_2 is perpendicular to a plane with equation $2x+y=7$, find $\mathbf{n}$. [4]

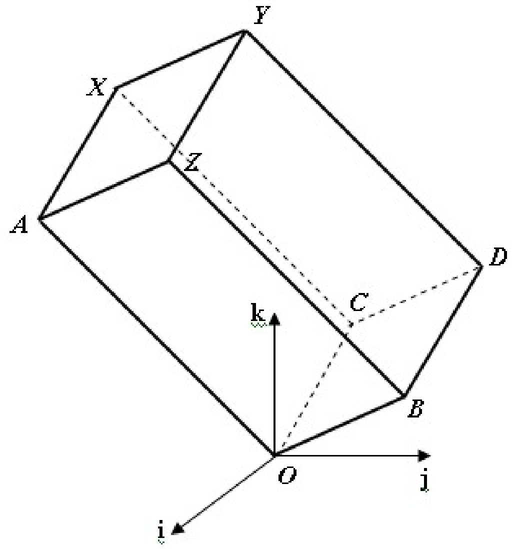
(i)	Equation of p_1 is $\underset{\sim}{r}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}=0$ and equation on line AB is $\underset{\sim}{r}=\begin{pmatrix}3\\1\\2\end{pmatrix}+s\begin{pmatrix}1\\-2\\2\end{pmatrix}, s\in\mathbb{R}$ Solving them, $\left(\begin{pmatrix}3\\1\\2\end{pmatrix}+s\begin{pmatrix}1\\-2\\2\end{pmatrix}\right)\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}=0$ $2+2s=0\Rightarrow s=-1$ $\overrightarrow{OB}=\begin{pmatrix}3\\1\\2\end{pmatrix}-\begin{pmatrix}1\\-2\\2\end{pmatrix}=\begin{pmatrix}2\\3\\0\end{pmatrix}$ The coordinates of B is (2, 3, 0).

(ii)	$\cos\left(\cos^{-1}\left(\frac{1}{9}\right)\right)=\frac{\begin{pmatrix}-4\\a\\1\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}}{\sqrt{4^2+a^2+1}\sqrt{1}}$ (Realise that the scalar product is positive. Else we should use $-\frac{1}{9}$.) $\frac{1}{9}=\frac{1}{\sqrt{17+a^2}}$ $17+a^2=81$ $a=8 \text{ or } -8 (\text{NA } \because a>0)$
(iii)	Method 1: $\overrightarrow{BC}=18\left(\frac{1}{\sqrt{16+64+1}}\right)\begin{pmatrix}-4\\8\\1\end{pmatrix}=\begin{pmatrix}-8\\16\\2\end{pmatrix}$ $\overrightarrow{OC}=\overrightarrow{OB}+\overrightarrow{BC}=\begin{pmatrix}2\\3\\0\end{pmatrix}+\begin{pmatrix}-8\\16\\2\end{pmatrix}=\begin{pmatrix}-6\\19\\2\end{pmatrix}$ Method 2: $\overrightarrow{BC}=k\begin{pmatrix}-4\\8\\1\end{pmatrix}$, $k>0$ because $\overrightarrow{BC}$ is in the direction of $\begin{pmatrix}-4\\8\\1\end{pmatrix}$ $BC=\left\|k\begin{pmatrix}-4\\8\\1\end{pmatrix}\right\|=9k=18 \Rightarrow k=2$ $\overrightarrow{OC}=\overrightarrow{OB}+\overrightarrow{BC}=\begin{pmatrix}2\\3\\0\end{pmatrix}+2\begin{pmatrix}-4\\8\\1\end{pmatrix}=\begin{pmatrix}-6\\19\\2\end{pmatrix}$
(iv)	Method 1: $\overrightarrow{CD}=\begin{pmatrix}0\\7\\-4\end{pmatrix}-\begin{pmatrix}-6\\19\\2\end{pmatrix}=\begin{pmatrix}6\\-12\\-6\end{pmatrix}=6\begin{pmatrix}1\\-2\\-1\end{pmatrix}$ A normal vector to the plane = $\begin{pmatrix}1\\-2\\-1\end{pmatrix}\times\begin{pmatrix}2\\1\\0\end{pmatrix}=\begin{pmatrix}1\\-2\\5\end{pmatrix}$

	$p_2 : \underset{\sim}{r} \cdot \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 7 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} = -34$, i.e. $\underset{\sim}{r} \cdot \begin{pmatrix} -1 \\ 2 \\ -5 \end{pmatrix} = 34$ $\therefore \underset{\sim}{n} = \begin{pmatrix} -1 \\ 2 \\ -5 \end{pmatrix}$ Method 2: Let $\underset{\sim}{n} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = 0 \Rightarrow 2x + y = 0$ $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 7 \\ -4 \end{pmatrix} = 34 \Rightarrow 7y - 4z = 34$ (since D is in p_2) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -6 \\ 19 \\ 2 \end{pmatrix} = 34 \Rightarrow -6x + 19y + 2z = 34$ (since C is in p_2) By GC, x = -1, y = 2, z = -5 $\therefore \underset{\sim}{n} = \begin{pmatrix} -1 \\ 2 \\ -5 \end{pmatrix}$

***4 TPJC 10/II/4**

The figure below shows a cuboid positioned on level ground so that it rests on one of its vertices, O. The vectors **i** and **j** are on the ground.



Given that $\overrightarrow{OA} = 3\mathbf{i} - 12\mathbf{j} + 3\mathbf{k}$, $\overrightarrow{OB} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $\overrightarrow{OC} = -2\mathbf{i} + 2\mathbf{k}$

(i) Show that the position vector of X is $\mathbf{i} - 12\mathbf{j} + 5\mathbf{k}$.

(ii) State the height of X above the ground. Hence find the angle between OX and the level ground, giving your answer to nearest 0.1°.

(iii) Find the equation of plane BDX in the form of $\mathbf{r} \bullet \mathbf{n} = \text{d}$

(iv) Find the acute angle between planes BDX and $OBDC$.

4	**(i)** $\overrightarrow{OA} = \begin{pmatrix} 3 \\ -12 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix}; \overrightarrow{OB} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}; \overrightarrow{OC} = \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix} = 2\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ $\overrightarrow{OX} = \overrightarrow{OA} + \overrightarrow{AX} = \overrightarrow{OA} + \overrightarrow{OC}$ $= \begin{pmatrix} 3 \\ -12 \\ 3 \end{pmatrix} + \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -12 \\ 5 \end{pmatrix}$ **(ii)** Height of X above the ground = 5 units Let θ be the required angle. Hence, $\sin\theta = \dfrac{5}{\lvert\overrightarrow{OX}\rvert} = \dfrac{5}{\sqrt{1+144+25}}$ $\Rightarrow \theta = 22.5^o$

(iii)

$$\overrightarrow{BD}=\overrightarrow{OC}=2\begin{pmatrix}-1\\0\\1\end{pmatrix}$$

$$\overrightarrow{BX}=\begin{pmatrix}1\\12\\-5\end{pmatrix}-\begin{pmatrix}2\\1\\2\end{pmatrix}=\begin{pmatrix}-1\\-13\\3\end{pmatrix}$$

$$\begin{pmatrix}-1\\0\\1\end{pmatrix}\times\begin{pmatrix}-1\\-13\\3\end{pmatrix}=\begin{pmatrix}13\\2\\13\end{pmatrix}$$

Therefore, equation of plane $BDX:\mathbf{r}\bullet\begin{pmatrix}13\\2\\13\end{pmatrix}=\begin{pmatrix}2\\1\\2\end{pmatrix}\bullet\begin{pmatrix}13\\2\\13\end{pmatrix}=54$

(iv) Normal vector to plane *OBDC* is $\overrightarrow{OA}=2\begin{pmatrix}1\\-4\\1\end{pmatrix}$

Let α be the required angle.

$$\cos\alpha=\frac{\begin{pmatrix}13\\2\\13\end{pmatrix}\bullet\begin{pmatrix}1\\-4\\1\end{pmatrix}}{\sqrt{342}\sqrt{18}}$$

$\therefore\alpha=76.7^o$

5. **JJC/2012/Prelim II/4**

The planes Π_1 and Π_2 have equations $\mathbf{r} \bullet (\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = 4$ and $\mathbf{r} = 3\mathbf{i} - \mathbf{j} + \mathbf{k} + \lambda(-\mathbf{i} + \mathbf{j} + \mathbf{k}) + \mu(4\mathbf{i} + \mathbf{j})$ respectively, and meet in a line l_1.

(i) Find the acute angle between Π_1 and Π_2. [3]

(ii) Find a vector equation of l_1. [2]

(iii) The points A and B have coordinates $(6,\ 3,\ -5)$ and $(2,\ 3,\ 1)$ respectively. Find the length of projection of $\overrightarrow{AB}$ onto the line l_1. [2]

The line l_2 passes through the point C with position vector $p\mathbf{i} + (2p+1)\mathbf{j} - 3\mathbf{k}$ and is parallel to $3q\mathbf{i} - 3\mathbf{j} + q\mathbf{k}$, where p and q are positive constants.

Given that the perpendicular distance from C to Π_1 is $\dfrac{15}{\sqrt{6}}$ and that the acute angle between l_2 and Π_1 is $\sin^{-1}\left(\dfrac{2}{\sqrt{6}}\right)$, find the values of p and q. [6]

(i)	$n_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ -5 \end{pmatrix}$
	Acute angle between Π_1 and $\Pi_2 = \cos^{-1} \dfrac{\left\| \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 4 \\ -5 \end{pmatrix} \right\|}{\left\| \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\| \left\| \begin{pmatrix} -1 \\ 4 \\ -5 \end{pmatrix} \right\|}$
	$= \cos^{-1}\left(\dfrac{14}{\sqrt{6}\sqrt{42}}\right)$ $= 28.1^\circ$ (to 1 dp)
(ii)	$\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 4 \\ -5 \end{pmatrix} = -12$ $\therefore \Pi_2 : \mathbf{r} \bullet \begin{pmatrix} -1 \\ 4 \\ -5 \end{pmatrix} = -12 \Rightarrow -x + 4y - 5z = -12$

	$\Pi_1 : x-2y+z=4$ $\Pi_2 : -x+4y-5z=-12$ Using GC, eqn of l_1 is $\mathbf{r}=\begin{pmatrix}-4\\-4\\0\end{pmatrix}+\alpha\begin{pmatrix}3\\2\\1\end{pmatrix}$, where $\alpha\in\mathbb{R}$.
(iii)	Given $\overrightarrow{OA}=\begin{pmatrix}6\\3\\-5\end{pmatrix}$ and $\overrightarrow{OB}=\begin{pmatrix}2\\3\\1\end{pmatrix}$, $\overrightarrow{AB}=\begin{pmatrix}2\\3\\1\end{pmatrix}-\begin{pmatrix}6\\3\\-5\end{pmatrix}=\begin{pmatrix}-4\\0\\6\end{pmatrix}$ Length of projection of $\overrightarrow{AB}$ onto the line l_1 $=\dfrac{\left\|\begin{pmatrix}-4\\0\\6\end{pmatrix}\bullet\begin{pmatrix}3\\2\\1\end{pmatrix}\right\|}{\sqrt{3^2+2^2+1^2}}$
	$=\dfrac{\|-6\|}{\sqrt{14}}$ $=\dfrac{3\sqrt{14}}{7}$
	$\Pi_1 : \mathbf{r}\bullet\begin{pmatrix}1\\-2\\1\end{pmatrix}=4$ $l_2 : \mathbf{r}=\begin{pmatrix}p\\2p+1\\-3\end{pmatrix}+t\begin{pmatrix}3q\\-3\\q\end{pmatrix}$ Let D be a point on the plane Π_1. Since $\begin{pmatrix}4\\0\\0\end{pmatrix}\bullet\begin{pmatrix}1\\-2\\1\end{pmatrix}=4$, D is (4, 0, 0). $\therefore\ \overrightarrow{DC}=\begin{pmatrix}p\\2p+1\\-3\end{pmatrix}-\begin{pmatrix}4\\0\\0\end{pmatrix}=\begin{pmatrix}p-4\\2p+1\\-3\end{pmatrix}$
	Perpendicular distance from C to Π_1 = Length of projection of $\overrightarrow{DC}$ onto the normal of Π_1

	$= \dfrac{\left\lvert \overrightarrow{DC}.\begin{pmatrix}1\\-2\\1\end{pmatrix}\right\rvert}{\left\lvert \begin{pmatrix}1\\-2\\1\end{pmatrix}\right\rvert} = \dfrac{\left\lvert \begin{pmatrix}p-4\\2p+1\\-3\end{pmatrix}.\begin{pmatrix}1\\-2\\1\end{pmatrix}\right\rvert}{\sqrt{6}} = \dfrac{15}{\sqrt{6}}$
	$\lvert(p-4)-2(2p+1)-3\rvert = 15$ $\lvert -3p-9\rvert = 15$ $-3p-9=15$ or $-3p-9=-15$ $p=-8$ (rej $\because p>0$) $p=2$
	Acute angle between l_2 and Π_1 $= \sin^{-1}\dfrac{\left\lvert \begin{pmatrix}3q\\-3\\q\end{pmatrix}.\begin{pmatrix}1\\-2\\1\end{pmatrix}\right\rvert}{\left\lvert \begin{pmatrix}3q\\-3\\q\end{pmatrix}\right\rvert \left\lvert \begin{pmatrix}1\\-2\\1\end{pmatrix}\right\rvert} = \sin^{-1}\left(\dfrac{2}{\sqrt{6}}\right)$
	$\dfrac{3q+6+q}{\sqrt{9q^2+9+q^2}\sqrt{6}} = \dfrac{2}{\sqrt{6}}$ $\left(\lvert 3q+6+q\rvert = 3q+6+q \text{ since } q>0\right)$
	$4q+6 = 2\sqrt{10q^2+9}$ $(4q+6)^2 = 4\left(10q^2+9\right)$ $q^2-2q=0$ $q=2$ (since $q>0$)

6. **AJC/2018/MYE/I/Q9**

The plane Π contains the origin *O* and is parallel to vectors $-\mathbf{i}+\mathbf{k}$ and $\mathbf{i}+2\mathbf{j}$.

(i) Find an equation of the plane Π in scalar product form. [2]

The point *P* has coordinates $(3\,,1\,,2)$.

(ii) By finding the foot of perpendicular, *N*, of point *P* to the plane Π, show that the position vector of the mirror image of *P* in Π is $-\mathbf{i}+3\mathbf{j}-2\mathbf{k}$. [3]

The point *Q* has coordinates $(4\,,0\,,1)$.

(iii) Find the exact length of projection of *PQ* on to the plane Π. [3]

(iv) Hence, or otherwise, find the exact area of the triangle *PNQ*. [2]

(i) Normal to the plane, Π is $\begin{pmatrix}-1\\0\\1\end{pmatrix}\times\begin{pmatrix}1\\2\\0\end{pmatrix}=\begin{pmatrix}-2\\1\\-2\end{pmatrix}$.

Equation of plane is $\Pi: \underset{\sim}{r}\bullet\begin{pmatrix}-2\\1\\-2\end{pmatrix}=0$.

(ii) Let *N* be the foot of perpendicular from *P* to Π and *P*' be the reflection of P about Π.

$$\overrightarrow{PN}=k\begin{pmatrix}-2\\1\\-2\end{pmatrix} \Rightarrow \overrightarrow{ON}-\overrightarrow{OP}=k\begin{pmatrix}-2\\1\\-2\end{pmatrix} \Rightarrow \overrightarrow{ON}=\begin{pmatrix}3\\1\\2\end{pmatrix}+k\begin{pmatrix}-2\\1\\-2\end{pmatrix}=\begin{pmatrix}3-2k\\1+k\\2-2k\end{pmatrix}$$

Since N lies in Π, $\begin{pmatrix}3-2k\\1+k\\2-2k\end{pmatrix}\bullet\begin{pmatrix}-2\\1\\-2\end{pmatrix}=0 \Rightarrow k=1 \quad \therefore \overrightarrow{ON}=\begin{pmatrix}1\\2\\0\end{pmatrix}$

$\overrightarrow{PP'}=2\overrightarrow{PN}$

$$\Rightarrow \overrightarrow{OP'}=2\overrightarrow{PN}+\overrightarrow{OP}=2\begin{pmatrix}-2\\1\\-2\end{pmatrix}+\begin{pmatrix}3\\1\\2\end{pmatrix}=\begin{pmatrix}-1\\3\\-2\end{pmatrix} \text{ (shown)}$$

(iii) $\overrightarrow{PQ}=\begin{pmatrix}4\\0\\1\end{pmatrix}-\begin{pmatrix}3\\1\\2\end{pmatrix}=\begin{pmatrix}1\\-1\\-1\end{pmatrix}$; Required length $\left|\begin{pmatrix}1\\-1\\-1\end{pmatrix}\times\widehat{\begin{pmatrix}-2\\1\\-2\end{pmatrix}}\right|=\frac{1}{\sqrt{9}}\left|\begin{pmatrix}3\\4\\-1\end{pmatrix}\right|=\frac{\sqrt{26}}{3}$ units

(iv) Area of $\triangle$ PNQ = $\frac{1}{2}\left(\frac{\sqrt{26}}{3}\right)(PN) = \frac{1}{2}\left(\frac{\sqrt{26}}{3}\right)(3)=\frac{\sqrt{26}}{2}$ units^2

7. **SRJC/2018/MYE/II/Q7(modified)**

The planes p_1, p_2 and p_3 have equations $2x+y+z=2$, $-2x+z=6$ and $\alpha x+3y+2z=\beta$ respectively, where $\alpha,\beta\in\mathbb{R}$. Given that the plane p_1 meets the plane p_2 at the line l,

(i) find the equation of l. [1]

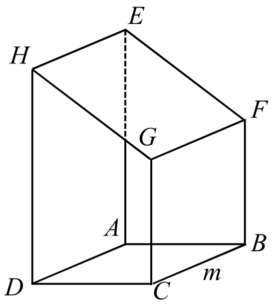
(ii) Find the values of α and β if p_3 also contains l. [2]

(iii) The plane p_4 is parallel to the plane p_2 and has equation $-2x+z=k$, where $k\in\mathbb{R}$. Find the possible values of k for which the plane p_4 is $\sqrt{5}$ units away from the plane p_2. [2]

(i) Solving $2x+y+z=2$ and $-2x+z=6$ using GC, $x=-3+\frac{1}{2}\mu$ $y=8-2\mu$ $z=\mu$ Hence, $l:\ \mathbf{r}=\begin{pmatrix}-3\\8\\0\end{pmatrix}+\mu\begin{pmatrix}\frac{1}{2}\\-2\\1\end{pmatrix},\ \mu\in\mathbb{R}$ $\left(\text{OR}\quad l:\ \mathbf{r}=\begin{pmatrix}-3\\8\\0\end{pmatrix}+\mu\begin{pmatrix}1\\-4\\2\end{pmatrix},\ \mu\in\mathbb{R}\right)$
(ii) $p_2:\ \mathbf{r}\bullet\begin{pmatrix}-2\\0\\1\end{pmatrix}=6,\quad p_3:\ \mathbf{r}\bullet\begin{pmatrix}\alpha\\3\\2\end{pmatrix}=\beta$ If p_1, p_2 and p_3 have a common line of intersection l, the normal of p_3 is perpendicular to the direction of l. $\begin{pmatrix}\alpha\\3\\2\end{pmatrix}\bullet\begin{pmatrix}1\\-4\\2\end{pmatrix}=0$ $\alpha-12+4=0$ $\alpha=8$ Since any point of l will be in plane p_3, $\begin{pmatrix}-3\\8\\0\end{pmatrix}\bullet\begin{pmatrix}8\\3\\2\end{pmatrix}=\beta$ $\beta=0$

(iii) $\left\|\dfrac{k-6}{\sqrt{(-2)^2+1^2}}\right\| = \sqrt{5}$ $\|k-6\| = 5$ $k-6=5$ or $k-6=-5$ $k=11$ or $k=1$

8. **ACJC/2018/MYE/Q7**



The diagram shows the structure of a building with a horizontal rectangular base *ABCD*, whereby *BC* is *m* units. The structure consists of four vertical columns, *AE*, *DH*, *CG* and *BF* which are parallel to the *z*-axis.

The roof *EFGH* has equation $x-y+\sqrt{2}\,z=2+4\sqrt{2}$ and the plane *ABFE* has equation $x+y=2$. The origin $(0,0,0)$ is a point within the rectangular base *ABCD*.

(i) Find the cartesian equation of the plane *DCGH* in terms of *m*. [2]

(ii) Find the acute angle between the roof *EFGH* and the base *ABCD*. [2]

To ensure that the roof is sturdier, a beam needs to be added to connect point $W(0,2,4)$, which lies on column *AE*, to the nearest point on the roof.

(iii) Show that this nearest point on the roof is $\left(1,1,4+\sqrt{2}\right)$. [3]

(iv) Hence find the exact cartesian equation of the beam *EF*. [3]

(i)	Plane *ABFE*: $\mathbf{r}.\begin{pmatrix}1\\1\\0\end{pmatrix}=2$ Let equation of plane *DCGH* be $\mathbf{r}.\begin{pmatrix}1\\1\\0\end{pmatrix}=p$, where $p<0$ since origin is within the base *ABCD*. Shortest distance between plane *DCGH* and plane *ABFE* = *m*

	$\Rightarrow \left\lvert \frac{2}{\sqrt{2}} - \frac{p}{\sqrt{2}} \right\rvert = m$ $\Rightarrow 2 - p = \sqrt{2}m$, since $2 - p > 0$ $\therefore p = 2 - \sqrt{2}m$ Hence, cartesian equation of Plane *DCGH*: $x + y = 2 - m\sqrt{2}$ (Note: $m > \sqrt{2}$)
(ii)	Let θ be the acute angle between the roof *EFGH* and the base *ABCD*. $\cos\theta = \left\lvert \frac{\begin{pmatrix} 1 \\ -1 \\ \sqrt{2} \end{pmatrix} . \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}{\left\lVert \begin{pmatrix} 1 \\ -1 \\ \sqrt{2} \end{pmatrix} \right\rVert \left\lVert \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rVert} \right\rvert$ $\cos\theta = \frac{\sqrt{2}}{\sqrt{4}}$ $\theta = 45^\circ$
(iii)	**Method 1:** Let point *N* be the foot of perpendicular from point *W* to the roof *EFGH*. Line *WN*: $\mathbf{r} = \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ \sqrt{2} \end{pmatrix}$ *where* $\lambda \in \mathbb{R}$ ------(1) Roof *EFGH*: $\mathbf{r} . \begin{pmatrix} 1 \\ -1 \\ \sqrt{2} \end{pmatrix} = 2 + 4\sqrt{2}$ ------(2) Sub (1) into (2): $\begin{pmatrix} \lambda \\ 2-\lambda \\ 4+\sqrt{2}\lambda \end{pmatrix} . \begin{pmatrix} 1 \\ -1 \\ \sqrt{2} \end{pmatrix} = 2 + 4\sqrt{2}$ $\lambda - 2 + \lambda + 4\sqrt{2} + 2\sqrt{2} = 2 + 4\sqrt{2}$ $\lambda = 1$ Sub. $\lambda = 1$ into (1), $\overrightarrow{ON} = \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ \sqrt{2} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 4+\sqrt{2} \end{pmatrix}$ Hence the nearest point on the roof from point *W* is $N\left(1, 1, 4+\sqrt{2}\right)$. (shown) **Method 2:** By trial and error, $M\left(2+4\sqrt{2}, 0, 0\right)$ is a point on the roof.

	$\overrightarrow{WM}=\begin{pmatrix}2+4\sqrt{2}\\0\\0\end{pmatrix}-\begin{pmatrix}0\\2\\4\end{pmatrix}=\begin{pmatrix}2+4\sqrt{2}\\-2\\-4\end{pmatrix}$ Let point N be the foot of perpendicular from point W to the roof $EFGH$ $\overrightarrow{WN}=\left(\overrightarrow{WM}\bullet\frac{\begin{pmatrix}1\\-1\\\sqrt{2}\end{pmatrix}}{\left\|\begin{pmatrix}1\\-1\\\sqrt{2}\end{pmatrix}\right\|}\right)\left(\frac{\begin{pmatrix}1\\-1\\\sqrt{2}\end{pmatrix}}{\left\|\begin{pmatrix}1\\-1\\\sqrt{2}\end{pmatrix}\right\|}\right)$ $=\left(\begin{pmatrix}2+4\sqrt{2}\\-2\\-4\end{pmatrix}\bullet\frac{\begin{pmatrix}1\\-1\\\sqrt{2}\end{pmatrix}}{\left\|\begin{pmatrix}1\\-1\\\sqrt{2}\end{pmatrix}\right\|}\right)\left(\frac{\begin{pmatrix}1\\-1\\\sqrt{2}\end{pmatrix}}{\left\|\begin{pmatrix}1\\-1\\\sqrt{2}\end{pmatrix}\right\|}\right)$ $=\begin{pmatrix}1\\-1\\\sqrt{2}\end{pmatrix}$ $\overrightarrow{WN}=\overrightarrow{ON}-\overrightarrow{OW}$ $\overrightarrow{ON}=\overrightarrow{WN}+\overrightarrow{OW}$ $=\begin{pmatrix}1\\-1\\\sqrt{2}\end{pmatrix}+\begin{pmatrix}0\\2\\4\end{pmatrix}$ $=\begin{pmatrix}1\\1\\4+\sqrt{2}\end{pmatrix}$ Hence the nearest point on the roof from point W is $N\left(1,1,4+\sqrt{2}\right)$. (shown)
(iv)	Direction vector of line EF $=\begin{pmatrix}1\\1\\0\end{pmatrix}\times\begin{pmatrix}1\\-1\\\sqrt{2}\end{pmatrix}=\begin{pmatrix}\sqrt{2}\\-\sqrt{2}\\-2\end{pmatrix}$ Line EF: $\mathbf{r}=\begin{pmatrix}1\\1\\4+\sqrt{2}\end{pmatrix}+\mu\begin{pmatrix}\sqrt{2}\\-\sqrt{2}\\-2\end{pmatrix}$ *where* $\mu\in\mathbb{R}$ Hence, a Cartesian equation of line EF is $\frac{x-1}{\sqrt{2}}=\frac{1-y}{\sqrt{2}}=\frac{4+\sqrt{2}-z}{2}$, i.e. $x-1=1-y=\frac{4+\sqrt{2}-z}{\sqrt{2}}$ ***(Qn: Why can't we simply solve the equations of plane EFGH and plane ABFE with GC to obtain equation of line EF? What is the limitation of this approach?)***

9. **2019 SAJC Prelim/P2/ Q3**

The plane π_1 has equation $\mathbf{r}\cdot\begin{pmatrix}1\\-1\\3\end{pmatrix}=10$, and the coordinates of A and B are (2, a, 2), (1, 0, 3) respectively, where a is a constant.

(i) Verify that B lies on π_1. [1]

(ii) Given that A does not lie on π_1, state the possible range of values for a. [1]

(iii) Given that $a=9$, find the coordinates of the foot of the perpendicular from A to π_1. Hence, or otherwise, find the vector equation of the line of reflection of the line AB in π_1. [5]

The plane π_2 has equation $\mathbf{r}\cdot\begin{pmatrix}1\\0\\1\end{pmatrix}=4$.

(iv) Find the acute angle between π_1 and π_2. [2]

(v) Find the cartesian equations of the planes such that the perpendicular distance from each plane to π_2 is $\frac{5\sqrt{2}}{2}$. [3]

(i)	$\overrightarrow{OB}\cdot\begin{pmatrix}1\\-1\\3\end{pmatrix}=1+0+9=10$ B lies on π_1.
(ii)	$\begin{pmatrix}2\\a\\2\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\3\end{pmatrix}\neq 10$ $2-a+6\neq 10$ $a\neq -2$ The range of values is $a\in\mathbb{R}, a\neq -2$.

(iii)	Let the foot of perpendicular be *F*. The line through *AF* has vector equation $l_{AF}: \mathbf{r} = \begin{pmatrix} 2 \\ 9 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix},\ \lambda \in \mathbb{R}.$ Since *F* lies on l_{AF}, $\overrightarrow{OF} = \begin{pmatrix} 2 \\ 9 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$ for some fixed $\lambda \in \mathbb{R}$ Since *F* lies on π_1, $\overrightarrow{OF} \cdot \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = 10$ $\therefore \left[\begin{pmatrix} 2 \\ 9 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \right] \cdot \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = 10$ $2 - 9 + 6 + 11\lambda = 10$ $\lambda = 1$ $\overrightarrow{OF} = \begin{pmatrix} 2 \\ 9 \\ 2 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 8 \\ 5 \end{pmatrix}$ The coordinates of *F* are $(3, 8, 5)$. Let the point of reflection of *A* about π_1 be *A*'. $\dfrac{\overrightarrow{OA} + \overrightarrow{OA'}}{2} = \overrightarrow{OF}$ $\overrightarrow{OA'} = 2\overrightarrow{OF} - \overrightarrow{OA} = \begin{pmatrix} 6 \\ 16 \\ 10 \end{pmatrix} - \begin{pmatrix} 2 \\ 9 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \\ 8 \end{pmatrix}$ $\overrightarrow{BA'} = \overrightarrow{OA'} - \overrightarrow{OB} = \begin{pmatrix} 4 \\ 7 \\ 8 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \\ 5 \end{pmatrix}$ Line of reflection, $l_{BA'}$, has vector equation $l_{BA'}: \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 7 \\ 5 \end{pmatrix},\ \mu \in \mathbb{R}$

(iv)	Acute angle between π_1 and π_2 is $\cos^{-1}\left\|\dfrac{\begin{pmatrix}1\\-1\\3\end{pmatrix}\cdot\begin{pmatrix}1\\0\\1\end{pmatrix}}{\left\|\begin{pmatrix}1\\-1\\3\end{pmatrix}\right\|\left\|\begin{pmatrix}1\\0\\1\end{pmatrix}\right\|}\right\| = \cos^{-1}\left\|\dfrac{4}{\sqrt{11}\sqrt{2}}\right\| = 31.5^\circ\ (1\text{ d.p.})$
(v)	The desired planes have equation $\mathbf{r}\cdot\begin{pmatrix}1\\0\\1\end{pmatrix} = D$, where *D* is the constant to be determined. Distance between the planes is given by $\left\|\dfrac{D}{\sqrt{2}}-\dfrac{4}{\sqrt{2}}\right\| = \dfrac{5}{\sqrt{2}}$ $\dfrac{D}{\sqrt{2}}-\dfrac{4}{\sqrt{2}} = \pm\dfrac{5}{\sqrt{2}}$ $D = 4\pm 5$ $D = 9$ or $D = -1$ The possible equations are $x+z=9$ or $x+z=-1$. **<u>Alternative Solution</u>** Let a point *D* on the desired plane have coordinates (x, y, z). Then $\dfrac{\left\|\overrightarrow{AD}\cdot\begin{pmatrix}1\\0\\1\end{pmatrix}\right\|}{\sqrt{2}} = \dfrac{5}{\sqrt{2}}$ $\left\|\begin{pmatrix}x-1\\y\\z-3\end{pmatrix}\cdot\begin{pmatrix}1\\0\\1\end{pmatrix}\right\| = 5$ $\|x-1+z-3\| = 5$ $x+z-4 = \pm 5$ $x+z = 4\pm 5$ The possible equations are $x+z=9$ or $x+z=-1$.

10. **2021 MYE SAJC/P1/Q1**

A factory manufactures machine components by using laser beams to cut through metal sheets which are placed in fixed positions. Laser beams travel in straight lines and the widths of the beams can be neglected.

The source of the laser is from a fixed point X with coordinates $(1,\ 2,\ 3)$. The initial ray of the laser beam is sent in the direction $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$. This laser beam makes an initial cut at point A on a metal sheet that is part of the plane $4x + y + z = 39$.

(i) Find the coordinates of A. [3]

(ii) B is a point on the metal sheet such that XB is perpendicular to the plane. Starting with its initial cut at A, the laser beam from X now changes its direction such that the laser beam cuts out a circle with center B on the metal sheet. Find the exact area of this circle. [3]

The metal sheet has been replaced with a new one that is put in the same fixed position as the previous one. The factory worker operating the machine intends to programme the laser beam such that there will be a straight line cut, l on the metal sheet given by the equation $x - 8 = \dfrac{y-5}{-2} = \dfrac{z-2}{-2}$.

(iii) Show that l lies on plane. [2]

(iv) To begin the cutting process, the laser beam has to make an initial point cut on the metal sheet that can lie on any point along the line l. Let C be the initial point cut on the metal sheet. Find the coordinates of C such that the length of laser beam XC is minimised. [4]

(v) It is required that a triangle be cut out from the metal sheet. The cutting process started from point C along line l towards the point D with coordinates (9, 3, 0). The worker then decides that the points D, C and B will be the vertices of the triangle, where B is the point on the metal sheet such that XB is perpendicular to the metal sheet. Find the exact area of the triangle cut out. [2]

(i)	Equation of beam: $r = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+2\lambda \\ 2+\lambda \\ 3+\lambda \end{pmatrix}, \lambda \in \mathbb{R}$ Since laser beam intersects plane, $\begin{pmatrix} 1+2\lambda \\ 2+\lambda \\ 3+\lambda \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} = 39$ $4+8\lambda+2+\lambda+3+\lambda=39$ $10\lambda = 30$ $\lambda = 3$

	$\overrightarrow{OA} = \begin{pmatrix} 7 \\ 5 \\ 6 \end{pmatrix}$ Coordinates of A = $(7,\ 5,\ 6)$
(ii)	$\overrightarrow{XB} = \mu \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}$ $\overrightarrow{OB} = \mu \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} + \overrightarrow{OX} = \begin{pmatrix} 4\mu \\ \mu \\ \mu \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1+4\mu \\ 2+\mu \\ 3+\mu \end{pmatrix}$ $\overrightarrow{OB} \cdot \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} = 39$ $\begin{pmatrix} 1+4\mu \\ 2+\mu \\ 3+\mu \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} = 39$ $4+16\mu+2+\mu+3+\mu=39$ $18\mu=30$ $\mu=\frac{5}{3}$ $\overrightarrow{OB} = \frac{1}{3}\begin{pmatrix} 23 \\ 11 \\ 14 \end{pmatrix}$ $\overrightarrow{AB} = \frac{1}{3}\begin{pmatrix} 23 \\ 11 \\ 14 \end{pmatrix} - \begin{pmatrix} 7 \\ 5 \\ 6 \end{pmatrix} = \frac{2}{3}\begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$ Radius of circle = $\left\|\overrightarrow{AB}\right\| = \left\|\frac{2}{3}\begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}\right\| = 2$ Area of circle = $\pi r^2 = 4\pi$ units^2
(iii)	$x-8 = \frac{y-5}{-2} = \frac{z-2}{-2} \Rightarrow r = \begin{pmatrix} 8 \\ 5 \\ 2 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$ Sub into equation of plane: $\left[\begin{pmatrix} 8 \\ 5 \\ 2 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}\right] \cdot \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} = 32+5+2+\lambda_1(4-2-2) = 39$ (shown)

(iv)	For minimum $\left\|\overrightarrow{XC}\right\|$, $\overrightarrow{XC}$ must be perpendicular to *l*. $\overrightarrow{XC}\cdot\begin{pmatrix}1\\-2\\-2\end{pmatrix}=0$ $\overrightarrow{OC}=\begin{pmatrix}8\\5\\2\end{pmatrix}+\lambda_1\begin{pmatrix}1\\-2\\-2\end{pmatrix}$ $\overrightarrow{XC}=\begin{pmatrix}8+\lambda_1\\5-2\lambda_1\\2-2\lambda_1\end{pmatrix}-\begin{pmatrix}1\\2\\3\end{pmatrix}=\begin{pmatrix}7+\lambda_1\\3-2\lambda_1\\-1-2\lambda_1\end{pmatrix}$ $\begin{pmatrix}7+\lambda_1\\3-2\lambda_1\\-1-2\lambda_1\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\-2\end{pmatrix}=0$ $7+\lambda_1-6+4\lambda_1+2+4\lambda_1=0$ $9\lambda_1+3=0$ $\lambda_1=-\frac{1}{3}$ $\overrightarrow{OC}=\frac{1}{3}\begin{pmatrix}23\\17\\8\end{pmatrix}$ Coordinates of $C=\left(\frac{23}{3},\ \frac{17}{3},\ \frac{8}{3}\right)$
(v)	Area of triangle ABC $=\frac{1}{2}\left\|\overrightarrow{BC}\times\overrightarrow{CD}\right\|$ $=\frac{1}{2}\left\|\begin{pmatrix}0\\2\\-2\end{pmatrix}\times\frac{4}{3}\begin{pmatrix}1\\-2\\-2\end{pmatrix}\right\|=\frac{2}{3}\left\|\begin{pmatrix}-8\\-2\\-2\end{pmatrix}\right\|=\frac{2\sqrt{72}}{3}=4\sqrt{2}\text{ units}^2$