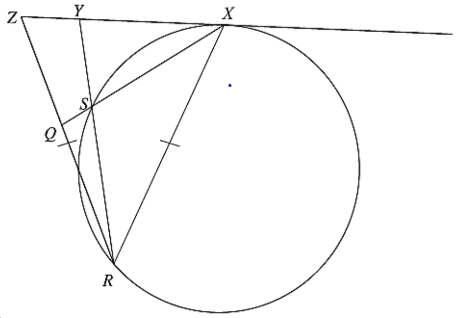
Plane Geometry

1 In the figure, XYZ is a straight line that is tangent to the circle at X. XQ bisects $\angle RXZ$ and cuts the circle at S. RS produced meets XZ at Y and $ZR = XR$.
Prove that

a) $SR = SX$,

b) a circle can be drawn passing through Z, Y, S and Q.

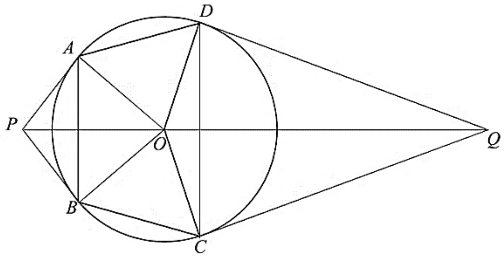


2 In the diagram, A, B, C and D are points on the circle centre O. AP and BP are tangents to the circle at A and B respectively. DQ and CQ are tangents to the circle at D and C respectively. POQ is a straight line.

(i)Prove that angle $CPD = 2 \times$ angle CDQ.

(ii)Make a similar deduction about angle AOB.

(iii)Prove that $2 \times$ angle $OAD =$ angle $CDQ +$ angle BAP



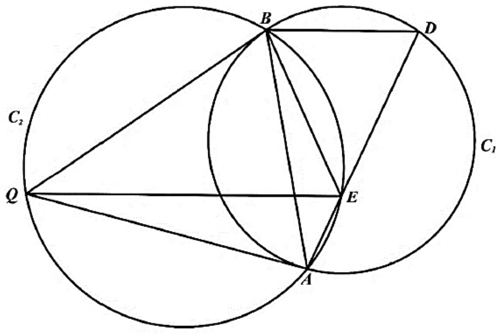
3 The diagram shows two intersecting circles, C_1 and C_2. C_1 passes through the vertices of the triangle ABD. The tangents to C_1 at A and B intersect at the point Q on C_2. A line os drawn from Q to intersect the line AD at E on C_2.

Prove that

(i)QE bisects angle AEB,

(ii)$EB = ED$,

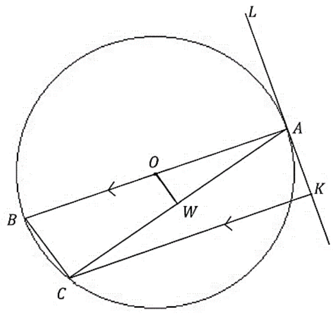
(iii)BD is parallel to QE.

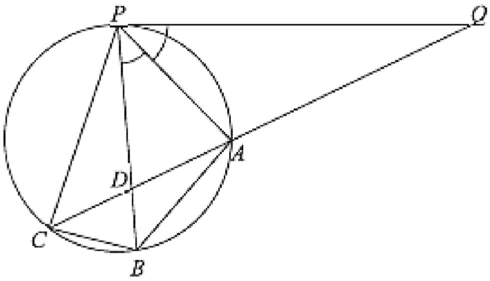
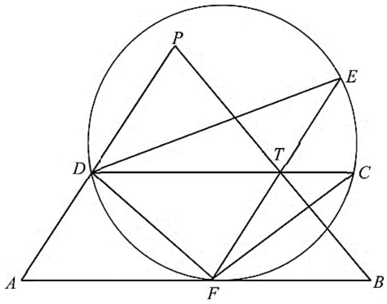
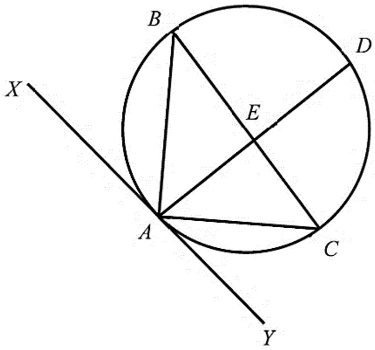
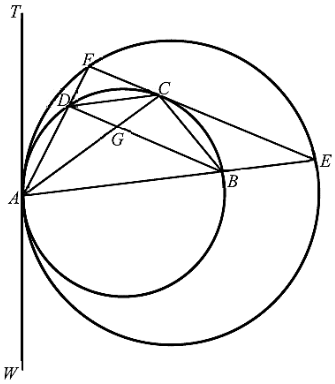
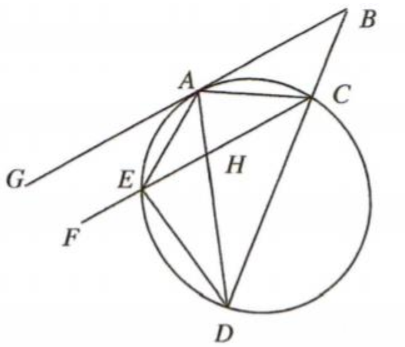


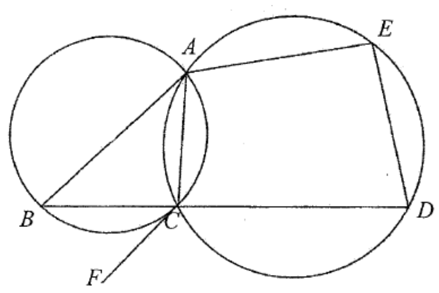
4 In the diagram, A, B and C are three points on the circle such that AB is the diameter of the circle and W is the midpoint of AC. AB and CK are parallel to each other and KL is a tangent to the circle at A

(i)Prove that OW is parallel to BC.

(ii)Prove that Angle $AWO =$ Angle AKC.



5	The diagram shows a point P on a circle and PQ is a tangent to the circle. Points A, B and C lie on the circle such that PA bisects angle QPB and QAC is a straight line. The lines QC and PB intersect at D.  (i) Prove that $AP = AB$. (ii) Prove that CD bisects angle PCB. (iii) Prove that triangles CDP and CBA are similar.
6	The diagram shows a circle passing through points D, E, C and F, where $FC = FD$. The point D lies on AP such that $AD = DP.DC$ and EF cut PB at T such that $PT = TB$.  (i) Show that AB is a tangent to the circle at point F. (ii) By showing that triangle DFT and triangle EFD are similar show that $DF^2 - FT^2 = FT \times ET$.
7	Given that AD and BC are straight lines, AC bisects angle DAY and AB bisects angle DAX, show that  (i) $AC^2 = EC \times BC$, (ii) BC is a diameter of the circle, (iii) AD and BC are perpendicular to each other.
8	In the diagram, two circles touch each other at A. TA is tangent to both circles at A and FE is a tangent to the smaller circle at C. Chords AE and AF intersect the smaller circle at B and D respectively. Prove that  (i) line BD is parallel to line FE, (ii) $\angle FAC = \angle CAE$.
9	In the diagram, $ACDE$ is a cyclic quadrilateral. Lines GAB and $FEHC$ are parallel, and line GAB is a tangent to the circle at A. Lines AD and EC meet at H. Prove that  (i) triangle ABD and triangle CBA are similar, (ii) triangle ACH and triangle ADC are similar, (iii) AD bisects angle CDE, (iv) $AB \times AH = AC \times BC$.

10	The diagram shows two circles that intersect each other at points A and C. The points E and D lie on the circumference of the larger circle. The point B lies on the circumference of the smaller circle such that BCD is a straight line. Line CF is a tangent to the smaller circle at C. $AC = BC$ and $AE = ED$. (i) Prove that AB and CF are parallel. (ii) Prove that ΔABC is similar to ΔADE and hence show that $AB \times DE = AD \times BC$.	

Answers

1	(a) $\angle ZXQ = \angle SRX$ (Alternate Segment Theorem) $\angle ZXQ = \angle QXR$ (XQ is the angle bisector of $\angle RXZ$) $\angle QXR = \angle SRX$ By base angles of isosceles triangles, $SR = SX$ (b) Let $\angle QXR$ be x $\angle RSX = 180^\circ - 2x$ (Isosceles Triangle) $\angle YSQ = 180^\circ - 2x$ (Vertically Opposite Angles) $\angle RZX = \angle ZXR = 2x$ (Base angles of Isosceles Triangle) $\angle RZX + \angle YSQ = 180^\circ - 2x + 2x = 180^\circ$ Since opposite angles are supplementary in cyclic quadrilaterals, a circle that passes through Z, Y, S and Q can be drawn. **Alternative** Similar but use of tangent secant theorem.
2	Let $\angle CDQ = a$ $\angle ODQ = 90^\circ$ (tan ⊥ rad) $\therefore \angle ODC = 90^\circ - a$ $\therefore \angle COD = 180^\circ - 2(90^\circ - a)(\angle \text{sum}, \Delta COD)$ $\angle AOB = 2 \times \angle BAP$ From (i) and (ii), $2(\angle CDQ + \angle BAP = \angle COD + \angle AOB$ $\angle CDQ + \angle BAP = \frac{1}{2}(\angle COD + \angle AOB)$ $= \angle AOP + \angle DOQ$ (⊥ prop of chord) $= 180^\circ - \angle AOD$ $= 2\angle OAD$
3	(i) Let $\angle QEA = x^\circ$ $\angle QBA = \angle QEA$ (angles in same segment in C_2) B1 $= x^\circ$ $QB = QA$ (tangents to C_1 from external point Q) B1 $\angle QAB = \angle QBA$ (base angles of isosceles triangle) B1 $= x^\circ$ $\therefore \angle QEB = \angle QEA$ Hence, QE bisects angle AEB. (ii) $\angle QBA = x^\circ$ (from (i)) $\angle ADB = \angle QBA$ (angles in alternate segment in C_1) either $= x^\circ$ $\angle AEB = 2x^\circ$ (from (i)) $\angle DBE = \angle AEB - \angle ADB$ (exterior angle of triangle BDE) or B1 $= 2x^\circ - x^\circ$ $= x^\circ$ $\therefore \angle ADB = \angle EDB = \angle DBE = x^\circ$ (base angles of isosceles triangle BDE) B1 Hence $EB = ED$ (iii) [2] From (i) $\angle EBD = \angle QEB = x$ B1 $\therefore \angle EBD$ and $\angle QEB$ are alternate angles of parallel lines. (alternate angles are equal) B1 BD is parallel to QE

4	O is the midpoint of AB and W is the midpoint of AC. By Midpoint Theorem, BC is parallel to OW. Angle AOW = Angle ABC (corr angles, $OW \| BC$) Angle ABC = Angle CAK (alt segment theorem) $\rightarrow Angle\ AOW = Angle\ CAK$ Angle $BAC = Angle\ ACK\ (alt\ angles, AB \| CK)$ ∴Angle AWO $= 180° - Angle\ BAC - Angle\ AOW\ (Angle\ sum\ of\ \triangle)$ $= 180° - Angle\ ACK - Angle\ CAK$ $= Angle\ AKC$ (shown)
5	(i) $\angle ABP = \angle APQ$ (alt. segment theorem) Since PA bisects $\angle QPB$, $\angle APQ = \angle APB$ ∴$\angle ABP = \angle APB$ (base $\angle s$ of isosceles triangle APB) Hence, $AP = AB$. (i) $\angle ACB = \angle APB$ ($\angle s$ in the same segment) $\angle ACP = \angle ABP$ ($\angle s$ in the same segment) $= \angle APB$ (shown) $\angle ACB = \angle ACP$ Hence, CD bisects $\angle PCB$. (ii) $\angle ACB = \angle ACP$ (from ii) $\angle CPD = \angle CAB$ ($\angle s$ in the same segment) Hence, △CDX and △CBA are similar.
6	(i) DT is parallel to AB. (Midpoint Theorem) $\angle AFD = \angle TDF$ (alt angles) $= \angle FED$ Since $\angle AFD$ and $\angle FED$ satisfies the alternate segment theorem, AB is a tangent at F. (ii) $\angle DFE$ is common. $\angle TDF = \angle DCF$ (base angles of an isos triangle) $\angle DCF = \angle DEF$ (angles in the same segment) ∴ DFT and EFD are similar triangles (AA) $\frac{DF}{EF} = \frac{FT}{FD}$ $DF^2 = FT \times EF$ $= FT \times (ET + TF)$ $= FT^2 + FT \times ET$ $DF^2 = FT^2 + FT \times ET$

7	(i) $\angle BCA = \angle ACE$ (Common angle) $\angle ABC = \angle CAY$ (Angles in the alternate segments) $= \angle EAC$ (AC bisects $\angle DAY$) $\therefore \triangle ABC$ and $\triangle AEC$ are similar. $\frac{AC}{EC} = \frac{BC}{AC}$ (corresponding sides of similar triangles) $AC^2 = EC \times BC$ (shown) (i) $\angle CAY = \angle EAC$ (AC bisects $\angle DAY$) $\angle BAX = \angle EAB$ (AB bisects $\angle BAX$) $\angle BAX + \angle EAB + \angle EAC + \angle CAY = 180^\circ$ (angles on a straight line) $2\angle EAB + 2\angle EAC = 180^\circ$ $\angle EAB + \angle BAC = 90^\circ$, BC is a diameter of the circle. (ii) $\angle ABE = \angle CAY$ (Angles in the alternate segments) $\angle CAY = \angle EAC$ (AC bisects $\angle BAY$) $\therefore \angle ABE = \angle EAC$ $\angle EAB + \angle EAC = \angle EAB + \angle ABE = 90^\circ$ (from (ii)) $\angle AEB = 90^\circ$ (sum of $\angle s$ in a triangle) $\therefore$ AD and BC are perpendicular.
8	(i) To prove: $BD \| FE$ Proof: Let $\angle TAF$ be $\boldsymbol{\theta}$. $\angle ABD = \angle TAF = \theta$ (alt seg thm) $\angle AEF = \angle TAF = \theta$ (alt seg thm) $\therefore \angle ABD = \angle AEF = \theta$ Using property of corresponding angles, $BD \| EF$ (shown) (ii)To prove: $\angle FAC = \angle CAE$ Proof: Let $\angle BCE = \alpha$ $\angle CBD = \angle BCE = \alpha$ (alt $\angle s$, $BD \| EF$) $\angle FAC = \angle CBD = \alpha$ ($\angle s$ in same segment) Also, $\angle CAE = \angle BCE = \alpha$ (alt seg thm) $\therefore \angle FAC = \angle CAE = \alpha$ (shown)
9	(i) $\angle CAB = \angle CDA$ (Alternate Segment Theorem) And $\angle BDA = \angle CDA$ (same angle) $\angle ABC = \angle ABD$ (Common angle) Triangle ABD is similar to triangle CBA. (AA) (ii) $\angle CAB = \angle CDA$ (Alternate Segment Theorem) $\angle CAB = \angle ACH$ (Alternate angles, $BAB \| FEHC$) Hence $\angle ACH = \angle CDA$ $\angle HAC = \angle DAC$ (Common angle) Triangle ACH is similar to triangle ADC. (AA) (iii) From (ii), $\angle ACH = \angle CDA$ $\angle ACH = ADE$ (Angles in the same segment)

	Hence $\angle ADE = \angle CDA$ Therefore, *AD* bisects angle *CDE*. (iv) Triangle *ABD* is similar to triangle *CBA*. $\frac{AB}{BC} = \frac{AD}{AC}$ Triangle *ACH* is similar to triangle *ADC*. $\frac{AC}{AH} = \frac{AD}{AC}$ Hence $\frac{AB}{BC} = \frac{AC}{AH}$ $AB \times AH = AC \times BC$
10	(i) Let $\angle ABC = x$ $\angle CAB = \angle ABC = x\ (AC = BC)$ $\angle CAB = \angle FCB = x$ (tangent chord theorem) Since $\angle FCB = \angle ABC = x, AB$ and CF are parallel, alternate angles. (ii) $\angle ACB = 180^\circ - 2x$ (angle sum of triangle) $\angle ACD = 2x$ (Supplementary angle) $\angle AED = 180^\circ - 2x$ (angle in opposite segment) $= \angle ACB$ Since, $\angle EAD = x$ (angle sum of isosceles triangle) $= \angle ABC$ Triangle *ABC* is similar to triangle *ADE* $\frac{AB}{AD} = \frac{BC}{DE}$ $AB \times DE = BC \times AD$