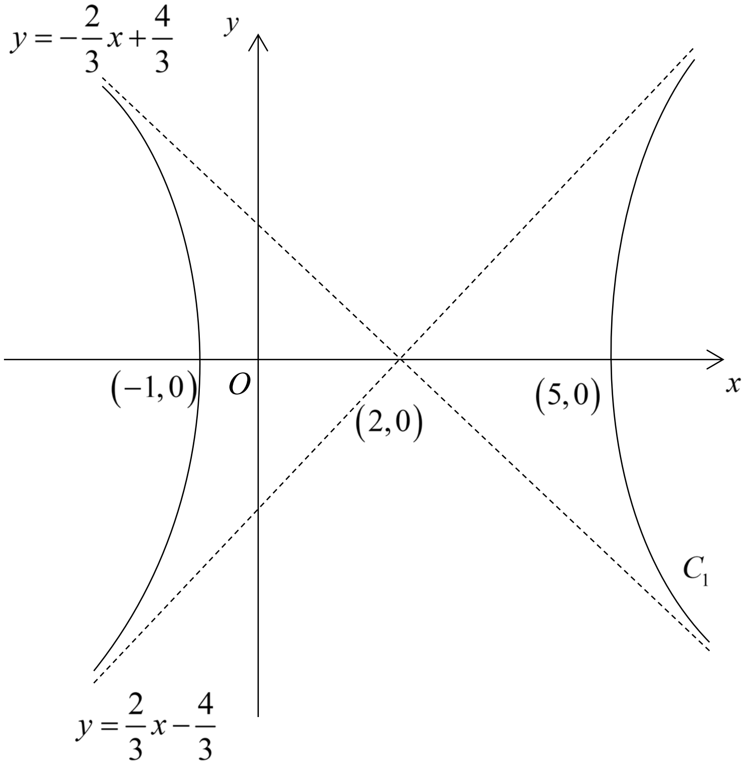
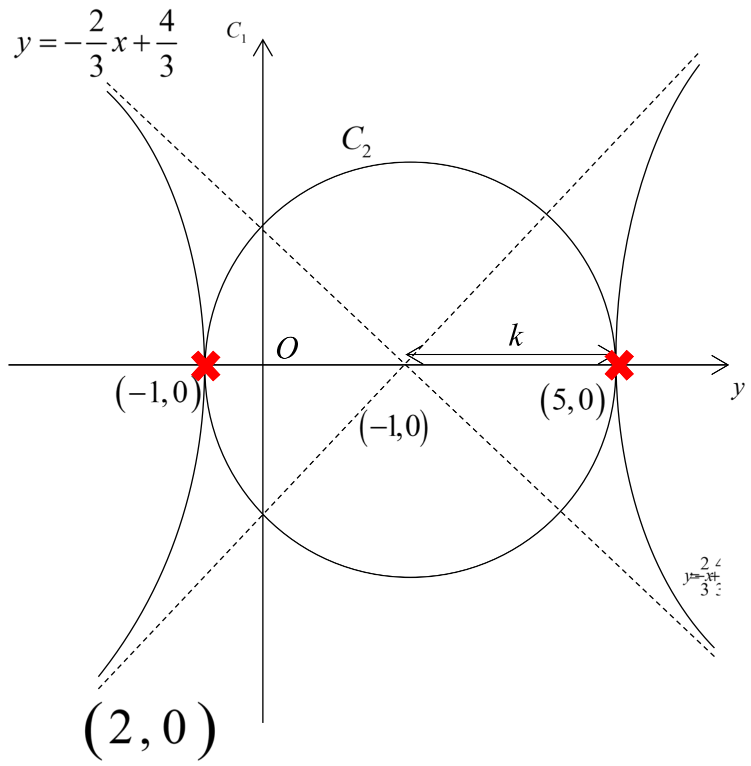
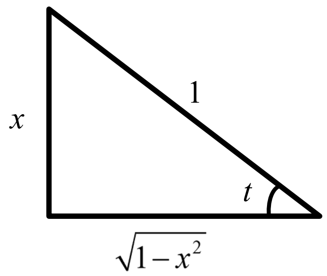
Qn	Solution
1	**Graphing Techniques**
(a)	Finding asymptotes for C_1: $\frac{(x-2)^2}{3^2}-\frac{y^2}{2^2}=0$ $y=\pm\frac{2}{3}(x-2)$ 

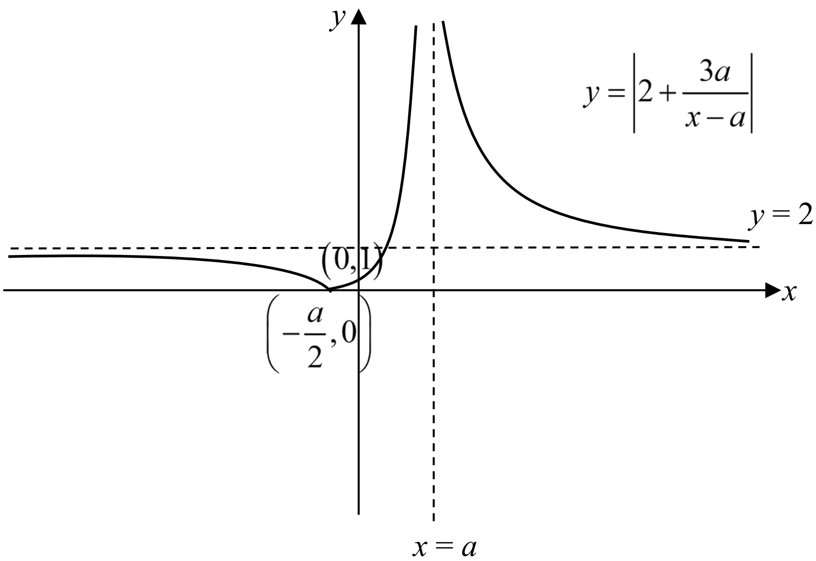
(b) C_2 is a circle centred at (2,0) with radius k.

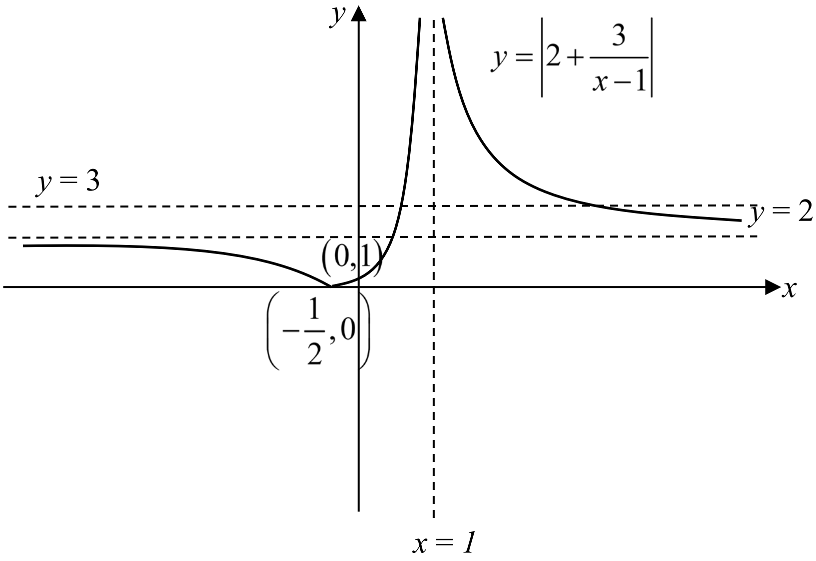
For C_1 and C_2 to intersect exactly twice, $k = 3$.



Qn	Solution
2	**Techniques of Integration**
(a)	$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{x^2+2x}\right)=\left(2x+2\right)\mathrm{e}^{x^2+2x}$ $\int_0^1\left(x+1\right)\mathrm{e}^{x^2+2x}\ \mathrm{d}x=\frac{1}{2}\int_0^1 2\left(x+1\right)\mathrm{e}^{x^2+2x}\ \mathrm{d}x$ $=\frac{1}{2}\left[\mathrm{e}^{x^2+2x}\right]_0^1$ $=\frac{1}{2}\left[\mathrm{e}^3-\mathrm{e}^0\right]$ $=\frac{1}{2}\left(\mathrm{e}^3-1\right)$
(b)	$x=\sin t\Rightarrow\frac{\mathrm{d}x}{\mathrm{d}t}=\cos t$ $\int\frac{1}{\left(1-x^2\right)^{\frac{3}{2}}}\ \mathrm{d}x$ $=\int\frac{1}{\left(1-\sin^2 t\right)^{\frac{3}{2}}}\cos t\ \mathrm{d}t$ $=\int\frac{1}{\left(\cos^2 t\right)^{\frac{3}{2}}}\cos t\ \mathrm{d}t$ $=\int\frac{1}{\left(\cos^3 t\right)}\cos t\ \mathrm{d}t$ $=\int\frac{1}{\left(\cos^2 t\right)}\ \mathrm{d}t$ $=\int\sec^2 t\ \mathrm{d}t$ $=\tan t+C\qquad ,C\in\mathbb{R}$ $=\frac{x}{\sqrt{1-x^2}}+C$  $\sin t=\frac{x}{1}$ $\tan t=\frac{x}{\sqrt{1-x^2}}$

Qn	Solution
3	**Arithmetic and Geometric Series**
(a)	$$\frac{n}{2}\left[2(5)+(n-1)(3)\right]\le 100$$ $$\frac{n}{2}\left[2(5)+(n-1)(3)\right]-100\le 0$$ Using GC, When $n=7$, $\frac{n}{2}\left[2(5)+(n-1)(3)\right]-100=-2\le 0$ When $n=8$, $\frac{n}{2}\left[2(5)+(n-1)(3)\right]-100=24>0$ Maximum number of squares Student A can form using the 100 cm wire is 7.
(b)	The circumference of the circles follow a geometric progression with common ratio $\frac{2}{3}$. $100=$ Total circumference of 12 circles $100=2\pi x+\frac{2}{3}(2\pi x)+\left(\frac{2}{3}\right)^2(2\pi x)+...+\left(\frac{2}{3}\right)^{11}(2\pi x)$ $100=\dfrac{2\pi x\left(1-\left(\frac{2}{3}\right)^{12}\right)}{1-\frac{2}{3}}$ Using GC, $x=5.3464$ $=5.35\ (3\text{ s.f.})$

Qn	Solution
4	**Graphing and Transformation**
(a)	$y=\frac{1}{x}$ $\downarrow$ Replace x by $x-a$ $y=\frac{1}{x-a}$ $\downarrow$ Replace y by $\frac{y}{3a}$ $y=\frac{3a}{x-a}$ $\downarrow$ Replace y by $y-2$ $y=2+\frac{3a}{x-a}$ Note $a > 0$. 1. Translation of a units in the positive x-direction. 2. Stretch by factor $3a$ parallel to the y-axis. 3. Translation of 2 units in the positive y-direction. OR 1. Stretch by factor $3a$ parallel to the x-axis. 2. Translation of a units in the positive x-direction. 3. Translation of 2 units in the positive y-direction.
(b)	

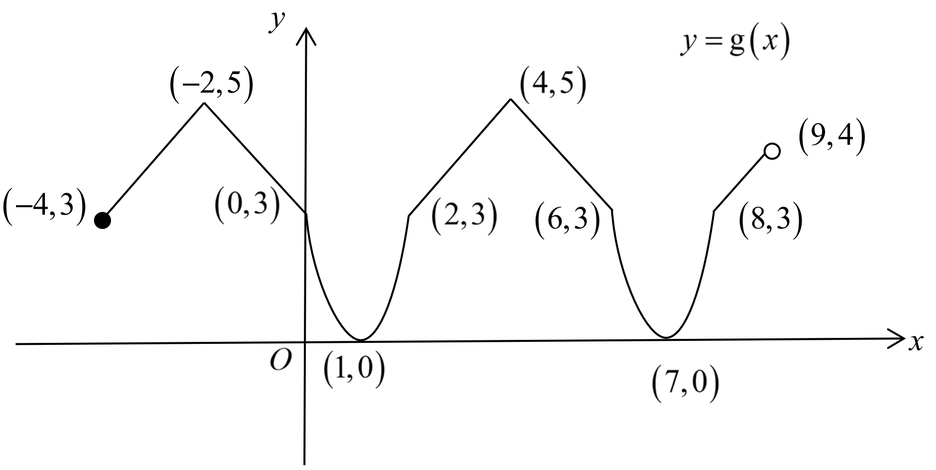
(c)	 $\left\|2+\dfrac{3}{x-1}\right\|=3$ $2+\dfrac{3}{x-1}=\pm 3$ $2+\dfrac{3}{x-1}=3$ or $2+\dfrac{3}{x-1}=-3$ $\dfrac{3}{x-1}=1$ $\dfrac{3}{x-1}=-5$ $x-1=3$ $-5(x-1)=3$ $x=4$ $x=\dfrac{2}{5}$ For $\left\|2+\dfrac{3}{x-1}\right\|<3$, from the graph in part **(b)**, $x<\dfrac{2}{5}$ or $x>4$

Qn	Solution
5	**Complex Numbers**
(a)	Since the coefficients of the polynomial are real, $\sqrt{3}+\text{i}$ is a root implies that $\sqrt{3}-\text{i}$ is also a root. A quadratic factor is: $\left[z-\left(\sqrt{3}+\text{i}\right)\right]\left[z-\left(\sqrt{3}-\text{i}\right)\right]$ $=\left[\left(z-\sqrt{3}\right)-\text{i}\right]\left[\left(z-\sqrt{3}\right)+\text{i}\right]$ $=\left(z-\sqrt{3}\right)^2-(\text{i})^2$ $=z^2-2\sqrt{3}z+3+1$ $=z^2-2\sqrt{3}z+4$ Let $z=k$ be the third root. $z^3-8z+a=\left(z^2-2\sqrt{3}z+4\right)(z-k)$ Comparing coefficient of z^2: $0=-2\sqrt{3}-k$ $k=-2\sqrt{3}$ Therefore $z=\sqrt{3}+\text{i}$ or $\sqrt{3}-\text{i}$ or $-2\sqrt{3}$.
	Alternative method (sub in $\sqrt{3}+\text{i}$ to find a first) (Not recommended in this question) $\text{P}(z)=z^3-8z+a$ Since $z=\sqrt{3}+\text{i}$ is a root, $\text{P}\left(\sqrt{3}+\text{i}\right)=0$. $\left(\sqrt{3}+\text{i}\right)^3-8\left(\sqrt{3}+\text{i}\right)+a=0$ $\left(\sqrt{3}\right)^3+3\left(\sqrt{3}\right)^2(\text{i})+3\left(\sqrt{3}\right)(\text{i})^2+(\text{i})^3-8\sqrt{3}-8\text{i}+a=0$ $a=8\sqrt{3}$ Since the coefficients of the polynomial are real, $\sqrt{3}+\text{i}$ is a root implies that $\sqrt{3}-\text{i}$ is also a root. A quadratic factor is: $\left[z-\left(\sqrt{3}+\text{i}\right)\right]\left[z-\left(\sqrt{3}-\text{i}\right)\right]$ $=\left[\left(z-\sqrt{3}\right)-\text{i}\right]\left[\left(z-\sqrt{3}\right)+\text{i}\right]$ $=\left(z-\sqrt{3}\right)^2-(\text{i})^2$ $=z^2-2\sqrt{3}z+3+1$ $=z^2-2\sqrt{3}z+4$ By comparing constant term, $z^3-8z+8\sqrt{3}=\left(z^2-2\sqrt{3}z+4\right)\left(z+2\sqrt{3}\right)$ $z=\sqrt{3}+\text{i}$ or $\sqrt{3}-\text{i}$ or $-2\sqrt{3}$

(b)	$\arg w = \frac{5\pi}{6}$ $\arg w^n = \frac{5\pi}{6} n$ For w^n to be purely imaginary, $\arg w^n = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}$ $\frac{5n\pi}{6} = \frac{\pi}{2} + k\pi$ $n = \frac{6k+3}{5}$ Using GC, the smallest three positive integers of *n*, $n = 3, 9, 15$.

Qn	Solution
6	**Functions and Equations and Inequality**
(a)	$a(-2)^2-2b+c=17 \quad \Rightarrow \quad 4a-2b+c=17 \quad$ ---(1) $a\left(\frac{1}{2}\right)^2+\frac{1}{2}b+c=\frac{3}{4} \quad \Rightarrow \quad \frac{1}{4}a+\frac{1}{2}b+c=\frac{3}{4} \quad$ ---(2) $a(5)^2+5b+c=3 \quad \Rightarrow \quad 25a+5b+c=3 \quad$ ---(3) Using GC, $a=1$, $b=-5$, $c=3$. $y=x^2-5x+3$
(b)	Let $y=\mathrm{f}(x)=x^2-5x+3, \quad x\le 0$ $y=x^2-5x+3$ $y=\left(x-\frac{5}{2}\right)^2-\left(\frac{5}{2}\right)^2+3$ $y+\frac{13}{4}=\left(x-\frac{5}{2}\right)^2$ $x-\frac{5}{2}=\pm\sqrt{y+\frac{13}{4}}$ $x=\frac{5}{2}\pm\sqrt{y+\frac{13}{4}}$ $x=\frac{5}{2}-\sqrt{y+\frac{13}{4}} \quad (\text{since } x\le 0)$ $\mathrm{f}^{-1}(x)=\frac{5}{2}-\sqrt{x+\frac{13}{4}}$

(c)



(d)

$$0 \le x \le \pi$$

$$-\frac{3}{2} \le \frac{3}{2}\cos x \le \frac{3}{2}$$

$$0 \le \frac{3}{2} + \frac{3}{2}\cos x \le 3$$

Since $R_h = [0,3] \subseteq [0,6) = D_g$, therefore the function gh exists.

$$D_h \xrightarrow{h} R_h \xrightarrow{g} R_{gh}$$

$[0,\pi]$ $\quad$ $[0,3]$ $\quad$ $[0,4]$

Restricted domain of g

$R_{gh} = [0,4]$

Qn	Solution
7	**Differentiation**

(a)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Differentiate w.r.t *x*:

$$\frac{2x}{a^2} + \frac{2y}{b^2}\left(\frac{dy}{dx}\right) = 0$$

$$\frac{2y}{b^2}\left(\frac{dy}{dx}\right) = -\frac{2x}{a^2}$$

$$\frac{dy}{dx} = -\frac{b^2 x}{a^2 y} \quad (\text{since } y \ne 0)(\text{shown})$$

(b)

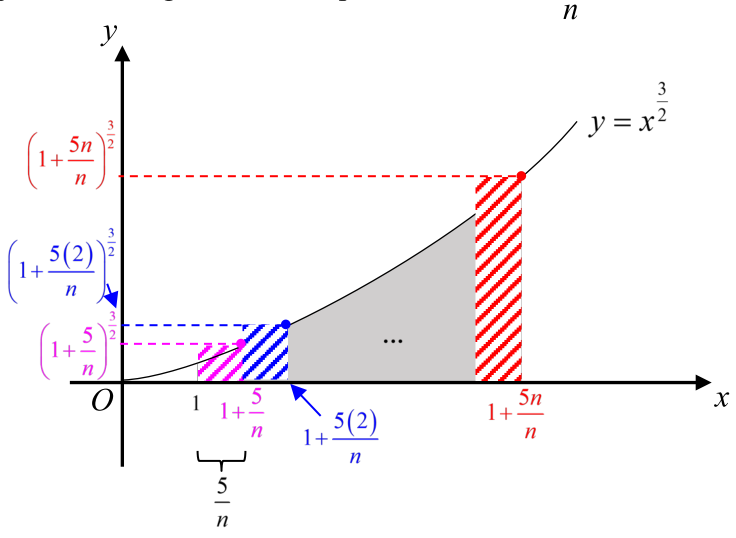
At $P(a\cos\theta, b\sin\theta)$,

$$\frac{dy}{dx} = -\frac{b^2(a\cos\theta)}{a^2(b\sin\theta)} = -\frac{b\cos\theta}{a\sin\theta}$$

Equation of tangent at P,

	$y-(b\sin\theta)=-\dfrac{b\cos\theta}{a\sin\theta}(x-a\cos\theta)$ $\dfrac{y}{b}-\sin\theta=-\dfrac{\cos\theta}{a\sin\theta}(x-a\cos\theta)$ $\dfrac{y}{b}\sin\theta-\sin^2\theta=-\dfrac{\cos\theta}{a}(x-a\cos\theta)$ $\dfrac{y}{b}\sin\theta-\sin^2\theta=-\dfrac{x}{a}\cos\theta+\cos^2\theta$ $\dfrac{x}{a}\cos\theta+\dfrac{y}{b}\sin\theta=\sin^2\theta+\cos^2\theta$ $\dfrac{x}{a}\cos\theta+\dfrac{y}{b}\sin\theta=1$ (shown)
(c)	When $x=0$, $\dfrac{y}{b}\sin\theta=1\Rightarrow y=\dfrac{b}{\sin\theta}$ When $y=0$, $\dfrac{x}{a}\cos\theta=1\Rightarrow x=\dfrac{a}{\cos\theta}$ Area of triangle ORS $=\dfrac{1}{2}\left(\dfrac{b}{\sin\theta}\right)\left(\dfrac{a}{\cos\theta}\right)=\dfrac{ab}{2\sin\theta\cos\theta}=\dfrac{ab}{\sin 2\theta}$
(d)	$0<\theta<\dfrac{\pi}{2}$ $0<\sin 2\theta\le 1$ $\dfrac{1}{\sin 2\theta}\ge 1$ $\dfrac{ab}{\sin 2\theta}\ge ab$ $\sin 2\theta=1$ $\theta=\dfrac{\pi}{4}\quad\left(\text{since } 0<\theta<\dfrac{\pi}{2}\right)$ Therefore, minimum area is ab (shown) and occurs when $\theta=\dfrac{\pi}{4}$.
	Alternative method Let area of triangle ORS be A. $A=\dfrac{ab}{\sin 2\theta}=ab(\sin 2\theta)^{-1}$ $\dfrac{dA}{d\theta}=-ab(\sin 2\theta)^{-2}(2\cos 2\theta)=\dfrac{-2ab\cos 2\theta}{(\sin 2\theta)^2}$ At stationary point, $\dfrac{dA}{d\theta}=0$

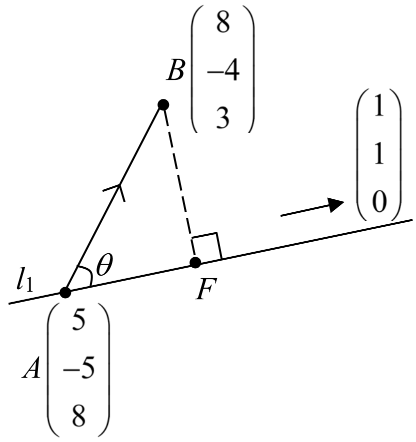
	$\frac{-2ab\cos 2\theta}{(\sin 2\theta)^2} = 0$ $\cos 2\theta = 0$ $\theta = \frac{\pi}{4}$ $(\because\ \theta \text{ is acute})$ Minimum area of triangle ORS $A = \frac{ab}{\sin 2\left(\frac{\pi}{4}\right)} = ab$ Therefore, minimum area is ab and occurs when $\theta = \frac{\pi}{4}$.

Qn	Solution
8	**Definite Integral**
(a)	$\pi\int_1^6 x^3 \, dx$ $= \pi\left[\frac{x^4}{4}\right]_1^6$ $= \frac{\pi}{4}\left(6^4 - 1^4\right)$ $= \frac{1295}{4}\pi$
(b)	The thickness of each circular disc is obtained by dividing x values from 1 to 6 into n equal parts, forming n discs of equal thickness of $\frac{5}{n}$.  $V = \pi\left[\left(1+\frac{5}{n}\right)^{\frac{3}{2}}\right]^2\left(\frac{5}{n}\right) + \pi\left[\left(1+\frac{5(2)}{n}\right)^{\frac{3}{2}}\right]^2\left(\frac{5}{n}\right) + \ldots + \pi\left[\left(1+\frac{5n}{n}\right)^{\frac{3}{2}}\right]^2\left(\frac{5}{n}\right)$ $= \frac{5\pi}{n}\left[\left(1+\frac{5}{n}\right)^3 + \left(1+2\left(\frac{5}{n}\right)\right)^3 + \ldots + \left(1+n\left(\frac{5}{n}\right)\right)^3\right]$ $= \frac{5\pi}{n}\sum_{r=1}^{n}\left(1+r\left(\frac{5}{n}\right)\right)^3 \qquad (\text{Shown})$

(c)	$V = \frac{5\pi}{n}\sum_{r=1}^{n}\left(1+r\left(\frac{5}{n}\right)\right)^3$ $= \frac{5\pi}{n}\sum_{r=1}^{n}\left[1+3\left(\frac{5r}{n}\right)+3\left(\frac{5r}{n}\right)^2+\left(\frac{5r}{n}\right)^3\right]$ $= \frac{5\pi}{n}\left\{\sum_{r=1}^{n}\left(1+\frac{15r}{n}\right)+\frac{75}{n^2}\sum_{r=1}^{n}r^2+\frac{125}{n^3}\sum_{r=1}^{n}r^3\right\}$ $= \frac{5\pi}{n}\left\{\frac{n}{2}\left(1+\frac{15(1)}{n}+1+\frac{15(n)}{n}\right)+\frac{75}{n^2}\left(\frac{1}{6}\right)n(n+1)(2n+1)+\frac{125}{n^3}\left(\frac{1}{4}\right)n^2(n+1)^2\right\}$ $= \frac{5\pi}{n}\left\{\frac{n}{2}\left(17+\frac{15}{n}\right)+\frac{25}{2n}(n+1)(2n+1)+\frac{125}{4n}(n+1)^2\right\}$ $= \frac{5\pi}{n}\left\{\left(\frac{17n}{2}+\frac{15}{2}\right)+\frac{25}{2n}(2n^2+3n+1)+\frac{125}{4n}(n^2+2n+1)\right\}$ $= \frac{5\pi}{n}\left\{\left(\frac{17n}{2}+\frac{15}{2}\right)+\left(25n+\frac{75}{2}+\frac{25}{2n}\right)+\left(\frac{125}{4}n+\frac{125}{2}+\frac{125}{4n}\right)\right\}$ $= \frac{5\pi}{n}\left(\frac{259}{4}n+\frac{215}{2}+\frac{175}{4n}\right)$ $= \frac{5\pi}{4}\left(259+\frac{430}{n}+\frac{175}{n^2}\right)$ $\therefore a = 430, \quad b = 175$
(d)	As $n\to\infty, \frac{430}{n}\to 0, \frac{175}{n^2}\to 0. \therefore V \to \frac{5\pi}{4}(259)$ Limit of $V = \frac{1295\pi}{4}$ Using integration, the volume of revolution of the solid formed when R is rotated through 2π radians about the x-axis is given by $\pi\int_1^6[f(x)]^2\ dx$. Thus, the volume is $\pi\int_1^6\left[x^{\frac{3}{2}}\right]^2 dx = \pi\int_1^6 x^3 dx = \frac{1295}{4}\pi$ as found in part **(a)** which is the same value as the limit of $V = \frac{1295\pi}{4}$. (Verified)

Qn	Solution
9	**Vectors**
(a)	

$\overrightarrow{OA} = \begin{pmatrix} 5 \\ -5 \\ 8 \end{pmatrix}, \overrightarrow{OB} = \begin{pmatrix} 8 \\ -4 \\ 3 \end{pmatrix}$



<u>Method 1: Scalar product = 0</u>

Since F lies on l_1,

$$\overrightarrow{OF} = \begin{pmatrix} 5 \\ -5 \\ 8 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \text{ for some } \lambda \in \mathbb{R}.$$

To find point F,

$$\left[\begin{pmatrix} 5 \\ -5 \\ 8 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 8 \\ -4 \\ 3 \end{pmatrix} \right] \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 0$$

$$\begin{pmatrix} -3+\lambda \\ -1+\lambda \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 0$$

$$-3+\lambda-1+\lambda = 0$$

$$\lambda = 2$$

$$\overrightarrow{OF} = \begin{pmatrix} 5 \\ -5 \\ 8 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 7 \\ -3 \\ 8 \end{pmatrix}$$

<u>Method 2: Vector Projection (Not recommended due to ease of making mistakes)</u>

$$\overrightarrow{AB} = \begin{pmatrix} 8 \\ -4 \\ 3 \end{pmatrix} - \begin{pmatrix} 5 \\ -5 \\ 8 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ -5 \end{pmatrix}$$

$$\overrightarrow{AF} = \left(\frac{\overrightarrow{AB} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}{\left| \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right|} \right) \frac{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}{\left| \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right|}$$

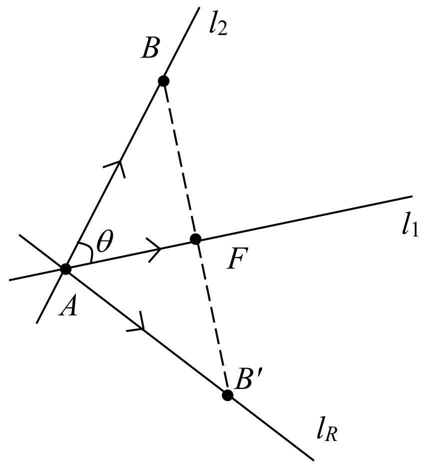
$$= \frac{4}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$$

$$\overrightarrow{OF} = \overrightarrow{OA} + \overrightarrow{AF}$$

$$= \begin{pmatrix} 5 \\ -5 \\ 8 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 7 \\ -3 \\ 8 \end{pmatrix}$$

(b)

Method 1: Midpoint theorem (w.r.t. point *A*)



$$\overrightarrow{AF} = \frac{\overrightarrow{AB} + \overrightarrow{AB'}}{2}$$

$$\overrightarrow{AB'} = 2\overrightarrow{AF} - \overrightarrow{AB}$$

$$= 2\left[\begin{pmatrix} 7 \\ -3 \\ 8 \end{pmatrix} - \begin{pmatrix} 5 \\ -5 \\ 8 \end{pmatrix}\right] - \left[\begin{pmatrix} 8 \\ -4 \\ 3 \end{pmatrix} - \begin{pmatrix} 5 \\ -5 \\ 8 \end{pmatrix}\right]$$

$$= \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$$

$$l_R : \mathbf{r} = \begin{pmatrix} 5 \\ -5 \\ 8 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}, \mu \in \mathbb{R}$$

Method 2: Midpoint theorem (w.r.t. point *O*)

$$\overrightarrow{OF} = \frac{\overrightarrow{OB} + \overrightarrow{OB'}}{2}$$

$$\overrightarrow{OB'} = 2\overrightarrow{OF} - \overrightarrow{OB}$$

$$= 2\begin{pmatrix} 7 \\ -3 \\ 8 \end{pmatrix} - \begin{pmatrix} 8 \\ -4 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ -2 \\ 13 \end{pmatrix}$$

$$\overrightarrow{AB'} = \begin{pmatrix} 6 \\ -2 \\ 13 \end{pmatrix} - \begin{pmatrix} 5 \\ -5 \\ 8 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$$

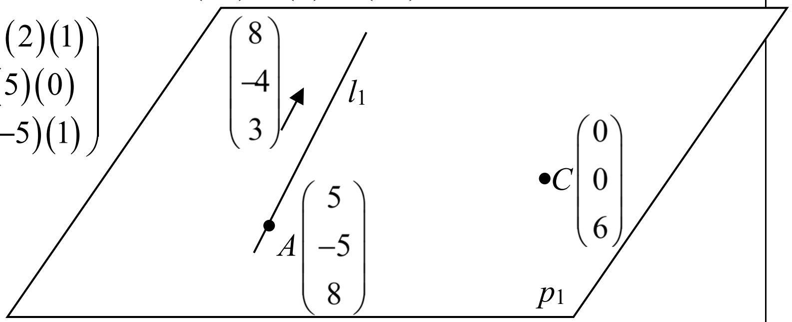
$$l_R : \mathbf{r} = \begin{pmatrix} 5 \\ -5 \\ 8 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}, \mu \in \mathbb{R}$$

(c)

$$\overrightarrow{CA}=\begin{pmatrix}5\\-5\\8\end{pmatrix}-\begin{pmatrix}0\\0\\6\end{pmatrix}=\begin{pmatrix}5\\-5\\2\end{pmatrix} \quad \text{or} \quad \overrightarrow{CF}=\begin{pmatrix}7\\-3\\8\end{pmatrix}-\begin{pmatrix}0\\0\\6\end{pmatrix}=\begin{pmatrix}7\\-3\\2\end{pmatrix}$$

$$n_1=\begin{pmatrix}5\\-5\\2\end{pmatrix}\times\begin{pmatrix}1\\1\\0\end{pmatrix}=\begin{pmatrix}(-5)(0)-(2)(1)\\(2)(1)-(5)(0)\\(5)(1)-(-5)(1)\end{pmatrix}$$

$$=\begin{pmatrix}-2\\2\\10\end{pmatrix}=2\begin{pmatrix}-1\\1\\5\end{pmatrix}$$



$$\mathbf{r}\cdot\begin{pmatrix}-1\\1\\5\end{pmatrix}=\begin{pmatrix}0\\0\\6\end{pmatrix}\cdot\begin{pmatrix}-1\\1\\5\end{pmatrix}=30$$

$p_1: -x+y+5z=30$ (Shown)

(d)

Method 1 (using GC)

$p_1: -x+y+5z=30 \qquad -(1)$

$p_2: -17x-37y+4z=24 \qquad -(2)$

Using GC,

$$l_S: \mathbf{r}=\begin{pmatrix}-21\\9\\0\end{pmatrix}+\alpha\begin{pmatrix}\frac{7}{2}\\\frac{-3}{2}\\1\end{pmatrix}, \alpha\in\mathbb{R}$$

Method 2 (not recommended for this question)

Direction vector of line of intersection

$$=\begin{pmatrix}-1\\1\\5\end{pmatrix}\times\begin{pmatrix}-17\\-37\\4\end{pmatrix}=\begin{pmatrix}189\\-81\\54\end{pmatrix}=27\begin{pmatrix}7\\-3\\2\end{pmatrix}$$

Check that $C(0,0,6)$ lies on both p_1 and p_2.

$$l_S: \mathbf{r}=\begin{pmatrix}0\\0\\6\end{pmatrix}+\alpha\begin{pmatrix}7\\-3\\2\end{pmatrix}, \alpha\in\mathbb{R}$$

(e)

$p_3 : -3x + \alpha y + 15z = \beta$

$$p_3 : \mathbf{r} \cdot \begin{pmatrix} -3 \\ \alpha \\ 15 \end{pmatrix} = \beta$$

Since $p_1 // p_3$,

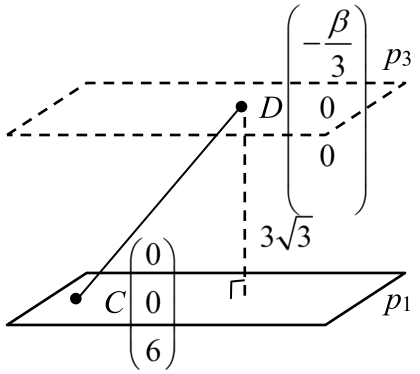
$$\begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix} = k \begin{pmatrix} -3 \\ \alpha \\ 15 \end{pmatrix}$$

$k = \frac{1}{3}, \ \alpha = 3$

Method 1: Length of Projection

Let a point on p_3 be D.

$$\overrightarrow{OD} = \begin{pmatrix} \frac{\beta}{3} \\ 0 \\ 0 \end{pmatrix}$$



$$\overrightarrow{CD} = \begin{pmatrix} \frac{\beta}{3} \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix} = \begin{pmatrix} \frac{\beta}{3} \\ 0 \\ -6 \end{pmatrix}$$

Length of projection =

$$\left| \frac{\begin{pmatrix} \frac{\beta}{3} \\ 0 \\ -6 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix}}{\left| \begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix} \right|} \right| = 3\sqrt{3}$$

$$\left| -\frac{\beta}{3} + 30 \right| = 3\sqrt{3}\left(\sqrt{27}\right)$$

$$-\frac{\beta}{3} + 30 = \pm 27$$

$-\frac{\beta}{3} = -57$ or $-\frac{\beta}{3} = -3$

$\beta = 171$ or $\beta = 9$

Method 2: Distance between planes

$$p_3 : \mathbf{r} \cdot \begin{pmatrix} -3 \\ 3 \\ 15 \end{pmatrix} = \beta \Rightarrow \quad p_3 : \mathbf{r} \cdot \begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix} = \frac{\beta}{3}$$

Since distance between p_1 and p_3 is exactly $3\sqrt{3}$ units,

$$\left| \frac{30 - \frac{\beta}{3}}{\sqrt{(-1)^2 + 1^2 + 5^2}} \right| = 3\sqrt{3}$$

$$\left| 30 - \frac{\beta}{3} \right| = 27$$

$$\frac{\beta}{3} = 30 \pm 27$$

$\therefore \beta = 9$ or $\beta = 171$.

Qn	Solution
10	**Differential Equations**
(a)	$y = ux^2$ Differentiate w.r.t. x, $\frac{\mathrm{d}y}{\mathrm{d}x} = u(2x) + x^2\frac{\mathrm{d}u}{\mathrm{d}x}$ ---------- (1) Substitute (1) and $y = ux^2$ into $\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{2y}{x} = x^3$, $2xu + x^2\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{2(ux^2)}{x} = x^3$ $\frac{\mathrm{d}u}{\mathrm{d}x} = x \quad (\text{since } x \neq 0)$ $u = \frac{1}{2}x^2 + C, \quad C \in \mathbb{R}$ $\frac{y}{x^2} = \frac{1}{2}x^2 + C$ $y = \frac{1}{2}x^4 + Cx^2$
(b)	$\frac{\mathrm{d}N}{\mathrm{d}t} = kN, \quad k > 0$ When $t = 0, N = 5000, \frac{\mathrm{d}N}{\mathrm{d}t} = 200$, $\frac{\mathrm{d}N}{\mathrm{d}t} = kN$ $200 = 5000k$ $k = \frac{1}{25}$ $\frac{1}{N}\frac{\mathrm{d}N}{\mathrm{d}t} = \frac{1}{25}$ $\int \frac{1}{N}\,\mathrm{d}N = \int \frac{1}{25}\,\mathrm{d}t$ $\ln\lvert N\rvert = \frac{1}{25}t + C$ $N = A\mathrm{e}^{\frac{t}{25}} \quad \text{where } A = \pm\mathrm{e}^{C}$ When $t = 0, N = 5000$, $5000 = A\mathrm{e}^{\frac{0}{25}}$ $A = 5000$ $\therefore N = 5000\mathrm{e}^{\frac{t}{25}}$ When $t = 50$, $N = 5000\mathrm{e}^{\frac{50}{25}}$ $= 36900 \quad (3\text{ s.f.})$

(c)	$\frac{dN}{dt} = kN(\ln M - \ln N)$ $\int \frac{1}{N(\ln M - \ln N)}\,dN = \int k\ dt$ $-\int \frac{-\frac{1}{N}}{(\ln M - \ln N)}\,dN = \int k\ dt$ $-\ln\left\|\ln M - \ln N\right\| = kt + D$ $\ln M - \ln N = \pm e^{-kt-D}$ $\ln\left(\frac{M}{N}\right) = Be^{-kt}$ where $B = \pm e^{-D}$ $\frac{M}{N} = e^{Be^{-kt}}$ $N = Me^{-Be^{-kt}}$ As $t \to \infty$, $e^{-kt} \to 0$, $e^{-Be^{-kt}} \to 1$, $N \to M$. Regardless of the initial population of the bacteria, the number of bacteria always tends towards M eventually.