	Suggested Solution	Remarks
1i	$\mathrm{F}_n(0)=\sum_{r=1}^{n}\frac{1}{r(r+1)}$ $=\sum_{r=1}^{n}\left(\frac{1}{r}-\frac{1}{r+1}\right)$ $=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\ldots+\left(\frac{1}{n}-\frac{1}{n+1}\right)$ $=1-\frac{1}{n+1}$ As $n\to\infty$, $\frac{1}{n+1}\to 0$ and so $\mathrm{F}_n(0)\to 1$.	
1ii (a)	$\mathrm{F}_n(x)=\sum_{r=1}^{n}\left(\frac{1}{r}-\frac{1}{r+1}+\frac{2}{(r-1)x+1}-\frac{2}{rx+1}\right)$ $=\left[\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\ldots\left(\frac{1}{n}-\frac{1}{n+1}\right)\right]+$ $\left[\left(\frac{2}{1}-\frac{2}{x+1}\right)+\left(\frac{2}{x+1}-\frac{2}{2x+1}\right)+\ldots\left(\frac{2}{(n-1)x+1}-\frac{2}{nx+1}\right)\right]$ $=1-\frac{1}{n+1}+2-\frac{2}{nx+1}$ $=3-\frac{1}{n+1}-\frac{2}{nx+1}$	
1ii (b)	Since $\lim_{n\to\infty}\left(\frac{1}{n+1}\right)=0$ and $\lim_{n\to\infty}\left(\frac{2}{nx+1}\right)=\begin{cases}2, \text{ if } x=0\\ 0, \text{ if } x\neq 0\end{cases}$, $\lim_{n\to\infty}\mathrm{F}_n(x)=\begin{cases}1, \text{ if } x=0\\ 3, \text{ if } x\neq 0\end{cases}$.	There's a need to distinguish the special case when $x=0$.
2i	Let $u=a-x$. $\int_0^a \mathrm{f}(a-x)\,\mathrm{d}x=\int_a^0 -\mathrm{f}(u)\,\mathrm{d}u=\int_0^a \mathrm{f}(u)\,\mathrm{d}u=\int_0^a \mathrm{f}(x)\,\mathrm{d}x$	
2ii	Since f is symmetrical about $x=\frac{1}{2}a$, $\mathrm{f}(x)=\mathrm{f}(a-x)$ for $0\le x\le a$. -------------------- (*) Since f is continuous and x is also continuous, the function given by $x\mathrm{f}(x)$ is also continuous on $[0,a]$. $\int_0^a x\,\mathrm{f}(x)\,\mathrm{d}x=\int_0^a (a-x)\,\mathrm{f}(a-x)\,\mathrm{d}x$ (by (i)) $=\int_0^a (a-x)\,\mathrm{f}(x)\,\mathrm{d}x$ (by (*)) $=a\int_0^a \mathrm{f}(x)\,\mathrm{d}x-\int_0^a x\,\mathrm{f}(x)\,\mathrm{d}x$ $2\int_0^a x\,\mathrm{f}(x)\,\mathrm{d}x=a\int_0^a \mathrm{f}(x)\,\mathrm{d}x$ $\int_0^a x\,\mathrm{f}(x)\,\mathrm{d}x=\frac{a}{2}\int_0^a \mathrm{f}(x)\,\mathrm{d}x$	

	Alternative Since f is symmetrical about $x=\frac{1}{2}a$, $\mathrm{f}\left(x+\frac{1}{2}a\right)=\mathrm{f}\left(-x+\frac{1}{2}a\right)$ for $-\frac{1}{2}a\le x\le\frac{1}{2}a$. Let $v=x-\frac{1}{2}a$. Note that $\mathrm{g}(v):=v\,\mathrm{f}\left(v+\frac{a}{2}\right)=-\left[-v\,\mathrm{f}\left(-v+\frac{a}{2}\right)\right]=-\mathrm{g}(-v)$, which shows that g is an odd function on $\left[-\frac{a}{2},\frac{a}{2}\right]$. $\int_0^a x\,\mathrm{f}(x)\,\mathrm{d}x=\int_{-\frac{a}{2}}^{\frac{a}{2}}\left(v+\frac{a}{2}\right)\mathrm{f}\left(v+\frac{a}{2}\right)\mathrm{d}v$ $=\int_{-\frac{a}{2}}^{\frac{a}{2}}v\,\mathrm{f}\left(v+\frac{a}{2}\right)\mathrm{d}v+\int_{-\frac{a}{2}}^{\frac{a}{2}}\frac{a}{2}\mathrm{f}\left(v+\frac{a}{2}\right)\mathrm{d}v$ $=0+\frac{a}{2}\int_{-\frac{a}{2}}^{\frac{a}{2}}\mathrm{f}\left(v+\frac{a}{2}\right)\mathrm{d}v \qquad (\text{since g is odd})$ $=\frac{a}{2}\int_0^a \mathrm{f}(x)\,\mathrm{d}x$	
2iii	Since $\sin x$ and $\frac{1}{1+\cos^2 x}$ are continuous on $[0,\pi]$, so is their product $\mathrm{h}(x):=\frac{\sin x}{1+\cos^2 x}$. We can also show that h is symmetrical about $x=\frac{\pi}{2}$: $$\mathrm{h}(\pi-x)=\frac{\sin(\pi-x)}{1+\cos^2(\pi-x)}=\frac{\sin x}{1+\cos^2 x}=\mathrm{h}(x)$$ Applying the result in **(ii)**: $\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}\mathrm{d}x=\frac{\pi}{2}\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}\mathrm{d}x=\frac{\pi}{2}\left[-\tan^{-1}(\cos x)\right]_0^{\pi}$ $=\frac{\pi}{2}\left[-\tan^{-1}(-1)+\tan^{-1}(1)\right]=\frac{\pi}{2}\left[-\left(-\frac{\pi}{4}\right)+\frac{\pi}{4}\right]=\frac{\pi^2}{4}$	
3	A triangle with sides of lengths p, q and r exists iff $p+q>r$, $q+r>p$ and $r+p>q$.	
3i	WLOG, let $c\ge b\ge a>0$. Note that $c\ge b\ge a\Rightarrow\frac{c}{1+c}\ge\frac{b}{1+b}\ge\frac{a}{1+a}$. It suffices to show that $\frac{a}{1+a}+\frac{b}{1+b}>\frac{c}{1+c}$. Indeed, $\frac{a}{1+a}+\frac{b}{1+b}\ge\frac{a}{1+c}+\frac{b}{1+c}=\frac{a+b}{1+c}>\frac{c}{1+c}$.	

3ii	WLOG, let $c \geq b \geq a > 0$. Note that $c \geq b \geq a \Rightarrow \sqrt{c} \geq \sqrt{b} \geq \sqrt{a}$. It suffices to show that $\sqrt{a} + \sqrt{b} > \sqrt{c}$. Indeed, we have $\left(\sqrt{a} + \sqrt{b}\right)^2 \geq a + b + 2\sqrt{ab} > a + b > c$ which implies that $\sqrt{a} + \sqrt{b} > \sqrt{c}$.	
3iii	By **(ii)**, if a triangle with sides of lengths $a\left(b+c-a\right)$, $b\left(c+a-b\right)$ and $c\left(a+b-c\right)$ exists, then so does a triangle with side of lengths $\sqrt{a\left(b+c-a\right)}$, $\sqrt{b\left(c+a-b\right)}$ and $\sqrt{c\left(a+b-c\right)}$. By symmetry, it suffices to show that $$a\left(b+c-a\right)+b\left(c+a-b\right)>c\left(a+b-c\right).$$ $a\left(b+c-a\right)+b\left(c+a-b\right)-c\left(a+b-c\right)$ $=ab+ac-a^2+bc+ab-b^2-ac-bc+c^2$ $=c^2+2ab-a^2-b^2$ $=c^2-\left(a-b\right)^2$ $=\left(c+a-b\right)\left(c+b-a\right)$ $>0 \quad \left(\because \triangle \text{ of sides of lengths } a, b, c \text{ exists}, \therefore\ c+a>b \text{ and } c+b>a\right)$	
4i (a)	Bijection: Place 7 identical balls (counters to select T-shirts) into 4 distinct boxes (T-shirts of different colours). Number of ways $= {}^{7+4-1}\mathrm{C}_{4-1} = {}^{10}\mathrm{C}_3 = 120$	
4i (b)	Bijection: With 1 ball in each of the four boxes, place 3 more identical balls (counters to select T-shirts) into the 4 distinct boxes (T-shirts of different colours). Number of ways $= {}^{3+4-1}\mathrm{C}_{4-1} = {}^{6}\mathrm{C}_3 = 20$	
4ii (a)	Number of ways $= 4^7 = 16384$	
4ii (b)	Number of ways $= 4 \times 3^6 = 2916$	
4ii (c)	Let A_i be the set of arrangements without using T-shirts of colour i. Number of ways $= 4^7 - \left\|A_1 \cup A_2 \cup A_3 \cup A_4\right\|$ $= 4^7 - \left[\sum_i \left\|A_i\right\| - \sum_{i \neq j} \left\|A_i \cap A_j\right\| + \sum_{i \neq j, i \neq k, j \neq k} \left\|A_i \cap A_j \cap A_k\right\|\right]$ $= 4^7 - {}^4\mathrm{C}_1\left(3^7\right) + {}^4\mathrm{C}_2\left(2^7\right) - {}^4\mathrm{C}_3\left(1^7\right)$ $= 8400$	

5i (a)	If a tessellation exists, then the $p \times q$ rectangle is formed completely by $a \times b$ rectangles. Hence, the area of the $p \times q$ rectangle, *pq*, is a multiple of the area of the $a \times b$ rectangle, *ab*. Hence, *ab* is a factor of *pq*.	
5i (b)	If a tessellation exists, then the leftmost column of *p* squares must be formed by columns and/or rows of $a \times b$ rectangles, i.e. $a \times 1$ and/or $b \times 1$. Hence, $p = \lambda a + \mu b$ for some $\lambda, \mu \in \mathbb{Z}_0^+$. Similarly, the bottommost row of *q* squares is also formed by columns and/or rows of $a \times b$ rectangles. Hence, $q = \lambda a + \mu b$ for some $\lambda, \mu \in \mathbb{Z}_0^+$.	
5i (c)	If a tessellation using $a \times b$ rectangles exists, then there is a tessellation using $a \times 1$ rectangles. Each of these $a \times 1$ rectangles has 1 shaded square. Similar to the argument in **(i)(a)**, the number of $a \times 1$ rectangles is $\frac{pq}{a}$, and hence, the number of shaded squares is $\frac{pq}{a}$.	
5ii (a)	Place the $r \times s$ rectangle in the bottom left corner of the $p \times q$ rectangle. This $r \times s$ rectangle will have *t* shaded squares, namely $(1,1)$, $(2,2)$, …, (t,t). The remainder of the $p \times q$ rectangle can be tessellated with $a \times 1$ rectangles that contain exactly 1 shaded square each. [The *s* columns above the $r \times s$ rectangle fitted with $a \times 1$ rectangles, and the remaining $p \times (q - s)$ rectangle to be fitted with $1 \times a$ rectangles.] Using the argument in **(i)(a)**, the number of such $a \times 1$ rectangles is $\frac{pq - rs}{a}$. Hence, the total number of shaded squares is given by $\frac{pq - rs}{a} + t$.	
5ii (b)	Using the results in **(i)(c)** and **(ii)(a)**, $$\frac{pq - rs}{a} + t = \frac{pq}{a}.$$ This yields $\frac{rs}{a} = t$ and we claim that $t = 0$. Suppose instead that $t \geq 1$, and since $t = r$ or $t = s$, then we must have $\frac{s}{a} = 1$ or $\frac{r}{a} = 1$, which are both impossible since $r, s < a$. Hence, $\frac{rs}{a} = t = 0$ which gives $r = 0$ or $s = 0$. Consequently, $p \equiv 0 \pmod{a}$ or $q \equiv 0 \pmod{a}$, i.e. *a* is a factor of either *p* or *q*.	

6a	Assumption: We will assume that friendship is symmetric relation (i.e. if *A* is a friend of *B*, then *B* is a friend of *A*). In a group of $n \geq 2$ students, if a student has 0 friends, there cannot be a student with $n-1$ friends; and similarly, if a student has $n-1$ friends, there cannot be a student with 0 friends. Hence, the number of friends each of the *n* students can possibly have must be from an $(n-1)$- element set ($\{1,2,...,n-1\}$ or $\{0,1,...,n-2\}$). By the pigeonhole principle, there must be at least 2 students with the same number of friends.	
6b	Let A_i be the interval $\left[\frac{i-1}{n},\frac{i}{n}\right)$, for $i=1,2,...,n$. These A_i's form the *n* pigeonholes, while the fractional parts form the *n* pigeons. <u>Case 1</u>: There is a fractional part, say $px-\lfloor px\rfloor$ (with $1\leq p\leq n$) in A_1. $$0\leq px-\lfloor px\rfloor<\frac{1}{n}\Rightarrow\left\lvert x-\frac{\lfloor px\rfloor}{p}\right\rvert<\frac{1}{pn}$$ Take $a=\lfloor px\rfloor$ and $b=p$. <u>Case 2</u>: There is no fractional part in A_1. By the pigeonhole principle, there must exist two fractional parts, say $px-\lfloor px\rfloor$ and $qx-\lfloor qx\rfloor$ (with $n\geq p>q\geq 1$) in some A_k. $$px-\lfloor px\rfloor\in\left[\frac{k-1}{n},\frac{k}{n}\right) \text{ and } qx-\lfloor qx\rfloor\in\left[\frac{k-1}{n},\frac{k}{n}\right)$$ $$\Rightarrow\left\lvert(p-q)x-(\lfloor px\rfloor-\lfloor qx\rfloor)\right\rvert=\left\lvert px-\lfloor px\rfloor-(qx-\lfloor qx\rfloor)\right\rvert<\frac{1}{n}$$ $$\Rightarrow\left\lvert x-\frac{\lfloor px\rfloor-\lfloor qx\rfloor}{p-q}\right\rvert<\frac{1}{(p-q)n}$$ As $1\leq p-q\leq n$, we may take $a=\lfloor px\rfloor-\lfloor qx\rfloor$ and $b=p-q$.	

7i	(1): $y\dfrac{\mathrm{d}y}{\mathrm{d}x}=x\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2+1,\ x>0$ and let $t=\dfrac{\mathrm{d}y}{\mathrm{d}x}$. $yt=xt^2+1$ Differentiating w.r.t. x on both sides: $\dfrac{\mathrm{d}y}{\mathrm{d}x}t+y\dfrac{\mathrm{d}t}{\mathrm{d}x}=t^2+2xt\dfrac{\mathrm{d}t}{\mathrm{d}x}$ $y\dfrac{\mathrm{d}t}{\mathrm{d}x}=2xt\dfrac{\mathrm{d}t}{\mathrm{d}x}$ $\quad\left(\because \dfrac{\mathrm{d}y}{\mathrm{d}x}t=t^2\right)$ $(y-2xt)\dfrac{\mathrm{d}t}{\mathrm{d}x}=0$ $y=2xt$ or $\dfrac{\mathrm{d}t}{\mathrm{d}x}=0$ $\left(\text{rej. } \because \dfrac{\mathrm{d}t}{\mathrm{d}x}=\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}\neq 0\right)$ $y=2x\dfrac{\mathrm{d}y}{\mathrm{d}x}$ $\int\dfrac{1}{x}\,\mathrm{d}x=\int\dfrac{2}{y}\,\mathrm{d}y$ $\ln\lvert x\rvert=2\ln\lvert y\rvert+C$ $\dfrac{y^2}{x}=A,$ where $A=\mathrm{e}^{-C}$ or $-\mathrm{e}^{-C}$ $y^2=Ax$ Hence, $2y\dfrac{\mathrm{d}y}{\mathrm{d}x}=A$ and by (1), $\dfrac{A}{2}\equiv x\left(\dfrac{A}{2y}\right)^2+1\equiv x\left(\dfrac{A^2}{4Ax}\right)+1\Rightarrow A=4$. Therefore, $y^2=4x$.	
7ii	"$\Rightarrow$" $S: y^2=4x\Rightarrow\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{2}{y}$ Equation of tangent: $y-y_0=\dfrac{2}{y_0}(x-x_0)$, where (x_0,y_0) lies on S This satisfies (1): $y\dfrac{\mathrm{d}y}{\mathrm{d}x}=\left(y_0+\dfrac{2}{y_0}(x-x_0)\right)\left(\dfrac{2}{y_0}\right)$ $\quad=2+x\left(\dfrac{2}{y_0}\right)^2-\dfrac{4x_0}{y_0^2}$ $\quad=x\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2+1$ $\quad\left(\because \dfrac{4x_0}{y_0^2}=1\right)$ "$\Leftarrow$" Suppose a straight line l with gradient m satisfies (1). Then $ym=xm^2+1$ which implies that $l: y=mx+\dfrac{1}{m}$ and $m\neq 0$. $S: y^2=4x\Rightarrow\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{2}{y}$	

	When $\frac{dy}{dx}=m$, $y=\frac{2}{m}$ and $x=\frac{1}{m^2}$. It suffices to check that $\left(\frac{1}{m^2},\frac{2}{m}\right)$ lies on *l*. Indeed, $\frac{2}{m}=m\left(\frac{1}{m^2}\right)+\frac{1}{m}$, so $\left(\frac{1}{m^2},\frac{2}{m}\right)$ satisfies $l: y=mx+\frac{1}{m}$.	
8i	$\sum_{r=1}^{3}\text{n}\left(\frac{11}{7}r\right)=\text{n}\left(1\frac{4}{7}\right)+\text{n}\left(3\frac{1}{7}\right)+\text{n}\left(4\frac{5}{7}\right)=2+3+5=10$	
8ii	$\sum_{r=1}^{5}\text{n}\left(\frac{7}{11}r\right)=\text{n}\left(\frac{7}{11}\right)+\text{n}\left(1\frac{3}{11}\right)+\text{n}\left(1\frac{10}{11}\right)+\text{n}\left(2\frac{6}{11}\right)+\text{n}\left(3\frac{2}{11}\right)$ $=1+1+2+3+3$ $=10$ $\sum_{r=1}^{5}\text{n}\left(\frac{7}{11}r\right)$ gives the total number of points underneath $y=\frac{7}{11}x+\frac{1}{2}$ for $1\le x\le 5$ (and $y\ge 1$).	
8iii	Let the equation of the new line be $y=\frac{7}{11}x+b$ since this line is parallel to $y=\frac{7}{11}x+\frac{1}{2}$. Furthermore, since $y=\frac{7}{11}x+\frac{1}{2}$ passes through $\left(0,\frac{1}{2}\right)$, the new line must pass through $\left(6,\frac{7}{2}\right)$ because $(3,2)$ is the midpoint of $\left(0,\frac{1}{2}\right)$ and $\left(3,\frac{7}{2}\right)$. [I.e. $\left(6,\frac{7}{2}\right)$ is the point when $\left(0,\frac{1}{2}\right)$ is rotated 180° about $(3,2)$.] Hence, $\frac{7}{2}=\frac{7}{11}(6)+b\Rightarrow b=-\frac{7}{22}$. Equation of the new line: $y=\frac{7}{11}x-\frac{7}{22}\Rightarrow x=\frac{11}{7}y+\frac{1}{2}$ **<u>Alternative</u>** Rotation about $(3,2)$ can be seen via the following transformations. $y=\frac{7}{11}x+\frac{1}{2}\xrightarrow{\text{translate by }\binom{-3}{-2}}y+2=\frac{7}{11}(x+3)+\frac{1}{2}$ $\xrightarrow{\text{rotate }180^\circ\text{ about origin}}-y+2=\frac{7}{11}(-x+3)+\frac{1}{2}$ $\xrightarrow{\text{translate by }\binom{3}{2}}-(y-2)+2=\frac{7}{11}(-(x-3)+3)+\frac{1}{2}$ $x=\frac{11}{7}y+\frac{1}{2}$ $\sum_{r=1}^{3}\text{n}\left(\frac{11}{7}r\right)$ counts the number of points to the left of $x=\frac{11}{7}y+\frac{1}{2}$, which are just the points beneath $y=\frac{7}{11}x+\frac{1}{2}$ in **(ii)** rotated 180° about $(3,2)$. Hence, $\sum_{r=1}^{3}\text{n}\left(\frac{11}{7}r\right)=\sum_{r=1}^{5}\text{n}\left(\frac{7}{11}r\right)$.	

8iv	We check that $y=\frac{q}{p}x+\frac{1}{2}$ and $x=\frac{p}{q}y+\frac{1}{2}$ are rotations of each other by 180° about $\left(\frac{p+1}{4},\frac{q+1}{4}\right)$: $$y=\frac{q}{p}x+\frac{1}{2}\xrightarrow{\text{translate by }\begin{pmatrix}-\frac{p+1}{4}\\-\frac{q+1}{4}\end{pmatrix}} y+\frac{q+1}{4}=\frac{q}{p}\left(x+\frac{p+1}{4}\right)+\frac{1}{2}$$ $$\xrightarrow{\text{rotate }180^\circ\text{ about origin}} -y+\frac{q+1}{4}=\frac{q}{p}\left(-x+\frac{p+1}{4}\right)+\frac{1}{2}$$ $$\xrightarrow{\text{translate by }\begin{pmatrix}\frac{q+1}{4}\\\frac{p+1}{4}\end{pmatrix}} -\left(y-\frac{q+1}{4}\right)+\frac{q+1}{4}=\frac{q}{p}\left(-\left(x-\frac{p+1}{4}\right)+\frac{p+1}{4}\right)+\frac{1}{2}$$ $$-y+\frac{q+1}{2}=\frac{q}{p}\left(-x+\frac{p+1}{2}\right)+\frac{1}{2}$$ $$x=\frac{p}{q}\left(y-\frac{q+1}{2}\right)+\frac{p+1}{2}+\frac{p}{2q}$$ $$x=\frac{p}{q}y+\frac{1}{2}$$ Each point (x,y) between the two lines has an image (x',y') also lying between the two lines, where (x',y') is a 180° rotation of (x,y) about $\left(\frac{p+1}{4},\frac{q+1}{4}\right)$. <u>Case 1</u>: $p\equiv q\equiv 3\pmod 4$ Then $\left(\frac{p+1}{4},\frac{q+1}{4}\right)$ is a point that lies between the two lines, and by the above argument, N is must be odd, i.e. $N\equiv 1\pmod 2$. $$\begin{aligned} N+\left(\frac{p-1}{2}\right)\left(\frac{q-1}{2}\right)&=N+\left(\frac{4m+2}{2}\right)\left(\frac{4n+2}{2}\right) && \text{for some } m,n\in\mathbb{Z}_0^+\\ &\equiv 1+1 && \pmod 2\\ &\equiv 0 && \pmod 2\end{aligned}$$ <u>Case 2</u>: $p\not\equiv 3\pmod 4$ or $q\not\equiv 3\pmod 4$ Then since p and q are odd, either $p=4m+1$ or $q=4n+1$ for some $m,n\in\mathbb{Z}_0^+$. Hence, $\frac{p-1}{2}=2m\equiv 0\pmod 2$ or $\frac{q-1}{2}=2n\equiv 0\pmod 2$. Also, $\left(\frac{p+1}{4},\frac{q+1}{4}\right)$ is not a point with integer coordinates, and by the above argument, N is must be even, i.e. $N\equiv 0\pmod 2$.	

	$\therefore N+\left(\frac{p-1}{2}\right)\left(\frac{q-1}{2}\right)\equiv 0 \pmod 2$ **<u>Alternative</u>** Using the same argument as in **(ii)**, both lines have gradient $\frac{p}{q}$ and $\left(0,\frac{1}{2}\right)$ lies on $y=\frac{q}{p}x+\frac{1}{2}$, while $\frac{p+1}{2}=\frac{p}{q}\left(\frac{q}{2}\right)+\frac{1}{2}$ which implies that $\left(\frac{p+1}{2},\frac{q}{2}\right)$ lies on $x=\frac{p}{q}y+\frac{1}{2}$. Since the midpoint of $\left(0,\frac{1}{2}\right)$ and $\left(\frac{p+1}{2},\frac{q}{2}\right)$ is $\left(\frac{p+1}{4},\frac{q+1}{4}\right)$, we can conclude that $y=\frac{q}{p}x+\frac{1}{2}$ and $x=\frac{p}{q}y+\frac{1}{2}$ are rotations of each other by 180° about $\left(\frac{p+1}{4},\frac{q+1}{4}\right)$. The total number of points is $\left(\frac{p-1}{2}\right)\left(\frac{q-1}{2}\right)$. This value is also gotten by taking the sum of the number of points beneath $y=\frac{q}{p}x+\frac{1}{2}$ and the number of points to the left of $x=\frac{p}{q}y+\frac{1}{2}$, then subtracting the number of points between these lines due to double counting. $\left(\frac{p-1}{2}\right)\left(\frac{q-1}{2}\right)=\sum_{r=1}^{\frac{p-1}{2}}\mathrm{n}\left(\frac{q}{p}r\right)+\sum_{r=1}^{\frac{q-1}{2}}\mathrm{n}\left(\frac{p}{q}r\right)-N$ $N+\left(\frac{p-1}{2}\right)\left(\frac{q-1}{2}\right)=2\sum_{r=1}^{\frac{p-1}{2}}\mathrm{n}\left(\frac{q}{p}r\right)$ $\equiv 0 \pmod 2$	