General Tips:

- Drawing a diagram helps, even if you don't have specific coordinates.
- Remember your tools:
    - Ratio Theorem
    - Comparing Coefficients
    - Parallel if and only if ka=b
    - $a \bullet b = |a||b|cos\theta$, $a \times b = |a||b|sin\theta\ (\hat{n})$
    - Be ready to use a combination of these tools.

- Remember properties of dot/cross product. ($\alpha \times b = -b \times a$, $2\alpha \times b = \alpha \times 2b$, distributiveness, etc.)
- Remember the angle form of your dot/cross products.
- More specific stuff: $a \bullet b =$**0** implies perpendicular vectors**,** $a \times b =$**0** implies parallel vectors

Basic Tools:

$a \bullet b = |a||b|cos(\Theta)$, the sign of the dot product tells you whether the angle between the two vectors is acute or obtuse. (Pointing roughly in the same direction or not).

$a \times b = |a||b|sin(\Theta)n$
Where n is the unit vector perpendicular to both vectors, whose direction is given by the right hand rule.

In particular:
If a and b are perpendicular, $a \bullet b = 0$
If a and b are parallel, $a \times b = 0$
$|a \times b|$ gives the area of the parallelogram whose sides are a and b. (Divide by 2 to get area of triangle OAB)

For any two vectors a, b, the angle Θ between them is given by:

$$cos(\Theta) = \frac{(a \bullet b)}{|a||b|}$$

Length of projection of a onto b is given by:

$$a \bullet \frac{b}{|b|} = a \bullet \hat{b}$$

Ratio Theorem: Given two points A and B with position vectors a and b, the position vector of the point C, that has the property AC:CB = $\lambda : \mu$, is given by:

$OC = (\lambda OB + \mu OA)/(\lambda + \mu)$

This can always be rescaled such that $\lambda + \mu = 1$, which gives:
$OC = (1 - \mu)OB + \mu OA$

Two vectors a and b are parallel if and only if:
$a = kb$, for some nonzero real number k

(Comparing Coefficients) For two non-parallel vectors a and b:
$\lambda a + \mu b = ua + vb \quad \leftrightarrow \quad \lambda = u, \mu = v$
This is useful when solving for intersections between two lines.

Useful Identities:

$|a|^2 = a \bullet a$, and in particular:
$|a + b|^2 = (a + b) \bullet (a + b) = |a|^2 + |b|^2 + 2(a \bullet b)$

$|a|^2 - |b|^2 = (a - b) \bullet (a + b)$ (Analogous to difference of 2 squares)

Overview:

The difficulty in abstract vector questions lies in identifying which formulae are useful for a given question. In applied vector questions, most question parts ask you to solve for things which are clearly associated with a certain formula/method that you are expected to know. (E.g. Foot of perpendicular, line equation, etc.) The various formulae required for applied vectors are different enough to not pose any issues in identifying which should be used(at least, after some practice). Phrased differently, if you have all the vectors formulae memorised, you shouldn't find any difficulty in most parts of an applied vectors question.

In contrast, an abstract vectors question tests your ability to identify which formulae will be useful. What they require you to solve for/prove often isn't something that is instantly given by a formula. There may even be multiple plausible ways to go about solving a question, which can contribute to the difficulty of considering what formula to use.

I will walk through my thought process for a few questions below, and hopefully it will shed some light on how to solve these types of questions in general.

General Principles:

- Almost always, every piece of information in the question will be useful, and is meant to inform you of what formulae to employ. Certain phrases/keywords can be identified and reliably associated with a certain formula to be used. (Note: This applies to all topics in H2 math, not just vectors.) After a good amount of practice, you can even foresee what questions will be asked after simply reading the information given.

- Knowing what quantities a formula relates together will help you identify which will be useful. (E.g. knowing the values of $|a|$, $a \bullet b$, and the angle between a and b, I can calculate $|b|$). So if a question asks for an angle between vectors, you ought to think of the formulae that contain angles between vectors.

- This may sound obvious, but having your formulae at your fingertips, or at least written down on rough paper, will help greatly.

- Drawing a diagram, even without specific coordinates, can only help you.

Note: I suggest trying out each question first, and to use pen and paper to follow the solutions.

Examples:

Example: RI 2020 Prelim P2

5 Referred to the origin O, points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively, where $\mathbf{a}$ and $\mathbf{b}$ are non-zero and non-parallel vectors. Point C lies on OA, between O and A, such that $OC:CA=2:1$. Point D lies on OB produced such that $OD:BD=3:2$.

(i) Find the position vectors $\overrightarrow{OC}$ and $\overrightarrow{OD}$, giving your answers in terms of $\mathbf{a}$ and $\mathbf{b}$. [2]

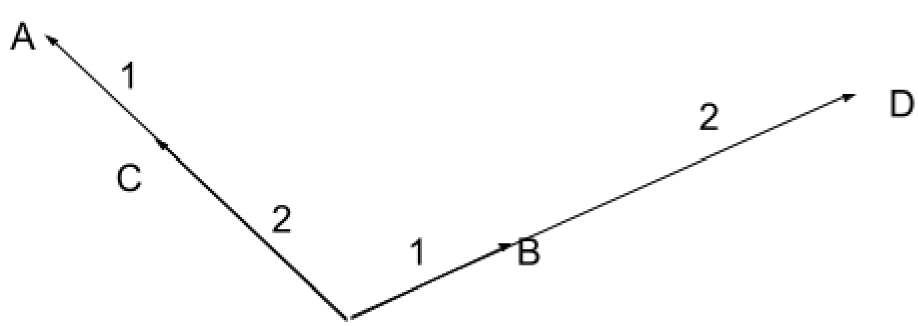
(ii) Show that the point E where the lines BC and AD meet has position vector $\frac{4}{3}\mathbf{a}-\mathbf{b}$. [4]

(iii) Show that the area of triangle CDE can be written as $k|\mathbf{a}\times\mathbf{b}|$, where k is a constant to be found. [3]

(iv) It is given that the point F is on BO produced, and OE bisects the angle AOF. Find the ratio $OA:OB$. [3]

I will walk through my thought process in doing this question:

i) C lies on OA, and D lies on OB, with $OC: CA = 2: 1$ and $OD: BD = 3: 2$. Drawing a diagram:



Thus, it should be clear that $OC = 2a/3$, and $OD = 3b$.

ii) Since the question mentioned the lines BC and and AD, we might as well find their equations: BC: $r = b + \lambda(c - b)$, and AD: $r = a + \mu(d - a)$.

We were asked to find their intersection, so we should equate these two equations we just constructed:

$a + \mu(d - a) = b + \lambda(c - b)$.

Additionally, we're meant to show the position vector of OE in terms of a and b, so we should remove c and d from our equation by substituting their values in from 5i).

$a + \mu(3b - a) = b + \lambda(2a/3 - b)$.

This seems closer to what we want. Now, the only thing left that we can do(and hopefully, the most natural thing to do,) is expand the brackets.

$(1 - \mu)a + 3\mu b = (2\lambda/3)a + (1 - \lambda)b)$

Upon seeing this, it should be natural to equate/compare coefficients. If it isn't natural, don't worry. It will become natural with practice.

Thus, we get: $1 - \mu = 2\lambda/3$ and $3\mu = 1 - \lambda$.
Which we can punch G.C. to solve and get: $\mu =- 1/3, \lambda = 2$.

Since we were asked to find OE, we sub these back in to get:

$OE = (4/3)a - 1b$, as required.

iii) The question wants us to show the area of triangle CDE is a multiple of $k|a \times b|$, so we ought to first write down an equation for the area of the triangle. And since <u>we aim to show it's a cross product</u>, we should at least <u>start by writing it in terms of a cross product.</u>

Area of Triangle CDE =
$(1/2)|CD \times CE| = (1/2)|(OD - OC) \times (OE - OC)| = |(3b - (2/3)a) \times ((2/3)a - b)|$

We could have taken a different pair of sides, but they will all give the same answer.

Now, since the end goal we're aiming for has the form k|a x b|, we should expand our brackets.

$(1/2)|(3b - (2/3)a) \times ((2/3)a - b)| = (1/2)|2(b \times a) + (2/3)a \times b|$
$= (1/2)|(-4/3)a \times b|$

Where we utilised the fact that $a \times a = 0$ and $a \times b = - b \times a$

Now, pulling out constants,

$(1/2)|CD \times CE| = (2/3)|a \times b|$

And we get our answer, $k = 2/3$.

5iv) Thus far, the questions have been quite standard as far as abstract vectors questions go. Like many other questions, the last part is the hardest, but can be done smoothly if you know what to consider.

We are given that F lies on OB produced, and OE bisects the angle AOF, and we are asked to find the ratio OA:OB. This is equivalent to finding $|a|/|b|$. Recall that bisecting means splitting into 2 equal parts, so angle AOE = angle EOF. In general <u>when dealing with angles</u>, the <u>dot product formula</u>/formula for angle between two vectors comes in useful, so it makes sense to write this information in terms of it.

$cos(AOE) = OA \bullet OE/|OA||OE|$
$cos(EOF) = OE \bullet OF/|OE||OF|$

Since the angles are equal, we may equate the two:

$(OA \bullet OE)/|OA||OE| = (OE \bullet OF)/|OE||OF|$

Notice that terms cancel out:

$(OA \bullet OE)/|OA| = (OE \bullet OF)/|OF|$

Since we need to get things in terms of $|a|$ and $|b|$, it makes sense to write the above line in terms of them. However, we face an issue when dealing with OF, as we only know that it lies on OB produced. Thus, we know $OF = - \lambda b$ for some positive constant, as it goes in the opposite direction of OB.

But, since we divide by |OF|, we have: $OF/|OF| =- \lambda b/|-\lambda b| =- b/|b|$, and we see that the constant cancels out on the top and bottom. Thus, we can proceed with writing the equality in terms of $|a|$ and $|b|$:

$a \bullet ((4/3)a - b)/|a| = ((4/3)a - b)) \bullet (- b/|b|)$
Expanding:
$(4/3)|a| - (a \bullet b)/|a| = (- 4/3)(a \bullet b)/|b| + |b|$

We seem to be stuck again, but a good rule of thumb when you've used all the information given to you is to check what formulas you can easily apply to whatever you've worked up to. In this case, we have yet to expand $(a \bullet b)$, and this can give us more terms in $|a|$ and $|b|$.

$(4/3)|a| - |b|cos(AOB) = (- 4/3)|a|cos(AOB) + |b|$

Since we want $|a|/|b|$, we can try dividing by $|b|$:
$(4/3)|a|/|b| - cos(AOB) = (- 4/3)|a|/|b|cos(AOB) + 1$

Now if we make it the subject:
$|a|/|b| \times (4/3)(1 + cos(AOB)) = (1 + cos(AOB))$

Thus: $|a|/|b| = 3/4$

And we are done.
(There is a one-liner solution that involves geometry, but this sort of thought process is more reliable.)

Example: ASRJC 2022 Prelim P2

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2 Relative to the origin O, the points A, B and C have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively. It is given that λ and μ are non-zero numbers such that $\lambda\mathbf{a} + \mu\mathbf{b} - \mathbf{c} = \mathbf{0}$ and $\lambda + \mu = 1$.

(i) Show that the points A, B and C are collinear. [3]

The angle between $\mathbf{a}$ and $\mathbf{b}$ is known to be obtuse and that $|\mathbf{a}| = 2$.

(ii) If k denotes the area of triangle OAB, show that $(\mathbf{a}\cdot\mathbf{b})^2 = 4\left(|\mathbf{b}|^2 - k^2\right)$. [3]

D is a point on the line segment AB with position vector $\mathbf{d}$.

(iii) It is given that area of triangle OAB is 6 units^2, $|\mathbf{b}| = 10$ and that AOD is 90^0. By finding the value of $\mathbf{a}\cdot\mathbf{b}$, find $\mathbf{d}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. [4]

2i) We need to show A,B,C are collinear. This requires us to show that any pair of AB, BC, or AC are related by scalar multiplication. That is, we can multiply any one by a scalar to get another. (Note that it is sufficient to show this holds for any pair of them. I chose AB and AC since they point in the same direction, which should make for marginally easier algebra.)

$AB = b - a$
$AC = c - a = (\lambda - 1)a + \mu b$, where this substitution for c is given in the question.

Additionally, we have $\lambda + \mu = 1$.
Thus:$AC = - \mu a + \mu b = \mu(b - a) = \mu AB$.
And thus they are collinear, as required.

A quicker method: Notice that OC is exactly the vector that divides AB into the ratio $\lambda : \mu$, by the ratio theorem. And if it divides the line segment AB, it must lie on AB. Hence A,B,C are collinear.

2ii) It might be completely unclear as to what the area of triangle OAB has to do with the identity, but we still have to try something. The first thing we can try is to write k in terms of a and b, since that was one of the few pieces of information they gave us.

$k = (1/2)|a \times b| = (1/2)|a||b|sin(AOB)$

Since they gave us $|a|$, we might as well sub it in:
$k = |b|sin(AOB)$.

Now, when we are required to prove something, we need to show the LHS=RHS. Since we did work with k, we should try subbing it into the RHS.

$RHS = 4(|b|^2 - k^2) = 4(|b|^2 - |b|^2 sin^2(AOB))$

Factorising: $RHS = 4|b|^2(1 - sin^2(AOB))$

At this point, we need to use our Pythagorean identities. Specifically: $cos^2x + sin^2x = 1$

Note: If anything of this form appears in a question, it is a huge sign to use these identities. Chances are, it's not a coincidence that such an expression shows up.

$RHS = 4|b|^2 cos^2(AOB)$

Now, it may not be obvious how to proceed from here(or maybe it is for some of you). One trick for proving questions is "Meeting in the middle," where you work separately on the LHS and RHS to show they're both equal to the same quantity. (That is, If we need to show $A = B$, we can try showing $A = C$ and $B = C$.)

So, since we don't see a way forward from the RHS, we swap over to the LHS and see what we can do to it.

$LHS = (a \bullet b)^2$

We can try expanding the dot product:

$LHS = |a|^2|b|^2 cos^2(AOB)$

Recall that we were given $|a| = 2$.

$LHS = 4|b|^2 cos^2(AOB)$

Which, luckily for us, is what we had before. We're technically done, but depending on how picky your school is on presentation, we may have to write these steps backwards, as such:

$RHS = 4|b|^2 cos^2(AOB) = |a|^2|b|^2 cos^2(AOB) = (a \bullet b)^2$

2iii) Once again, we are given random pieces of information and are tasked to find something that seems completely unrelated. Regardless, we have to attempt it.

Among the information they gave, what stands out is that AOD is a right angle. This heavily hints at using a dot product, as the dot product can act as a test for perpendicular vectors. Thus, we can expect to use the fact:

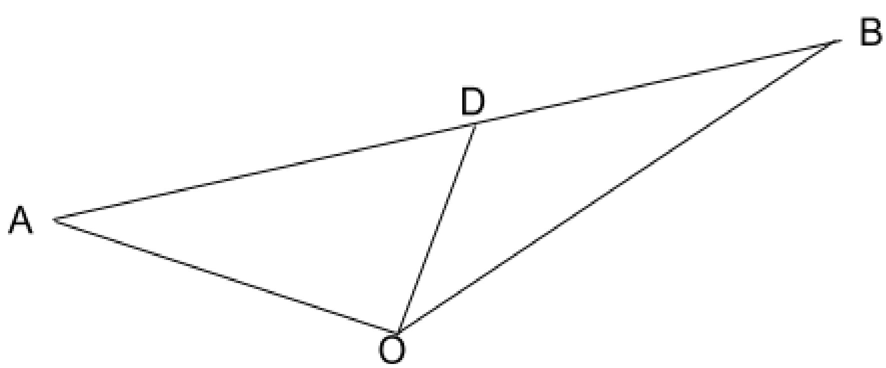
$d \bullet a = 0$

To help solve for d. Besides that, we are given the area of triangle OAB, which if you recall, was the value of k previously. Additionally, we have the value of $|b|$. This together with the value of k suggests that we should use the equation above in 2ii). (Moreover, they want us to calculate $a \bullet b$, which was also in the equation above on the LHS.)

Plugging these values into 2ii) and solving for $a \bullet b$, we obtain:

$a \bullet b = \pm\ 16.$

Now we have a problem. We don't know which sign to take. Recall that the sign of the dot product tells us whether the angle between the vectors is acute(+) or obtuse(-). So, we need to figure out which is the case here. This is best done with a diagram.



Recall that we were told that D lies on the line segment AB. This, together with the fact that AOD is a right angle, forces AOB to be an obtuse angle. (If you aren't convinced, try drawing AOB with an acute angle. D will be forced to lie outside the line segment AB.)

Thus, we can conclude that $a \bullet b = -16.$

Now, we still need to obtain some expression for OD. Considering that we just mentioned that D lies on AB, we ought to write it in terms of the general position vector of a point on a line. (Note: This idea of writing an unknown position vector as a point on a line turns out to be quite a common trick.)

$d = a + \lambda(b - a),\ \lambda \in R$ (In fact, $0 \le \lambda \le 1$. See if you can reason why.)

Now, we finally try taking the dot product, as we suggested at the beginning:
$d \bullet a = 0$
$(a + \lambda(b - a)) \bullet a = 0$
$|a|^2 + \lambda(b \bullet a - |a|^2) = 0$

Subbing in the values we know(we know all of them):

$4 + \lambda(-16 - 4) = 0$
$\lambda = 1/5$

Hence: $d = (4/5)a + (1/5)b$.

Example: DHS 2023 Prelim P1

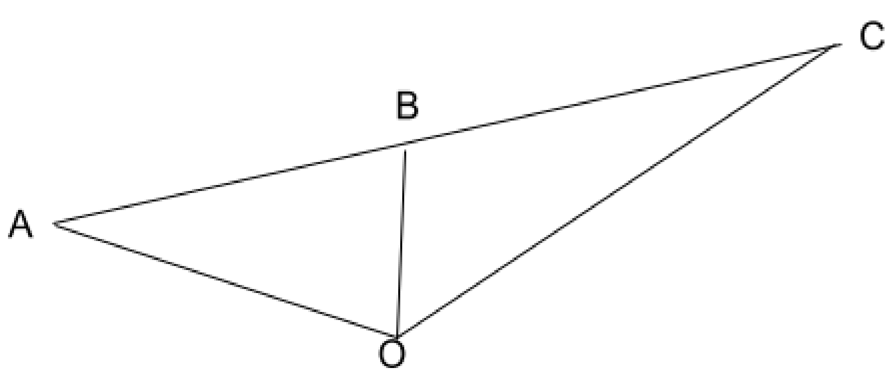
5 With respect to an origin O, the points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively where $\mathbf{a}$ and $\mathbf{b}$ are non-parallel. It is given that B lies on the line segment AC such that $\overrightarrow{BC} = 3\mathbf{b} - \mu\mathbf{a}$.

(a) Find the value of μ. Hence find $\overrightarrow{OC}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. [3]

(b) Q is a point on line segment OC where $\overrightarrow{OQ} = t\overrightarrow{OC}$. The line segment AQ meets line segment OB at point P. Given that $AP : PQ = \lambda : 1$, deduce the value of λt. [4]

(c) By using your result in part **(b)**, find the ratio $OP : PB$ when $\lambda = 5$. [2]

5a) Drawing a diagram(always a good first step):



Since C lies on AB produced, we know BC is parallel to AB, and hence they are related by scalar multiplication.

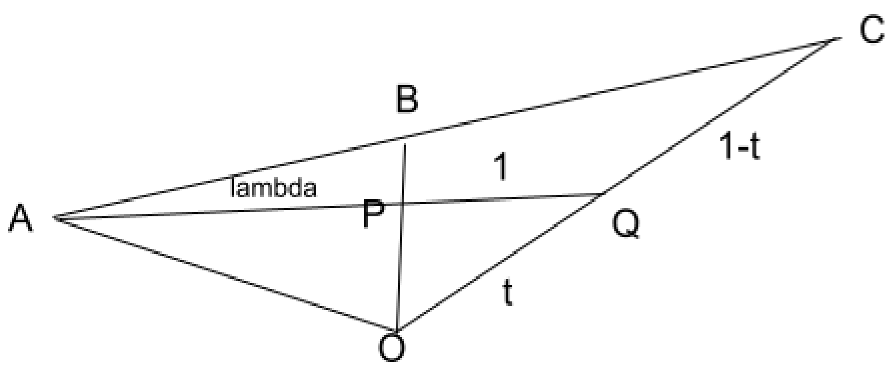
$BC = \lambda AB$, for some scalar.

$3b - \mu a = \lambda b - \lambda a$

Now, we compare coefficients:
$3 = \lambda, \lambda = \mu$

And it follows that $\mu = 3$, and that $OC = b + BC = 4b - 3a$

5b) There's a lot to unpack here. Again, we draw a diagram.



(Google docs won't let me insert special symbols into the drawing haiz)

Now, we were given that P divides AP into the <u>ratio</u> λ:1. This is a sign to <u>use the ratio</u> theorem. (Basically, if you see a ratio, you should consider using the ratio theorem.)

$OP = (\lambda OQ + a)/(\lambda + 1)$

Additionally, we were given that $OQ = tOC$:

$OP = (\lambda tOC + a)/(\lambda + 1)$, which is promising as we now have the λt the question was asking for. All that remains is to find a way to equate the RHS to something useful, so we can get a value for λt.

This next step is a bit more challenging, but referring to the diagram will make thinking of it easier. Notice that P lies on OB(as was stated in the question). Hence, we may write:

$OP = \gamma b$ for some real scalar γ.

If we sub this into the above:
$\gamma b = (\lambda t OC + a)/(\lambda + 1)$

And we solved for OC before, so we may try subbing that in:

$\gamma b = (4\lambda t b + (1 - 3\lambda t)a)/(\lambda + 1)$

Now, we use the same method as in the previous part:
$\gamma b = (4\lambda t/(1 + \lambda))b$, $0a = (1 - 3\lambda t)/(\lambda + 1)a$
$1 - 3\lambda t = 0$
$\lambda t = 1/3$

A few things of note:
-If it wasn't already clear, diagrams help tremendously.

-A lot of information required in this part was built up by previous parts. This is a common trend in math questions in general. <u>Information obtained in earlier parts can assist in later parts.</u>

- In my experience, this is actually quite a common question type. They give you some intersection of two lines, and by writing the position vector in two ways and comparing coefficients, you can get some information about some unknowns.

5c) Since they gave us a value for λ, we ought to sub it in. This gives us:

$t = 1/15$

Since we need to solve for OP:PB, we should start by first solving for OP:

From the previous part:

$\gamma b = (4(\lambda t)/(1 + \lambda))b = (2/9)b$

$OP = (2/9)b$
Thus: $BP = (7/9)b$
And hence, $OP{:}BP = 2{:}7$

While abstract vectors questions can be quite difficult, just like other topics in H2 Math, practice will make you familiar with common question types. Additionally, after each question, if you think about what methods worked and what didn't, it will help you solve similar questions in the future, by making you quicker at identifying the correct formulae.