Name: **Register Number:** **Class:**

For Marker's Use
90

Parents' signature:

南僑中學

NAN CHIAU HIGH SCHOOL

PRELIMINARY EXAMINATION 2022
SECONDARY FOUR EXPRESS

ADDITIONAL MATHEMATICS **4049/01**
Paper 1 **24 August 2022, Wednesday**

Candidates answer on the Question Paper. **2 hours 15 minutes**

No Additional Materials are required.

READ THESE INSTRUCTIONS FIRST

Write your name, class and index number in the spaces at the top of this page.
Write in dark blue or black pen.
You may use a pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved scientific calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

The total number of marks for this paper is 90.

Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Binomial expansion

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \cdots + \binom{n}{r} a^{n-r} b^r + \cdots + b^n,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{(n-r)!r!} = \frac{n(n-1)\ldots(n-r+1)}{r!}$

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\Delta = \frac{1}{2} bc \sin A$$

1 In triangle ABC, the lengths of AB and BC are $(\sqrt{3}+\sqrt{5})$ cm and $(4\sqrt{3}-2\sqrt{5})$ cm respectively. Given that angle ABC is $120°$, without using a calculator, find the value of AC^2. Leave your answer in the form of $(a+b\sqrt{15})$, where a and b are integers. [3]

2 A section of roller coaster track is parabolic in shape. The height, h m, of the first carriage above the ground at time t seconds is given by $h = 2t^2 - 36t + 164$.

(a) Explain why this carriage cannot go below the height of 2 m. [2]

(b) Find the total duration for the carriage to be below 20 m for this section of the ride. [2]

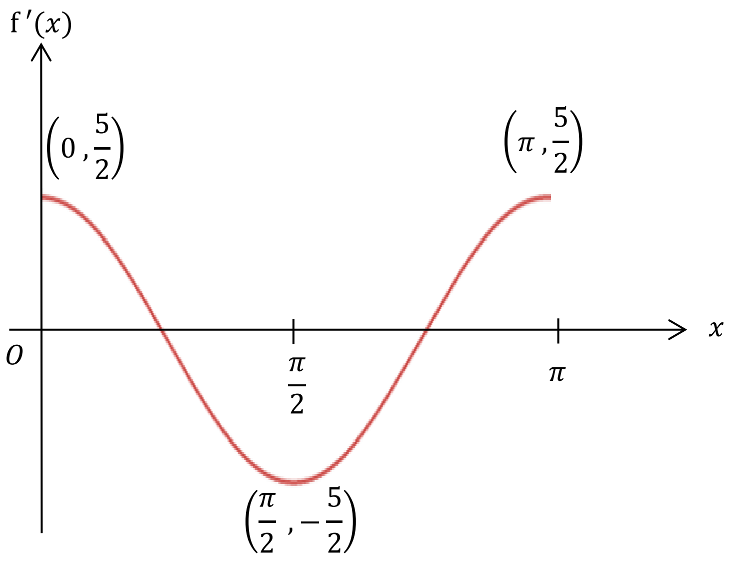
3 **(a)** Without using a calculator, show that $\cot 75° = 2 - \sqrt{3}$. [3]

(b) Hence, show that $\text{cosec}^2\, 75° = 4 \cot 75°$. [2]

4 It is given that $\mathrm{f}(x)$ is the equation of the curve such that $\mathrm{f}'(x) = \frac{5}{x^2} + \cos^2 x + \tan^2 2x$. Given that $\mathrm{f}\left(\frac{\pi}{2}\right) = -\frac{\pi}{4}$, find the equation of the curve $\mathrm{f}(x)$. [5]

5 Solve the equation $(\log_5 y)^2 + \log_5 \frac{1}{y^3} = 28$. [4]

6



The diagram shows the graph of $f'(x)$ of a function $y = f(x)$.

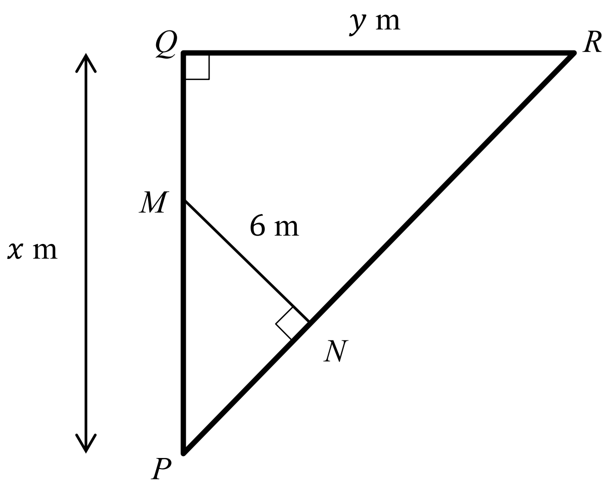
$f'(x) = a\cos bx + c$ for $0 \leq x \leq \pi$ radians. The graph has maximum points at $\left(0, \frac{5}{2}\right)$ and $\left(\pi, \frac{5}{2}\right)$ and a minimum point at $\left(\frac{\pi}{2}, -\frac{5}{2}\right)$.

(a) Explain why $c = 0$. [2]

(b) Explain why $b = 2$. [2]

(c) Without finding the function $f(x)$ and using the diagram above, state and explain at which x-coordinate does the maximum point of $f(x)$ occurs. [2]

7



The diagram shows a triangular field *PQR*, in which *PQ* is x m, *QR* is y m and angle $PQR = 90°$. Points *M* and *N* lie on *PQ* and *PR* respectively, such that *MN* = *MQ* = 6 m and angle *MNP* = 90°.

(a) Show that the area of the triangular field *PQR*, *A* m², is given by

$$A = \frac{3x^2}{\sqrt{x^2 - 12x}}\ .$$

[3]

(b) Given that x can vary, find the stationary value of A and determine its nature. [5]

8 **(a)** Prove the identity $\sec^4 x - \tan^4 x = \frac{1+\sin^2 x}{\cos^2 x}$. [3]

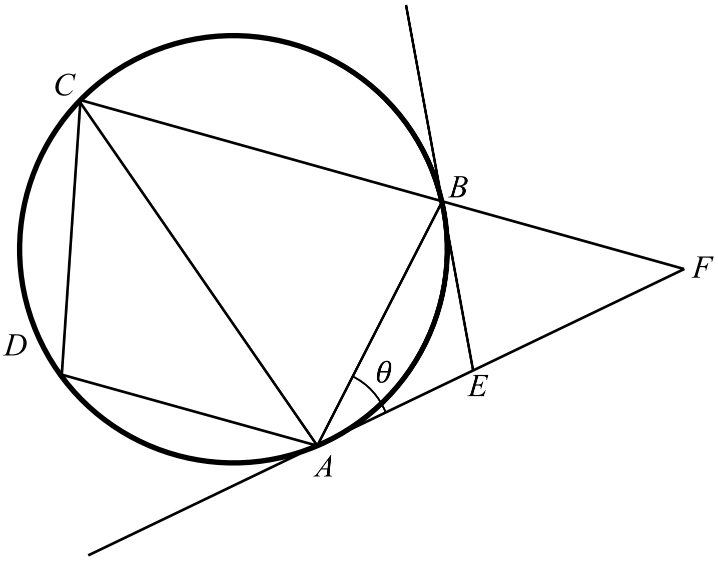
(b) Hence solve the equation $\sec^4(2x - 70°) - \tan^4(2x - 70°) = 4$ for $0° \leq x \leq 180°$. [4]

9 A container with a capacity of 1200 cm^3 was initially filled with water to the brim. When the depth of water is h cm, the volume, V cm^3, of the water in the container is given by $V = h^2 - 10h$. Due to a leakage at the bottom of the container, the depth of water decreases at a rate of $e^{\frac{1}{2}t}$ cm/s, where t is the time in seconds after the leakage started.

(a) Find the initial depth of water in the container. [1]

(b) Find the rate of change of the volume of water when $h = 15$ cm. [6]

10



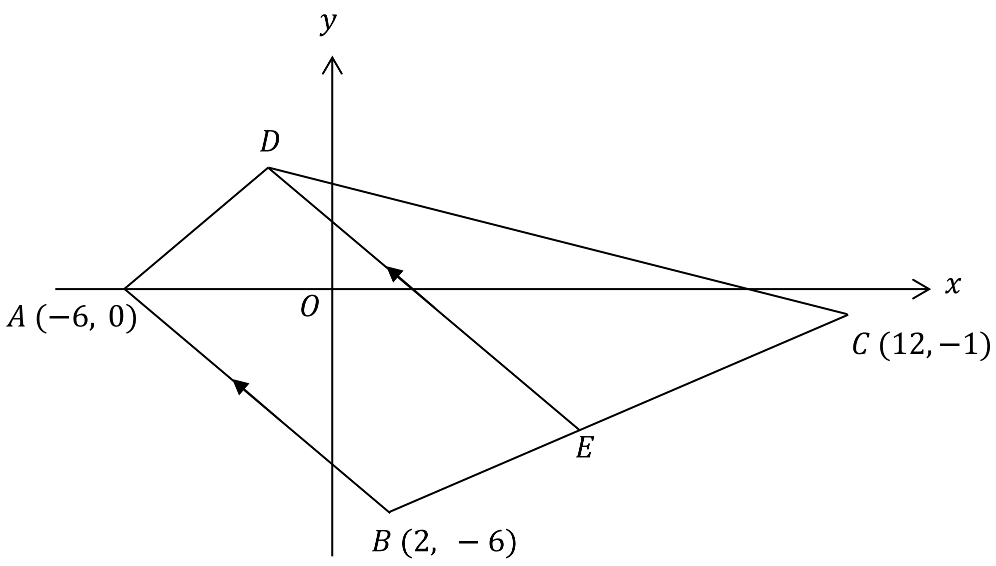
In the diagram, triangle ACD is an isosceles triangle with $AD = CD$. Triangle ACF is also an isosceles triangle with $AC = AF$. The line CF intersects the circle at B. The tangent to circle at A meets CB produced at F. The tangent to the circle at B meets AF at E. Angle $BAE = \theta$.

(a) Given that the line AC bisects angle DCB, what is the name of the special quadrilateral $ABCD$? Show your workings clearly. [3]

(b) Prove that triangle *ACB* and triangle *BFE* are similar. [2]

(c) Show that $BC \times BE = \mathrm{AB} \times AF - AB \times AE$. [2]

11



The diagram shows a quadrilateral *ABCD* with vertices $A\ (-6, 0)$, $B\ (2, -6)$, $C\ (12, -1)$ and *D*. Point *E* lies along the line *BC* such that *BE* : *BC* is 2 : 5 . The line *AB* is parallel to line *DE* and angle *DAO* is equal to angle *OAB*.

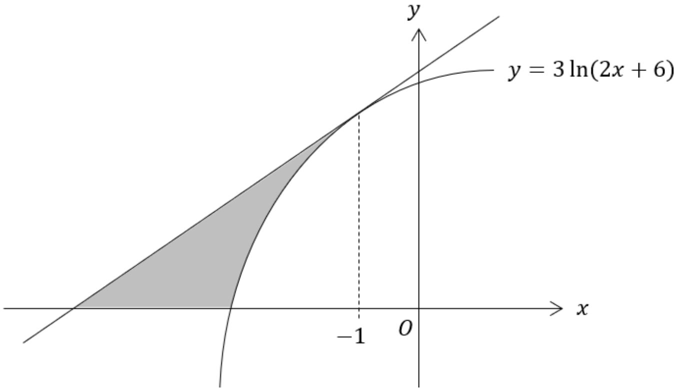
Find

(a) the equation of *DE*, [3]

(b) the coordinates of *D*, [4]

(c) the area of the quadrilateral *ABCD*. [2]

12



The diagram above shows the curve with equation $y = 3\ln(2x + 6)$ and the tangent to the curve at $x = -1$.

(a) Show that the equation of the tangent to the curve $y = 3\ln(2x + 6)$ at $x = -1$ is $2y = 3x + 6\ln 4 + 3$. [3]

(b) Find the area of the shaded region bounded by the curve, the tangent to the curve at $x = -1$ and the x-axis, leaving your answer to 2 decimal places. [5]

13 A polynomial, $\mathrm{P}(x)$, is $2x^3 + 3x^2 - 8x + k$, where k is a constant.

(a) Find the value of k given that $\mathrm{P}(x)$ leaves a remainder of 14 when divided by $x - 2$. [2]

(b) In the case where $k = -12$,

(i) find the value of the constant a, given that $x^2 + a$ is a factor of $\mathrm{P}(x)$. [3]

(ii) hence, solve the equation $\frac{1}{4}x^3 + \frac{3}{4}x^2 - 4x + k = 0$. [3]

14 **(a)** Express $\frac{2x^2-18x+44-6x^3}{(x^2+4)(2-3x)}$ in partial fractions. [5]

(b) Differentiate $\ln(x^2 + 4)$ with respect to x. [1]

(c) Hence, find $\int \frac{2x^2-18x+44-6x^3}{(x^2+4)(2-3x)}\,\mathrm{d}x$. [3]

- End of Paper -

1	Using cosine rule, $AC^2 = (\sqrt{3}+\sqrt{5})^2 + (4\sqrt{3}-2\sqrt{5})^2 - 2(\sqrt{3}+\sqrt{5})(4\sqrt{3} - 2\sqrt{5})\cos 120°$ $= 3 + 2\sqrt{15} + 5 + 48 - 16\sqrt{15} + 20 + \frac{1}{2}(24 - 4\sqrt{15} + 8\sqrt{15} - 20)$ $= 78 - 12\sqrt{15}$
2a)	$h = 2t^2 - 36t + 164$ $= 2(t-9)^2 + 2$
2b)	$h \leq 20$ $2t^2 - 36t + 164 \leq 20$ $2t^2 - 36t + 144 \leq 0$ $t^2 - 18t + 72 \leq 0$ $(t-6)(t-12) \leq 0$ $6 \leq t \leq 12$ Total duration $= 12 - 6$ $= 6\ h$
3a)	$\cot 75° = \frac{1}{\tan(45°+30°)}$ $= \frac{1 - \tan 45° \tan 30°}{\tan 45° + \tan 30°}$ $= \frac{1 - \frac{\sqrt{3}}{3}}{1 + \frac{\sqrt{3}}{3}}$ $= \frac{3-\sqrt{3}}{3+\sqrt{3}} \times \frac{3-\sqrt{3}}{3-\sqrt{3}}$ $= \frac{9 - 6\sqrt{3} + 3}{6}$ $= 2 - \sqrt{3}$
3b)	$\mathrm{cosec}^2\, 75° = 1 + \cot^2 75°$ $= 1 + (2-\sqrt{3})^2$ $= 1 + 4 - 4\sqrt{3} + 3$ $= 8 - 4\sqrt{3}$ $= 4(2-\sqrt{3})$ $= 4\cot 75°$
4	$\mathrm{f}(x) = \int \frac{5}{x^2} + \cos^2 x + \tan^2 2x \ \mathrm{d}x$ $= \int 5x^{-2} + \frac{\cos 2x + 1}{2} + \sec^2 2x - 1\, \mathrm{d}x$ $= \frac{5x^{-1}}{-1} + \frac{1}{2}\frac{\sin 2x}{2} + \frac{1}{2}x + \frac{\tan 2x}{2} - x + c$ $= -\frac{5}{x} + \frac{1}{4}\sin 2x + \frac{1}{2}\tan 2x - \frac{1}{2}x + c$ When $x = \frac{\pi}{2}, \mathrm{f}(x) = -\frac{\pi}{4}$, $-\frac{\pi}{4} = -\frac{5}{\left(\frac{\pi}{2}\right)} + \frac{1}{4}\sin\pi + \frac{1}{2}\tan\pi - \frac{1}{2}\left(\frac{\pi}{2}\right) + c$ $-\frac{\pi}{4} = -\frac{10}{\pi} - \frac{\pi}{4} + c$ $c = \frac{10}{\pi}$ $\therefore\ \mathrm{f}(x) = -\frac{5}{x} + \frac{1}{4}\sin 2x + \frac{1}{2}\tan 2x - \frac{1}{2}x + \frac{10}{\pi}$
5	$(\log_5 y)^2 + \log_5 1 - \log_5 y^3 = 28$ $(\log_5 y)^2 - 3\log_5 y = 28$ Let $u = \log_5 y$, $u^2 - 3u - 28 = 0$ $(u-7)(u+4) = 0$ $u = 7$ $\log_5 y = 7$ $y = 5^7$ $= 78125$ $u = -4$ $\log_5 y = -4$ $y = 5^{-4}$ $= \frac{1}{625}$

6a)

c is the centre of the maximum and minimum range of the $f'(x)$ graph.

$$c = \frac{\frac{5}{2} + \left(-\frac{5}{2}\right)}{2}$$
$$= 0$$

6b)

1 period of the graph = π

$$\frac{2\pi}{b} = \pi$$
$$b = 2$$

6c)

Maximum point occurs when $f'(x) = 0, f''(x) < 0$.

From the graph, $f'(x) = 0$ at $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$,

Since gradient of $f'(x)$ at $x = \frac{\pi}{4} < 0$, thus the maximum occurs at $x = \frac{\pi}{4}$.

7a)

Using similar triangles,

$$\frac{6}{y} = \frac{\sqrt{(x-6)^2 - 6^2}}{x}$$
$$\frac{6}{y} = \frac{\sqrt{x^2 - 12x}}{x}$$
$$y = \frac{6x}{\sqrt{x^2 - 12x}}$$

$$A = \frac{1}{2}xy$$
$$= \frac{1}{2}x\left(\frac{6x}{\sqrt{x^2 - 12x}}\right)$$
$$= \frac{3x^2}{\sqrt{x^2 - 12x}}$$

7b)

$$\frac{dA}{dx} = \frac{(x^2 - 12x)^{\frac{1}{2}}(6x) - 3x^2\frac{1}{2}(x^2 - 12x)^{-\frac{1}{2}}(2x - 12)}{x^2 - 12x}$$
$$= \frac{(x^2 - 12x)^{-\frac{1}{2}}[6x(x^2 - 12x) - 3x^2(x - 6)]}{x^2 - 12x}$$
$$= \frac{6x^3 - 72x^2 - 3x^3 + 18x^2}{(x^2 - 12x)^{\frac{3}{2}}}$$
$$= \frac{3x^3 - 54x^2}{(x^2 - 12x)^{\frac{3}{2}}}$$

When $\frac{dA}{dx} = 0$,

$$3x^3 - 54x^2 = 0$$
$$3x^2(x - 18) = 0$$
$$x = 0 \text{ (rej)} \quad or \quad x = 18$$

$$\therefore A = \frac{3(18)^2}{\sqrt{(18)^2 - 12(18)}} = 93.5 \text{ m}^2$$

Using first derivative test,

	18^-	18	18^+
$\frac{dA}{dx}$	$-$ve	0	$+$ve
Tangent sketch	\	—	/

$\therefore$ A is minimum value

8a)

$$\text{LHS} = \sec^4 x - \tan^4 x$$
$$= (\sec^2 x - \tan^2 x)(\sec^2 x + \tan^2 x)$$
$$= (1)\left(\frac{1}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x}\right)$$
$$= \frac{1 + \sin^2 x}{\cos^2 x}$$
$$= \text{RHS}$$

8b)

$$\frac{1 + \sin^2(2x - 70°)}{\cos^2(2x - 70°)} = 4$$

$$0° \le x \le 180°$$

	$-70° \leq (2x - 70°) \leq 290°$ $1 + \sin^2(2x - 70°)$ $= 4(1 - \sin^2(2x - 70°))$ $5\sin^2(2x - 70°) = 3$ $\sin(2x - 70°) = \pm\sqrt{\frac{3}{5}}$ $\alpha = \sin^{-1}\sqrt{\frac{3}{5}}$ $= 50.768°$ $(2x - 70°)$ lies in all quad. $(2x - 70°)$ $= 50.768°, 129.2315°, 230.768°, -50.768°$ $x = 9.6°, 60.4°, 99.6°, 150.4°$
9a)	$1200 = h^2 - 10h$ $0 = h^2 - 10h - 1200$ $0 = (h - 40)(h + 30)$ $h = 40$ or $h = -30$ (rej)
9b)	$\frac{dV}{dh} = 2h - 10$ When h = 15, $\frac{dV}{dh} = 20$ $\frac{dh}{dt} = -e^{\frac{1}{2}t}$ $h = -2e^{\frac{1}{2}t} + c$ When t = 0, h = 40, $c = 42$ $h = -2e^{\frac{1}{2}t} + 42$ When h = 15, $15 = -2e^{\frac{1}{2}t} + 42$ $t = 5.20538$ s

	When $t = 5.20538$, $\frac{dh}{dt} = -e^{\frac{1}{2}(5.20538)} = -\frac{27}{2}$ $\therefore \frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$ $= 20 \times \left(-\frac{27}{2}\right)$ $= -270 \text{ cm}^3/s$
10a)	$\angle ACB$ $= \angle BAF$ (alternate segment theorem) $= \theta$ $\angle DCA = \angle ACB$ (given) $= \theta$ $\angle CAD = \angle DCA$ (base $\angle$s of isos. Δ) $= \theta$ Since $\angle BCA = \angle CAD = \theta$, by **property of alternate angles, $BC \parallel AD$**, $\therefore ABCD$ is a trapezium.
10b)	$\angle ACB = \angle BFE$ (base $\angle$s of isos. Δ) $= \theta$ Let $\angle PBC = \alpha$, $\angle CAB = \alpha$ (alternate segment theorem) $\angle FBE = \alpha$ (vertically opposite angles) $\therefore \angle CAB = \angle FBE$ $\therefore$ ΔACB is similar to ΔBFE. (AA Similarity)
10c)	$\frac{BC}{FE}$ $= \frac{AB}{BE}$ (corresponding sides of similar tria $BC \times BE = AB \times FE$ $= AB \times (AF - AE)$ $= AB \times AF - AB \times AE$
11a)	grad $AB = \frac{0 - (-6)}{-6 - 2}$ $= -\frac{3}{4}$ grad $DE = -\frac{3}{4}$ Given BE : BC is 2 : 5, $E(6, -4)$ <u>Equation DE</u>

	$y-(-4)=-\frac{3}{4}(x-6)$ $y=-\frac{3}{4}x+\frac{1}{2}$
10b)	grad $AB=\frac{3}{4}$ <u>Equation AD</u> $y-0=\frac{3}{4}(x-(-6))$ $y=\frac{3}{4}x+\frac{9}{2}\quad -(1)$ $y=-\frac{3}{4}x+\frac{1}{2}\quad -(2)$ (1) = (2) $\frac{3}{4}x+\frac{9}{2}=-\frac{3}{4}x+\frac{1}{2}$ $\frac{3}{2}x=-4$ $x=-\frac{8}{3}$ $y=\frac{5}{2}$ $\therefore D\left(-\frac{8}{3},\frac{5}{2}\right)$
10c)	area of $ABCD$ $=\frac{1}{2}\begin{vmatrix}-6 & 2 & 12 & -\frac{8}{3} & -6\\ 0 & -6 & -1 & \frac{5}{2} & 0\end{vmatrix}$ $=\frac{1}{2}\left(64-\left(-\frac{253}{3}\right)\right)$ $=\frac{445}{6}\text{ units}^2$

11a)	$y=3\ln(2x+6)$ $\frac{dy}{dx}=3\left(\frac{2}{2x+6}\right)$ $=\frac{3}{x+3}$ When $x=-1$, $\frac{dy}{dx}=\frac{3}{2}$ $y=3\ln 4$ <u>Equation of tangent</u> $y-3\ln 4=\frac{3}{2}(x-(-1))$ $2y-6\ln 4=3x+3$ $2y=3x+6\ln 4+3$
10b)	$y=3\ln(2x+6)$ $\ln(2x+6)=\frac{y}{3}$ $x=\frac{e^{\frac{y}{3}}-6}{2}=\frac{1}{2}e^{\frac{y}{3}}-3$ Area $=-\int_0^{3\ln 4}\frac{1}{2}e^{\frac{y}{3}}-3\ dx$ $=-\left[\frac{3}{2}e^{\frac{y}{3}}-3y\right]_0^{3\ln 4}$ $=-\left[\left(\frac{3}{2}(4)-3(3\ln 4)\right)-\left(\frac{3}{2}-0\right)\right]$ $=-\left(\frac{9}{2}-9\ln 4\right)$ When $y=0$, $0=3x+6\ln 4+3$ $x=-2\ln 4-1$

	Area of trapezium $= \frac{1}{2} \times 3\ln 4$ $\times \left(1 + (2\ln 4 + 1)\right)$ $= 9.9243$ $\therefore$ Shaded region $= 9.9243 - \left(9\ln 4 - \frac{9}{2}\right)$ $= 1.94767$ $= 1.95 \text{ units}^2$
13a)	$\mathrm{P}(2) = 2(2)^3 + 3(2)^2 - 8(2) + k$ $14 = 12 + k$ $k = 2$
13b)i)	$\mathrm{P}(x) = 2x^3 + 3x^2 - 8x + 12$ $2x^3 + 3x^2 - 8x + 12$ $= (x^2 + a)(Bx + C)$ Comparing coefficients: x^3: $2 = B$ x^2: $3 = C$ k: $-12 = ac$ $-12 = a(3)$ $a = -4$
13b)ii)	$2\left(\frac{1}{2}x\right)^3 + 3\left(\frac{1}{2}x\right)^2 - 8\left(\frac{1}{2}x\right) + k = 0$ $\mathrm{P}(x) = (x^2 - 4)(2x + 3)$ $0 = (x^2 - 4)(2x + 3)$ $x = \pm 2$ or $x = -\frac{3}{2}$ $\frac{1}{2}x = \pm 2$ $\frac{1}{2}x = -\frac{3}{2}$ $x = \pm 4$ $x = -3$

14a)	$(x^2 + 4)(2 - 3x)$ $= 2x^3 - 3x^3 - 12x + 8$ Long division: Quotient: 2 $2x^3 - 3x^3 - 12x + 8 \;\overline{\big)\; -6x^3 + 2x^2 - 18x + 44}$ $-(-6x^3 + 4x^2 - 24x + 16)$ $-2x^2 + 6x + 28$ $\therefore \frac{2x^2 - 18x + 44 - 6x^3}{(x^2+4)(2-3x)} = \frac{Ax + B}{x^2 + 4} + \frac{C}{2 - 3x}$ $2x^2 - 18x + 44 - 6x^3$ $= (Ax + B)(2 - 3x)$ $+ C(x^2 + 4)$ When $x = \frac{2}{3}$, $\frac{280}{9} = \frac{40}{9}C$ $C = 7$ Comparing coefficients, x^2: $-2 = -3A + C$ $3A = 7 + 2$ $A = 3$ k: $28 = 2B + 4C$ $2B = 28 - 4(7)$ $B = 0$ $\therefore 2 + \frac{3x}{x^2 + 4} + \frac{7}{2 - 3x}$
14b)	$\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln(x^2 + 4)\right) = \frac{2x}{x^2 + 4}$

14c)	Hence,

	$= 2x + \frac{3}{2}\ln(x^2+4) - \frac{7}{3}\ln(2-3x)$ $+ C$