FYE P1

1

$$\frac{(2x+1)^2}{x^2+x+1} > 0$$

$$\frac{(2x+1)^2}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}} > 0$$

Since $(2x+1)^2 \geq 0$ and $\left(x+\frac{1}{2}\right)^2+\frac{3}{4} > 0$, for all $x \in \mathbb{R}$

$\therefore x \in \mathbb{R} \setminus \{-\frac{1}{2}\}$

$$\frac{x^2+4x+4}{x^2+x+1} > 0$$

$$\frac{(x+2)^2}{x^2+x+1} > 0$$

$$\frac{4\left(\frac{1}{x}\right)^2+4\left(\frac{1}{x}\right)+1}{\left(\frac{1}{x}\right)^2+\left(\frac{1}{x}\right)+1} > 0$$

Replace x with $\frac{1}{x}$:

$\therefore x \in \mathbb{R}, \quad \frac{1}{x} \neq -\frac{1}{2} \to x \neq -2$

2

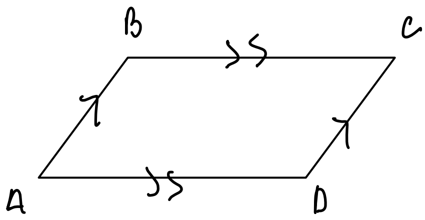
(i)

$$\overrightarrow{AB} = \overrightarrow{DC}$$

$$\overrightarrow{OB} - \overrightarrow{OA} = \overrightarrow{OC} - \overrightarrow{OD}$$

$$\boldsymbol{b} - \boldsymbol{a} = \boldsymbol{c} - \overrightarrow{OD}$$

$$\overrightarrow{OD} = \boldsymbol{a} - \boldsymbol{b} + \boldsymbol{c}$$

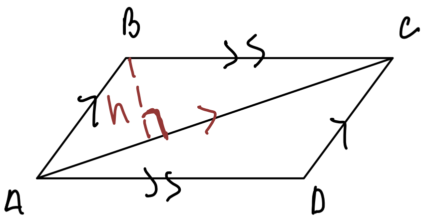


(i) Area of Parallelogram $ABCD = \left|\overrightarrow{AB} \times \overrightarrow{BC}\right|$

$$= \left|(\boldsymbol{b}-\boldsymbol{a}) \times (\boldsymbol{c}-\boldsymbol{b})\right| = \left|\boldsymbol{b}\times\boldsymbol{c} - \boldsymbol{b}\times\boldsymbol{b} - \boldsymbol{a}\times\boldsymbol{c} + \boldsymbol{a}\times\boldsymbol{b}\right|$$

$$= \left|\boldsymbol{b}\times\boldsymbol{c} + \boldsymbol{a}\times\boldsymbol{b} - \boldsymbol{a}\times\boldsymbol{c}\right|$$

$$= \left|\boldsymbol{a}\times\boldsymbol{b} + \boldsymbol{b}\times\boldsymbol{c} + \boldsymbol{c}\times\boldsymbol{a}\right| \quad \text{(proved)}$$



(ii) Area of Triangle $ACD = \frac{1}{2}$ Area of Parallelogram $ABCD = \frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{BC}\right|$

$$\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{BC}\right| = \frac{\left|\boldsymbol{a}\times\boldsymbol{b}+\boldsymbol{b}\times\boldsymbol{c}+\boldsymbol{c}\times\boldsymbol{a}\right|}{2} = \left[\frac{1}{2}\times\overrightarrow{AC}(\text{base})\times\text{height}\right]$$

$$\left|\boldsymbol{a}\times\boldsymbol{b}+\boldsymbol{b}\times\boldsymbol{c}+\boldsymbol{c}\times\boldsymbol{a}\right| = \text{height}\times\overrightarrow{AC}$$

$\therefore$ Shortest distance from D to AC = $\dfrac{\left|\boldsymbol{a}\times\boldsymbol{b}+\boldsymbol{b}\times\boldsymbol{c}+\boldsymbol{c}\times\boldsymbol{a}\right|}{\left|\boldsymbol{c}-\boldsymbol{a}\right|}$ (shown)

3

$t = e^{1-x}$

$\ln t = 1 - x$ or	$\frac{dt}{dx} = -e^{1-x}$
$\frac{1}{t}\frac{dt}{dx} = -1$	$\frac{dt}{dx} = -t$
$\frac{dx}{dt} = \frac{-1}{t}$	$\frac{dx}{dt} = \frac{1}{-t}$

When $x = 0,\ t = e$

When $x = 1,\ t = 1$

$$\int_0^1 e^{1-x}\tan^{-1}\left(e^{1-x}\right)\mathrm{d}x = \int_e^1 t\tan^{-1}t\left(\frac{-1}{t}\mathrm{d}t\right)$$

$$= -\int_e^1 \tan^{-1}t\,\mathrm{d}t$$

$$= \int_1^e \tan^{-1}t\,\mathrm{d}t \quad \text{(shown)}$$

$$u = \tan^{-1}t \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = 1$$

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{1}{1+t^2} \qquad v = \int 1\mathrm{d}t$$

$$v = t$$

$$= \left[t\tan^{-1}t\right]_1^e - \int_1^e \frac{t}{1+t^2}\mathrm{d}t$$

$$= \left[t\tan^{-1}t\right]_1^e - \frac{1}{2}\int_1^e \frac{2t}{1+t^2}\mathrm{d}t$$

$$= \left[t\tan^{-1}t - \frac{1}{2}\ln\left(1+t^2\right)\right]_1^e$$

$$= \left[\left((e)\tan^{-1}(e) - \frac{1}{2}\ln(1+e^2)\right) - \left((1)\tan^{-1}(1) - \frac{1}{2}\ln(2)\right)\right]$$

$$= \left[\left(e\tan^{-1}(e) - \ln(\sqrt{1+e^2})\right) - \left(\frac{\pi}{4} - \ln(\sqrt{2})\right)\right]$$

$$= \left[e\tan^{-1}(e) - \ln(\sqrt{1+e^2}) - \frac{\pi}{4} + \ln(\sqrt{2})\right]$$

$$= e\tan^{-1}(e) - \frac{\pi}{4} - \frac{1}{2}\ln\left(1+e^2\right) + \frac{1}{2}\ln 2$$

4

Let P_n be the statement

$$\sum_{r=1}^{n}\cos r\theta = \frac{\sin(n+\frac{1}{2})\theta - \sin\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta}, \qquad n = 1, 2, 3, \ldots$$

When $n = 1$,

LHS = $\sum_{r=1}^{1}\cos r\theta = \cos\theta$

RHS = $\dfrac{\sin\frac{3}{2}\theta - \sin\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta} = \dfrac{2\cos\theta\sin\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta} = \cos\theta$

Thus, P_1 is true.

Assume that P_k is true for some k, $k = 1,2,3,\dots$

$$\sum_{r=1}^{k}\cos r\theta = \frac{\sin(k+\frac{1}{2})\theta - \sin\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta}$$

To prove that P_{k+1} is also true, i.e.

$$\sum_{r=1}^{k+1}\cos r\theta = \frac{\sin(k+\frac{3}{2})\theta - \sin\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta}$$

$$\text{LHS } = \sum_{r=1}^{k+1}\cos r\theta$$

$$= \sum_{r=1}^{k}\cos r\theta + \cos(k+1)\theta$$

$$= \frac{\sin(k+\frac{1}{2})\theta - \sin\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta} + \cos(k+1)\theta$$

$$= \frac{\sin(k+\frac{1}{2})\theta + 2\cos(k+1)\theta\sin\frac{1}{2}\theta - \sin\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta}$$

$$= \frac{\sin(k+\frac{1}{2})\theta + \left[\sin(k+1+\frac{1}{2})\theta - \sin(k+1-\frac{1}{2})\theta\right] - \sin\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta}$$

$$= \frac{\sin(k+\frac{1}{2})\theta + \sin(k+\frac{3}{2})\theta - \sin(k+\frac{1}{2})\theta - \sin\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta}$$

$$= \frac{\sin(k+\frac{3}{2})\theta - \sin\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta}$$

= RHS

Thus, P_{k+1} is also true.

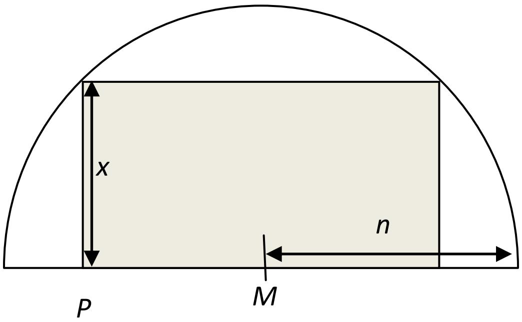
Since P_1 is true and P_k is true $\Rightarrow P_{k+1}$ is also true, then by mathematical induction, P_n is true for $n = 1, 2, 3, \dots$.

Let M be the center of the hemisphere.

By Pythagoras Theorem,

$PM = \sqrt{n^2 - x^2}$

$$\begin{aligned} A &= \pi \times \text{height} \times \text{diameter} \\ &= 2\pi\left(\frac{x}{2}\right)\left(2\sqrt{n^2-x^2}\right) \\ &= \pi(x)\left(2\sqrt{n^2-x^2}\right) \\ &= 2\pi x\sqrt{n^2-x^2} \ \text{ cm}^2 \ \text{(shown)} \end{aligned}$$



As x varies,

$$\frac{dA}{dx} = 2\pi\sqrt{n^2-x^2} + \frac{2\pi x(-2x)}{2\sqrt{n^2-x^2}}$$

When $\frac{dA}{dx} = 0$, for stationary points,

$$2\pi\left(n^2-x^2\right) - 2\pi x^2 = 0$$

$$\left(n^2-x^2\right) - x^2 = 0$$

$$n^2 - 2x^2 = 0$$

$$x^2 = \frac{n^2}{2}$$

$$x = \frac{n}{\sqrt{2}} \ \text{(reject -ve)}$$

To show *A* maximum,

x	$\left(\frac{n}{\sqrt{2}}\right)^-$	$\frac{n}{\sqrt{2}}$	$\left(\frac{n}{\sqrt{2}}\right)^+$
$\frac{dA}{dx}$	+	0	-

$$\frac{\text{Diameter of cylinder}}{\text{Height of cylinder}} = \frac{x}{2\sqrt{n^2 - x^2}}$$

$$= \frac{\frac{n}{\sqrt{2}}}{2\sqrt{n^2 - \frac{n^2}{2}}}$$

$$= \frac{\frac{n}{\sqrt{2}}}{2\sqrt{\frac{n^2}{2}}}$$

$$= \frac{\frac{\sqrt{2}}{2}}{\sqrt{2}}$$

$$= \frac{1}{2}$$

$\therefore k = 2$

6.

Solutions:

(i)

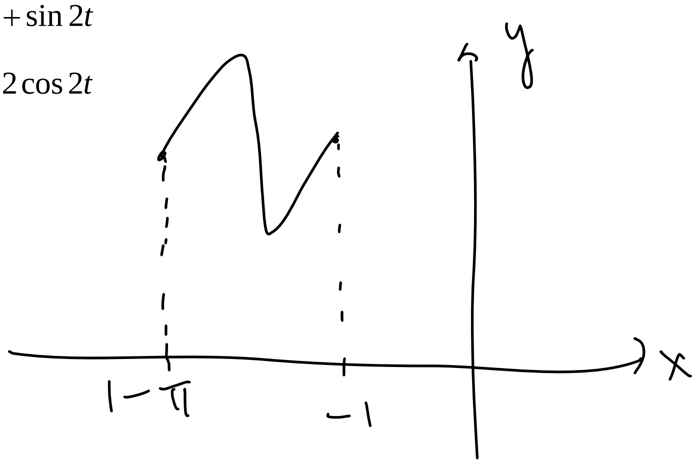
$x = t - \cos t$

$\frac{\mathrm{d}x}{\mathrm{d}t} = 1 + \sin t$

$y = 3 + \sin 2t$

$\frac{\mathrm{d}y}{\mathrm{d}t} = 2\cos 2t$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \times \frac{\mathrm{d}t}{\mathrm{d}x}$$
$$= \frac{2\cos 2t}{1 + \sin t}$$



For turning point, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$

$$\frac{2\cos 2t}{1+\sin t} = 0$$

$$\cos 2t = 0$$

$$2t = -\frac{\pi}{2} \text{ or } -\frac{3\pi}{2}$$

$$t = -\frac{\pi}{4} \text{ or } -\frac{3\pi}{4}$$

When $t = -\frac{\pi}{4}$, $\quad x = -\frac{\pi}{4} - \cos\left(-\frac{\pi}{4}\right) = -\frac{\pi}{4} - \frac{\sqrt{2}}{2}$, $\qquad y = 3 + \sin 2\left(-\frac{\pi}{4}\right) = 2$

$\therefore$ The coordinates of the turning point is $\left(-\frac{\pi}{4} - \frac{\sqrt{2}}{2}, 2\right)$

When $t = -\frac{3\pi}{4}$, $\quad x = -\frac{3\pi}{4} - \cos\left(-\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2} - \frac{3\pi}{4}$, $\quad y = 3 + \sin 2\left(-\frac{3\pi}{4}\right) = 4$

$\therefore$ The coordinates of the turning point is $\left(\frac{\sqrt{2}}{2} - \frac{3\pi}{4}, 4\right)$

ii)

When $t = -\frac{\pi}{2}$, $\quad \frac{dy}{dx} = \frac{2\cos 2\left(-\frac{\pi}{2}\right)}{1+\sin\left(-\frac{\pi}{2}\right)} = \frac{-2}{1+(-1)} = \frac{-2}{0} = -\infty$

$\therefore \quad x = -\frac{\pi}{2} - \cos\left(-\frac{\pi}{2}\right) = -\frac{\pi}{2}$

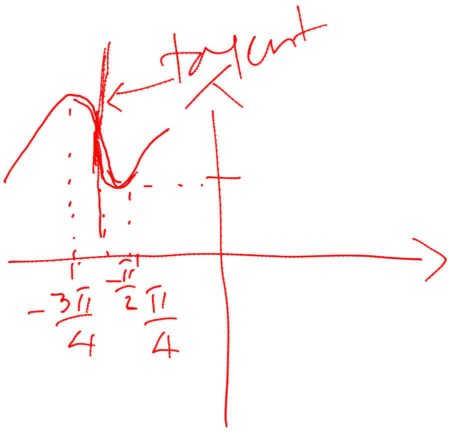
Equation of tangent: $x = -\frac{\pi}{2}$

iii)

$$\int_{-\frac{\pi}{2}}^{-\frac{\pi}{4}-\frac{\sqrt{2}}{2}} y\,dx = \int_{-\frac{\pi}{2}}^{-\frac{\pi}{4}} (3+\sin 2t)(1+\sin t\,dt)$$

$= 0.16584$

$= 0.166$ (3 s.f.) (By G.C.)



7.

$$\frac{2}{r(r+2)} \equiv \frac{A}{r} + \frac{B}{r+2}$$

$2 = A(r+2) + Br$

$r = 0: \quad 2 = 2A \rightarrow \therefore A = 1$

$r = -2: \quad 2 = B(-2) \rightarrow B = -1$

$$\therefore \frac{2}{r(r+2)} \equiv \frac{1}{r} - \frac{1}{(r+2)}$$

$$\sum_{r=1}^{N} \frac{2}{r(r+2)} = \sum_{r=1}^{N} \left(\frac{1}{r} - \frac{1}{(r+2)} \right)$$

$$= \left(\begin{array}{c} \frac{1}{1} - \frac{1}{3} \\ + \frac{1}{2} - \frac{1}{4} \\ + \frac{1}{3} - \frac{1}{5} \\ + \frac{1}{4} - \frac{1}{6} \\ \ldots\ldots \\ + \frac{1}{N-2} - \frac{1}{N} \\ + \frac{1}{N-1} - \frac{1}{N+1} \\ + \frac{1}{N} - \frac{1}{N+2} \end{array} \right)$$

$$= \left(1 + \frac{1}{2} - \frac{1}{N+1} - \frac{1}{N+2} \right)$$

$$= \frac{3}{2} - \frac{1}{(N+1)} - \frac{1}{(N+2)}$$

$$\sum_{r=1}^{N} \frac{2}{r(r+2)} = \frac{2}{(1)(3)} + \frac{2}{(2)(4)} + \frac{2}{(3)(5)} + \ldots + \frac{2}{(N-3)(N-1)} + \frac{2}{(N-2)N} + \frac{2}{(N-1)(N+1)} + \frac{2}{N(N+2)}$$

$$\sum_{r=6}^{N+3}\frac{2}{(r-4)(r-2)}=\frac{2}{(2)(4)}+\frac{2}{(3)(5)}+\ldots+\frac{2}{(N-3)(N-1)}+\frac{2}{(N-2)N}+\frac{2}{(N-1)(N+1)}$$

$$=\sum_{r=1}^{N}\frac{2}{r(r+2)}-\frac{2}{3}-\frac{2}{N(N+2)}$$

$$=\left(\frac{3}{2}-\frac{1}{(N+1)}-\frac{1}{(N+2)}\right)-\frac{2}{3}-\frac{2}{N(N+2)}$$

$$=\frac{5}{6}-\left(\frac{1}{(N+1)}+\frac{1}{(N+2)}+\frac{2}{N(N+2)}\right)$$

$$=\frac{5}{6}-\left(\frac{N(N+2)+N(N+1)+2(N+1)}{N(N+1)(N+2)}\right)$$

$$=\frac{5}{6}-\left(\frac{2N^2+5N+2}{N(N+1)(N+2)}\right)$$

$$=\frac{5}{6}-\left(\frac{(2N+1)(N+2)}{N(N+1)(N+2)}\right)$$

$$=\frac{5}{6}-\frac{2N+1}{N(N+1)}$$

Alternatively,

$$\sum_{r=6}^{N+3}\frac{2}{(r-4)(r-2)}=\frac{2}{(2)(4)}+\frac{2}{(3)(5)}+\ldots+\frac{2}{(N-3)(N-1)}+\frac{2}{(N-2)N}+\frac{2}{(N-1)(N+1)}$$

$$=\sum_{r=1}^{N-1}\frac{2}{r(r+2)}-\frac{2}{3}$$

$$=\frac{3}{2}-\frac{1}{(N)}-\frac{1}{(N+1)}-\frac{2}{3}$$

$$=\frac{5}{6}-\frac{2N+1}{N(N+1)}$$

$$\sum_{r=6}^{\infty}\frac{2}{(r-4)(r-2)}=\lim_{N\to\infty}\left(\sum_{r=6}^{N}\frac{2}{(r-4)(r-2)}\right)$$

$$=\lim_{N\to\infty}\left(\frac{5}{6}-\frac{2N+1}{N(N+1)}\right)$$

$$=\frac{5}{6}$$

8.

(i)

$$\frac{\mathrm{d}F}{\mathrm{d}t} \propto \left(F-(-20)\right)$$

$$\frac{\mathrm{d}F}{\mathrm{d}t} \propto \left(F+20\right)$$

$$\frac{\mathrm{d}F}{\mathrm{d}t} = -k\left(F+20\right), \qquad \text{for some constant } k>0$$

$$\frac{1}{\left(F+20\right)}\frac{\mathrm{d}F}{\mathrm{d}t} = -k$$

$$\int \frac{1}{\left(F+20\right)}\frac{\mathrm{d}F}{\mathrm{d}t}\,\mathrm{d}t = \int -k\,\mathrm{d}t$$

$$\ln\left(F+20\right) = -kt + C$$

When $t=0$, $F=20^{\circ}C$,

$$C = \ln(20+20)+0 = \ln 40$$

Hence, $\ln(F+20) = -kt + \ln 40$

When $t=10$, $F=-10^{\circ}C$,

$$\ln\left(-10+20\right) = -k\left(10\right) + \ln 40$$

$$\ln 10 - \ln 40 = -10k$$

$$\ln\left(\frac{1}{4}\right) = -10k$$

$$k = -\frac{1}{10}\ln\left(\frac{1}{4}\right)$$

$$\therefore \ln\left(F+20\right) = -\left(-\frac{1}{10}\ln\frac{1}{4}\right) + \ln 40$$

$$\ln\left(F+20\right) = \ln\left(40\left(\frac{1}{4}\right)^{\frac{1}{10}}\right)$$

$$F+20 = 40\left(\frac{1}{4}\right)^{\frac{1}{10}}$$

$$F = 40\left(\frac{1}{4}\right)^{\frac{1}{10}} - 20 \text{ (Shown)}$$

(ii)

When $F = -15$,

$$-15 = 40\left(\frac{1}{4}\right)^{\frac{t}{10}} - 20$$

$$\left(\frac{1}{4}\right)^{\frac{t}{10}} = \frac{1}{8}$$

$$\frac{t}{10}\ln\frac{1}{4} = \ln\frac{1}{8}$$

$$\frac{t}{10} = \frac{\ln\frac{1}{8}}{\ln\frac{1}{4}}$$

$$t = 15$$

$\therefore$ 5 more minutes are required for the temperature of the meat to reach $-15^{o}C$

9.

(i)

Let n be the total # of weeks taken for the owner to sell all his hamsters.

$$\therefore n = \frac{500}{k}$$

(ii)

Week	**# of Hamsters of sold**	**Amount**
1	k	$10k$
2	k	$0.95(10)k$
3	k	$0.95^2(10)k$
4	k	$0.95^3(10)k$
$\cdots$	$\cdots$	$\cdots$
n	k	$0.95^{n-1}(10)k$

Total selling price at the end of n weeks

$$= 10k + 0.95(10)k + 0.95^2(10)k + 0.95^3(10)k + ... + 0.95^{n-1}(10)k$$
$$= \frac{10k\left(1-0.95^n\right)}{1-0.95}$$
$$= 200k\left(1-0.95^n\right)$$

Sub $n = \frac{500}{k}$: Total selling price at the end of n weeks $= 200k\left(1-0.95^{\frac{500}{k}}\right)$

(iii)

Week	Hamsters Remaining	Feeding Cost
1	500	$0.5(500)$
2	$500-k$	$0.5(500-k)$
3	$500-2k$	$0.5(500-2k)$
4	$500-3k$	$0.5(500-3k)$
$\cdots$	$\cdots$	$\cdots$
n	$500-(n-1)k$	$0.5(500-(n-1)k)$

Total feeding cost at the end of n weeks

$$= 0.5(500) + 0.5(500-k) + 0.5(500-2k) + 0.5(500-3k) + \ldots + 0.5(500-(n-1)k)$$
$$= 0.5(500 + (500-k) + (500-2k) + (500-3k) + \ldots + (500-(n-1)k))$$
$$= 0.5((500+500+\ldots+500) - k - 2k - 3k - \ldots - (n-1)k)$$
$$= 0.5(500n - k(1+2+3+\ldots+(n-1)))$$
$$= 0.5\left(500n - k\left(\frac{n-1}{2}(1+n-1)\right)\right)$$
$$= 0.5\left(500n - k\left(\frac{n(n-1)}{2}\right)\right)$$
$$= \frac{n}{2}\left(500 - \frac{k(n-1)}{2}\right)$$
$$= \frac{n}{4}(1000 - k(n-1))$$

Sub $n = \frac{500}{k}$: Total feeding cost at the end of n weeks $= \frac{500}{4k}\left(1000 - k\left(\frac{500}{k} - 1\right)\right)$

$$= \frac{500}{4k}(1000 - 500 + k)$$
$$= \frac{125}{k}(500 + k)$$

(iv)

For profit:

$$200k\left(1-0.95^{\frac{500}{k}}\right)>\frac{125}{k}\left(500+k\right)$$

From G.C. (Use of graph or use of list are both acceptable)

$k>21.68$

least value of $k=25$ (Since $k>0$ and k is a factor of 500)

$\therefore$ He must sell at least 25 hamsters per week in order to make a profit.

10

a.) $2p-qi=2$ -- (1)

$p^2-q+8+2i=0--(2)$

$(2):\quad q=p^2+8+2i--(3)$

(3) in (1):

$2p-(p^2+8+2i)i=2$

$2p-p^2i-8i-2i^2=2$

$2p-p^2i-8i+2-2=0$

$p^2i+8i-2p=0$

$ip^2-2p+8i=0$

$\left(p+4i\right)\left(ip+2\right)=0$

$p=-4i$ or $p=-\frac{2}{i}$

$p=-4i$ or $p=2i$

when $p=-4i$: $\quad q=\left(-4i\right)^2+8+2i=2i-8$

when $p=2i$: $\quad q=\left(2i\right)^2+8+2i=2i+4$

Alternatively,

$$(2): \quad p=\frac{2+qi}{2} ----(3)$$

(3) in (1):

$$\left(\frac{2+qi}{2}\right)^2-q+8+2i=0$$

$$\frac{1}{4}(2+qi)^2-q+8+2i=0$$

$$\left(4+4qi-q^2\right)-4q+32+8i=0$$

$$q=\frac{-4+4i\pm\sqrt{144}}{2}=4+2i \quad \text{or} \quad -8+2i$$

$$\text{when } q=4+2i: \qquad p=\frac{2+(4+2i)i}{2}=2i$$

$$\text{when } q=-8+2i: \qquad p=\frac{2+(-8+2i)i}{2}=-4i$$

b.) $z=\dfrac{-1+i}{\sqrt{3}-i}\times\dfrac{\sqrt{3}+i}{\sqrt{3}+i}$

$$=\frac{-\sqrt{3}-i+\sqrt{3}i+i^2}{3-i^2}$$

$$=\frac{-\sqrt{3}-1+\sqrt{3}i-i}{4}$$

$$=\frac{-\sqrt{3}-1}{4}+\frac{\left(\sqrt{3}-1\right)}{4}i$$

$$|z|=\frac{|-1+i|}{\left|\sqrt{3}-i\right|}=\frac{\sqrt{2}}{2}=\frac{1}{\sqrt{2}}$$

$$\arg(z)=\arg\left(\frac{-1+i}{\sqrt{3}-i}\right)=\arg(-1+i)-\arg\left(\sqrt{3}-i\right)=\frac{3\pi}{4}-\left(-\frac{\pi}{6}\right)=\frac{11\pi}{12}$$

$$\therefore z=\frac{1}{\sqrt{2}}\left[\cos\left(\frac{11\pi}{12}\right)+i\sin\left(\frac{11\pi}{12}\right)\right]$$

By comparison,

$$\frac{1}{\sqrt{2}}\cos\left(\frac{11\pi}{12}\right)=\frac{-\sqrt{3}-1}{4}$$

$$\cos\left(\frac{11\pi}{12}\right)=\frac{\sqrt{2}\left(-\sqrt{3}-1\right)}{4}$$

$$\cos\left(\frac{11\pi}{12}\right)=-\frac{\left(\sqrt{6}+\sqrt{2}\right)}{4}\text{(shown)}$$

$$\frac{1}{\sqrt{2}}\sin\left(\frac{11\pi}{12}\right)=\frac{\left(\sqrt{3}-1\right)}{4}$$

$$\sin\left(\frac{11\pi}{12}\right)=\frac{\sqrt{2}\left(\sqrt{3}-1\right)}{4}$$

$$\sin\left(\frac{11\pi}{12}\right)=\frac{\sqrt{6}-\sqrt{2}}{4}$$

$$\tan\left(\frac{11\pi}{12}\right)=\frac{\sin\left(\frac{11\pi}{12}\right)}{\cos\left(\frac{11\pi}{12}\right)}$$

$$=\frac{\sqrt{6}-\sqrt{2}}{4}\times\left(-\frac{4}{\sqrt{6}+\sqrt{2}}\right)$$

$$=\frac{\sqrt{2}-\sqrt{6}}{\sqrt{2}+\sqrt{6}}$$

11

$(i) c^2x^2 - b^2y^2 - a^2 = 0 \Rightarrow 2c^2x - 2b^2y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = 0 \Rightarrow x = 0 \Rightarrow y^2 = -\frac{a^2}{b^2} \Rightarrow undefined \Rightarrow$ no turning points

(ii) $c^2x^2 - b^2y^2 - a^2 = 0 \Rightarrow y = \pm\frac{\sqrt{c^2x^2 - a^2}}{b}$

As $x \to \pm\infty, y \to \pm\frac{cx}{b}$

$\therefore$ asymptotes are $y = \pm\frac{cx}{b}$

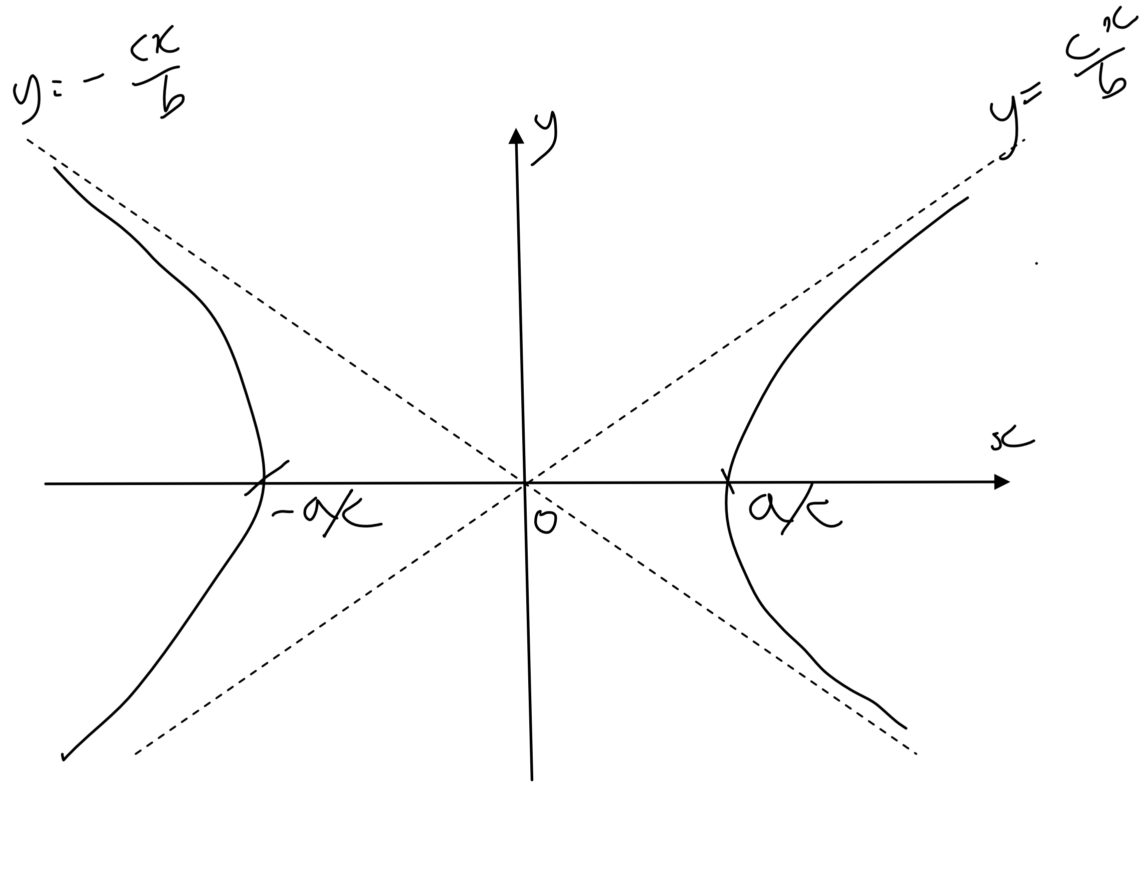
(iii)

y is undefined if $c^2x^2 - a^2 < 0 \Rightarrow x^2 < \frac{a^2}{c^2} \Rightarrow x \in \left(\frac{-a}{c}, \frac{a}{c}\right)$

Therefore x cannot take any value in interval $\left(\frac{-a}{c}, \frac{a}{c}\right)$ (or possible x-values are $x \le \frac{-a}{c}$ or $x \ge \frac{a}{c}$)

Axes of symmetry : y=0, x=0

(iv)



(v)

$$\left(c-bk\right)\left(c+bk\right)x^{2}=a^{2}----(*)$$
$$\Rightarrow\left(c^{2}-b^{2}k^{2}\right)x^{2}=a^{2}$$
$$\Rightarrow c^{2}x^{2}-a^{2}=b^{2}k^{2}x^{2}$$
$$\Rightarrow\frac{c^{2}x^{2}-a^{2}}{b^{2}}=\left(kx\right)^{2}$$
$$\Rightarrow\pm\frac{\sqrt{c^{2}x^{2}-a^{2}}}{b}=kx\Rightarrow\text{ additional graph to draw }y=kx$$

From sketch, eqn (*) has 2 real roots

$\Rightarrow y=kx$ cuts original sketch at 2 points

$$\Rightarrow-\frac{c}{b}<k<\frac{c}{b}$$
$$\Rightarrow-1<-\frac{c}{b}<k<\frac{c}{b}<1\ \left(\text{since }0<c<b\Rightarrow\ 0<\frac{c}{b}<1\right)\text{ (Shown)}$$

<u>12</u>

<u>Solution:</u>

(i) $p_{1}:r\bullet\begin{pmatrix}1\\-1\\0\end{pmatrix}=5$

$$\begin{pmatrix}1\\2\\-3\end{pmatrix}\bullet\begin{pmatrix}1\\-1\\0\end{pmatrix}=\sqrt{14}\sqrt{2}\cos\theta$$

$\theta=100.8933946°$

Since the angle obtained is an obtuse; find θ' first;

$\theta'=180°-100.8933946°=79.10660535°$

$\therefore\alpha=90°-79.10660535°=10.89339465°\approx10.9°$(1 dp)

(ii) Let N be the foot of the perpendicular from A to p_1

$$\overrightarrow{AN} = \lambda \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

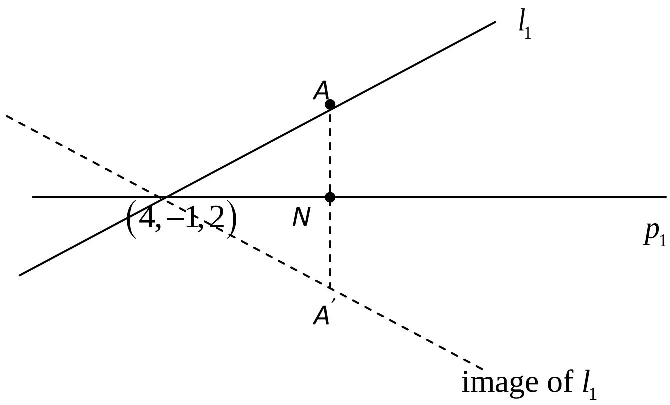
$$\overrightarrow{ON} = \begin{pmatrix} \lambda \\ -\lambda \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} \lambda+1 \\ -\lambda+2 \\ -3 \end{pmatrix}$$

Point N lies in $p_1 \Rightarrow \begin{pmatrix} \lambda+1 \\ -\lambda+2 \\ -3 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = 5$

$\Rightarrow \lambda + 1 + \lambda - 2 = 5$

$\therefore \lambda = 3$

$$\therefore \overrightarrow{ON} = \begin{pmatrix} 3+1 \\ -3+2 \\ -3 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix}$$



(iii)

To show B lies in $p_1 \Rightarrow$ B satisfies the equation of p_1

$r \bullet \mathbf{n} = a \bullet \mathbf{n}$

$$\begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = 4 + 1 = 5 \text{ (Shown)}$$

Since N is the foot of perpendicular from A to p_1

$\therefore$ $\overrightarrow{AN}$ is perpendicular to p_1, Let A' be the image of A when reflected about p_1.

By ratio theorem : $\overrightarrow{ON} = \dfrac{\overrightarrow{OA} + \overrightarrow{OA'}}{2}$.

$$\overrightarrow{OA}' = 2\overrightarrow{ON} - \overrightarrow{OA}$$

$$= 2\begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$$

$$= \begin{pmatrix} 7 \\ -4 \\ -3 \end{pmatrix}$$

Direction vector of image of l_1

$$\overrightarrow{BA'} = \begin{pmatrix} 7 \\ -4 \\ -3 \end{pmatrix} - \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ -5 \end{pmatrix}$$

Equation of the image of l_1: $r = \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -3 \\ -5 \end{pmatrix}, \mu \in \mathbb{R}$ or

$$r = \begin{pmatrix} 7 \\ -4 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -3 \\ -5 \end{pmatrix}, \mu \in \mathbb{R}$$

Alternatively,

By ratio theorem : $\overrightarrow{BN} = \dfrac{\overrightarrow{BA} + \overrightarrow{BA'}}{2}$.

$$\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} - \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \\ -5 \end{pmatrix} \qquad \overrightarrow{BN} = \overrightarrow{ON} - \overrightarrow{OB} = \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix} - \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -5 \end{pmatrix}$$

$$\overrightarrow{BN} = \frac{\overrightarrow{BA} + \overrightarrow{BA'}}{2}$$

$$\overrightarrow{BA}' = 2\overrightarrow{BN} - \overrightarrow{BA}$$

$$= 2\begin{pmatrix} 0 \\ 0 \\ -5 \end{pmatrix} - \begin{pmatrix} -3 \\ 3 \\ -5 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ -3 \\ -5 \end{pmatrix}$$

$$\overrightarrow{OA}' = \begin{pmatrix} 3 \\ -3 \\ -5 \end{pmatrix} + \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ -4 \\ -3 \end{pmatrix}$$

Equation of the image of l_1: $r = \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -3 \\ -5 \end{pmatrix}, \mu \in \mathbb{R}$ or

$$r = \begin{pmatrix} 7 \\ -4 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -3 \\ -5 \end{pmatrix}, \mu \in \mathbb{R}$$

(iv)

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 5 \end{pmatrix} \qquad \overrightarrow{BN} = \overrightarrow{ON} - \overrightarrow{OB} = \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix} - \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -5 \end{pmatrix}$$

$$\mathbf{n_2} = \mathbf{n_1} \times \overrightarrow{AB} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 3 \\ -3 \\ 5 \end{pmatrix} = \begin{pmatrix} -5 \\ -5 \\ 0 \end{pmatrix} = -5\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Alternatively,

$$\mathbf{n_2} = \overrightarrow{BN} \times \overrightarrow{AB} = \begin{pmatrix} 0 \\ 0 \\ -5 \end{pmatrix} \times \begin{pmatrix} 3 \\ -3 \\ 5 \end{pmatrix} = \begin{pmatrix} -15 \\ -15 \\ 0 \end{pmatrix} = -15\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$\therefore$ Equation of p_2 is $\mathrm{r} \bullet \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 3$

Line of intersection of p_1 and p_2

Use of GC / Solve using simultaneous equation -

Solving 2 equations

$x - y = 5 \qquad (1)$

$x + y = 3 \qquad (2)$

$\therefore x = 4; y = -1$

$$\overrightarrow{BN} = \begin{pmatrix} 0 \\ 0 \\ -5 \end{pmatrix} = -5\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Since both p_1 and p_2 contain the position vector $\begin{pmatrix} 4 \\ -1 \\ 0 \end{pmatrix}$, *B* and *N* **<u>AND</u>** their line of intersection passes through the *B* and *N*.

$\therefore$ Vector equation of line of intersection of p_1 and p_2 is l: $\mathbf{r} = \begin{pmatrix} 4 \\ -1 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \alpha \in \mathbb{R}$

(v)

Since l lies in both p_1 and p_2, for all 3 planes to have no point in common would imply l is parallel to p_3

$$\begin{pmatrix}2\\3\\a\end{pmatrix}\bullet\begin{pmatrix}0\\0\\1\end{pmatrix}=0\Rightarrow\therefore a=0$$

Equation of $p_3 = \mathrm{r}\bullet\begin{pmatrix}2\\3\\0\end{pmatrix}=d$

Let D be the point on P_3 with coordinates $(\frac{d}{2},0,0)$ and $B(4,-1,2)$

$$\overrightarrow{BD}=\begin{pmatrix}\frac{d}{2}\\0\\0\end{pmatrix}-\begin{pmatrix}4\\-1\\2\end{pmatrix}=\begin{pmatrix}\frac{d}{2}-4\\1\\-2\end{pmatrix}$$

$$\left|\overrightarrow{BD}\bullet n_3\right|=\sqrt{13}$$

$$\frac{\left|\begin{pmatrix}\frac{d}{2}-4\\1\\-2\end{pmatrix}\bullet\begin{pmatrix}2\\3\\0\end{pmatrix}\right|}{\left|\begin{pmatrix}2\\3\\0\end{pmatrix}\right|}=\sqrt{13}$$

$$|d-8+3|=13$$

$$d-5=\pm 13$$

$$\therefore d=-8 \text{ or } 18$$