Mathematics

Lecture 2: Logicism, Intuitionism, Formalism and the 3 crises

Overview

- Logicism
- Intuitionism
- Formalism
- 3 crises

Recap

- Examined how Math is constructed and its nature (deductive, a priori)
- Two of the usual candidates on the nature of Math – Platonism and Empiricism

Logicism

- Another Realist, Rationalist view of Math, closely linked to Platonism
- Main proponents: Frege, Bertrand Russell, Peano and Whitehead
- Purpose: give math a firm foundation by showing that classical math is **part of logic**
- Why? So that mathematics can be shown to be **free of contradictions** and therefore a body of knowledge that is **certain** and **absolute**.
  - Foundationalist tendencies are present here.
  - Indeed, it seems <u>weird</u> to think of Math as <u>anything but foundational</u>.
  - It's how Euclid started out in his *Elements of Mathematics*
  - Thinking of Math as coherent or reliabilist is just <u>plain weird</u>.
- Also, if successful, questions like "why is classical math free of contradictions?" can be reduced to "why is logic free of contradictions?"
- The latter Qn is something that philosophers can at least have a thorough handle on.

Logicism

- Russell and Whitehead showed in their *Principia Mathematica* that all classical math, known in their time, can be derived from set theory and hence from the axioms of the Principia (or other formal set theories like that developed by Zermelo and Fraenkel)
- (Using the ZF theory) Since the ZF theory has only 9 axioms, the logicist program is thus to show that <u>all 9 ZF axioms belong to logic</u>, i.e. can be reduced to logical propositions.
- Logical Proposition: one that has **complete generality** and is **true in virtue of its form** rather than its content.
- E.g. Law of the excluded middle: if p is a proposition, then either p or its negation ~p is true
- This law does not hold because of any special content of the proposition *p*; **it doesn't matter** whether *p* is a proposition of math or physics etc
- Rather, this law holds with **"complete generality"**, i.e. for any proposition *p* whatsoever.
- Why then does this hold? Logicists: because of its *form*!

Counterargument

- Did the logicists succeed in reducing all 9 ZF axioms to logical propositions?
- No; at least 2 axioms (infinity and choice) cannot be considered as logical propositions
- E.g. the axiom of infinity states that there exist infinite sets.
- Why do we accept it? Because we are familiar with quite a few infinite sets, say, the set of natural no. or the set of points in Euclidean 3-space.
- But then this shows that we accept this axiom by virtue of its **content** and not by virtue of its form!
- In general, when an axiom claims the existence of objects with which we are familiar <u>on grounds of our common everyday experience</u>, it is pretty certain that this axiom is not a logical proposition in the sense of Logicism.

Problem

- 1st crisis in math: at least 2 out of the 9 ZF axioms are not logical propositions in the sense of Logicism.
- Hence, this school failed by about 20% in its effort to give math a firm foundation!
- Without this firm foundation, math is then **open to sceptical attacks** like the infinite regress.

Intuitionism

- Anti-Realist, A Priori
- Main proponent: Brouwer
- Radically different from the logicists who thought that there was never anything wrong with classical math
- Intuitionists thought that there was plenty wrong with classical math
- E.g. Cantor's set theory had several paradoxes/contradictions – the logicists thought of them as common errors caused by erring mathematicians and not by a faulty mathematics; the intuitionists thought otherwise.
- Hence, they thought that math had to be *rebuilt* from the bottom on up.

Intuitionism

- This bottom, this beginning of math, is their explanation of what the natural no. 1,2,3… are (doesn't include 0).
- For the intuitionists, all human beings have an **innate primordial intuition** for the natural numbers (not unlike Kant) (c.f. Article C, 243-4)
  - i.e. that we have an **immediate certainty** as to what is meant by the number 1
  - and that the mental process which goes into the formation of the number 1 can be **repeated** to get 2, and on and on and on…
- This process is both **inductive** and **effective**
  - Inductive: if one wants to construct 3, then one must go through constructing 1 and then 2.
  - Effective: once the construction of a natural no. is finished, that natural no. is entirely constructed and a complete finished mental construct, ready for our study of it.

Intuitionism

- Math for the Intuitionists is a **mental activity** (of inductive and effective constructions) and not a set of theorems (which is what Logicism held)
- i.e. Math is the mental activity which consists in **carrying out constructs** one after the other and is thus **a priori** (and therefore necessary).
- Upshot? All intuitionistic math, like proofs, theorems and definitions, is **constructed**.
- Furthermore, any mathematical theorem, proof etc that is not composed of constructs are seen as **meaningless combinations of words**.
  - This also includes some logical rules such as the law of excluded middle
- In this way, the intuitionists have come up with their own intuitionistic arithmetic, algebra, analysis, set theory etc

Arguments for Intuitionism

- Seems to provide "the best account of the relations between arithmetic and the human brain" among other theories on the nature of math (244, Article C)
- Modern psychology seems to support the idea of a primordial mathematical intuition – that we have certain "innate categories according to which we apprehend the world" like colour, space, time AND NUMBERS (recall Kant's mathematical filter)
  - The human baby is born with innate mechanisms for individuating objects and for extracting the numerosity of small sets
  - In children, numerical estimation, comparison, counting, simple addition and subtraction all emerge spontaneously without much explicit instruction

Arguments for Intuitionism

- Accounts for ‘Universality’ of Math – all human beings share this same faculty/intuition/filter of consciousness and thus we construct the same math.
  - Use of place-value systems by the Chinese, Mayans, Indians and Babylonians, even though 3,000 years stand between them, and the use of place-value system was lost with the collapse of Babylonian civilisation
  - Independent development of Calculus by Leibniz and Newton
  - Creation of abstract symbols to represent any physical object, i.e. numbers
  - Use of Round numbers across all languages (and thus cultures) probably because all human beings hare the same mental apparatus and are therefore confronted with the difficulty of conceptualising large quantities; the larger a number, the less accurate our mental representation of it.
  - Special status to the 1st 3 numbers in our language: 1st, 2nd, 3rd (all different) or primary, secondary and tertiary but then 4th, 6th, 10th, 12th, etc. Why? Probably because those are the most used numbers in our history (don’t forget the development of arithmetic to keep tally of things)

Arguments for Intuitionism

- Accounts for how we obtain Math knowledge – we don't obtain it from some abstract realm of non-causal, non-spatiotemporal entities which sounds too mystical and superstitious; **rather, we construct it**.
- Accounts for Certainty of Math – because it is still a mental activity, it is a priori
- Accounts for the "unreasonable effectiveness" of math in the natural world – because the phenomenal world or the world we study has already been constructed in mathematical terms!
  - So even though complex numbers might seem like a purely mathematical invention, a 'game', it actually isn't as it still came out of our understanding of the phenomenal world.

Counterargument I

- Problem! There are classical theorems which are not composed of constructs and are hence meaningless combinations of words according to Intuitionism.
- Hence, one cannot say that intuitionists have reconstructed all of classical math!
- Rejoinder: bite the bullet! Not a problem since it is meaningless anyway!
- Recall: Intuitionists never set out to justify classical math.
- Their purpose is to give a *valid definition* of math and then see what math comes out of it.
- So if much of classical math is to be jettisoned, then so be it! (not so for the logicists)
- In fact, intuitionist philosophy is able to show why intuitionistic math is **free of contradictions** because constructs can never give rise to contradictions.
- Indeed, all other problems of a foundational nature seem to receive **perfectly satisfactory solutions** in intuitionism.

Counterargument II

- But the mathematical community themselves do not want to do away with the many beautiful theorems in classical math which the intuitionists view as meaningless combinations of words.
- E.g. the Brouwer (the same proponent of intuitionism) fixed point theorem of topology which the intuitionists reject because the fixed point cannot be constructed but can only be shown to exist on grounds of an existence proof.
- **If your own members don't seem to accept it, then it's hard for others to accept it.**
- A good argument?

Counterargument III

- There are many theorems which can be proven both classically and intuitionistically.
- But doing so intuitionistically usually turns out to be about **10 times as long** as the classical proof and often seems, to the classical mathematicians, to have <u>lost all of its elegance</u>.
- E.g. the fundamental theorem of algebra is proved in about half a page classically but takes about 10 pages for the intuitionistic method.
- Here, the classical mathematicians refuse to believe that their clever proofs are meaningless while their bulky intuitionistic cousins aren’t.
- But why trust elegance and cleverness to be criteria of truth?
- Yet this is one way we do choose scientific theories – Occam’s Razor!

Problem

- 2^{nd} crisis in math – the **failure** of intuitionism to make it **acceptable** to at least the majority of mathematicians.
- The reasons for rejecting intuitionism by classical mathematicians aren't rational nor scientific, and not pragmatic
- In fact, they are all **emotional** reasons, grounded in a deep sense as to what mathematics is all about.
- Ironic? Math and Emotion?
- Is this a good reason for not accepting Intuitionism?

Counterargument IV

- Universality of Math isn't technically accounted for, as least not to the extent of Platonism and Logicism
- There, Math is *truly* universal, i.e. it holds true in all possible worlds
- Under Intuitionism however, Math is merely a **human** construct such that while this faculty/intuition/filter of consciousness is universal <u>to mankind</u>, there is no guarantee that this is true for other sentient beings (say if aliens exist)
- In other words, it is entirely possible that these other sentient beings construct a different kind of math due to a different filter of consciousness
- Furthermore, all knowledge, including math, is knowledge only of the Phenomenal World
- i.e. the Noumenal world is not mathematical.
- Hence, unlike Platonism and Logicism, Math **no longer** holds that special status of allowing us a special insight into the structure of reality

Intuitionism and Philosophy

- Just as Logicism is related to Realism, Intuitionism is related to Conceptualism
- Conceptualism is *Anti-Realist*, i.e. abstract entities exist only insofar as they are constructed by the human mind.
- This is very much the attitude of Intuitionism – abstract entities in math are all mental *constructions*
- This is why one does not find in Intuitionism the staggering collection of abstract entities which occur in classical math and hence in Logicism (and Platonism)
- A plus point?

Formalism

- NON-realist - "…the issue of existence of mathematical objects is meaningless and void" (243, Article C)
  - Note: anti-realism is a subset of non-realism
- Main proponent: David Hilbert
- Math is "only a game in which one manipulates symbols according to precise formal rules"
- Seems quite true! - "a large part of mathematics is a purely formal game. Indeed, numerous questions in pure mathematics have arisen from what, at first sight, may seem to be fanciful ideas."
- E.g. Complex numbers! Riemannian Geometry! – Essentially, the "what if…" move.
- Math is also **a priori** here (actually, all the main positions holds that Math is a priori)
- This also accounts for the **necessity** of Math

Formalism

- Strategy to provide math with a solid foundation and show that math was free of contradictions? Formalise it!
- To formalise math is essentially to **reduce** math to <u>first order language</u>
- In doing so, math is then given a **precise syntax**, so precise that it itself can then be studied as a mathematical object.
- This then allows one to ask <u>whether one can run into contradictions</u> if one proceeds entirely formally within the first order language *L*, using only the axioms of an axiomatised theory *T* and those of classical logic (all of which have been expressed in *L*).
- If the answer is no, then there is now a mathematical proof that theory *T* is free of contradictions!
- This is basically what the famous "Hilbert program" was all about.

Formalism

- Note that this formalisation is similar to what is going on in Logicism.
- However, the aim is different.
- The logicists wanted to use such a formalisation to show that the branch of math in question belongs to logic (which the philosophers can then proceed to show how logic, and thus math, is free of contradictions).
- The formalists wanted to use it to prove **mathematically** that that branch is free of contradictions.

Arguments For Formalism

- Explains differences in mathematical systems across time and space (as seen earlier)
  - E.g. Different symbols for numbers in different cultures depending on the medium with which the people were working – Vs and Xs for the Romans as it was harder to cut across the length of a piece of wood; Sumerians used cylindrical notches instead because they worked on soft clay with a pencil; Indians used 10 arbitrary digits whose shapes are unrelated to the numerical quantities they represent (unlike the others like the Sumerians, Chinese and Mayans, who used strokes to represent physical quantities)
  - The invention (or discovery?) of Zero by the Indians which only seems intuitive to us now but not so to the ancients who did not have it, like the Babylonians
  - Different bases – like base 2, base 5, etc.
- Why? The differences exist because different cultures had different needs and/or played the game differently

Arguments For Formalism

- Explains why we can continually have interesting mathematics!
- "…the mathematician could formulate only a handful of interesting theorems without defining concepts beyond those contained in the axioms and that the concepts outside those contained in the axioms are defined with a view of permitting ingenious logical operations which appeal to our aesthetic sense both as operations and also in their results of great generality and simplicity." (Unreasonable Effectiveness, 2)
- E.g. Complex Numbers – nothing in our experience suggests the introduction of such a concept
- But mathematician can justify his interest in complex numbers by pointing to the many beautiful theorems in the theory of equations, of power series, and of analytic functions in general.

Counterarguments

- But why would such a concept, if only invented as part of a game, prove to be so useful in quantum physics later on? Formalism has no answer
- Formalism might explain the recent evolution of pure math but does not provide an adequate explanation for the *origins* of Math, i.e. how Math was developed historically to be useful in our daily lives.
  - E.g. Arithmetic, Geometry! They didn't begin as formal games but as **tools** to deal with life
  - It was only *later on* that they became formalized into what we know today.
- Formalism, being a priori, preserves the certainty of Math, but it is unable to account for the universality of Math
  - If Math is merely a construct meant to show our ingenuity, it is weird to think that we would come up with similar concepts and systems. And yet we do!

Counterarguments

- Recall: Formalism aimed to prove that math is free of contradictions by formalising math into a first order language *L*.
- However, if no sentence of *L* which can be interpreted as asserting that theorem *T* is free of contradictions can be proven formally within *L*, then <u>math is not able to prove its own freedom of contradictions</u>, i.e. the Hilbert program cannot be carried out.
- This is known as Gödel’s first incompleteness theorem (cf. TOK 208)
- Gödel essentially proved that any effectively generated theory capable of expressing elementary arithmetic **cannot** be both **consistent and complete**.
- In particular, for any consistent, effectively generated formal theory that proves certain basic arithmetic truths, **there is an arithmetical statement that is true, but not proven in the theory**.

Watch this NOW

- https://www.youtube.com/watch?v=O4ndIDcDSGc
- Numberphile - Gödel's Incompleteness Theorem

Gödel's 1st incompleteness theorem

- Roughly speaking, for each theory *T*, the corresponding Gödel sentence (the implied true but unprovable statement) *G* asserts: "*G* **cannot be proved** within the theory *T*".
- 1st step: Assume that the statement is false, i.e. that G *can* be provable within T.
- But **if *G* were provable** under the axioms and rules of inference of *T*, **then it would be true**, i.e. *T* would have a theorem, *G*.
- This then means that G is true, which **effectively contradicts** itself, and thus the theory *T* would be **inconsistent**.
- (Contradiction: we began with the assumption that the statement is false but then showed it to be true.)
- This means that **if the theory *T* is consistent, G has to be true**, and thus *G* cannot be proved within it.
- This means that *G*'s claim about its own unprovability is **correct**; in this sense *G* is **not only unprovable but true**.
- Thus provability-within-the-theory-*T* is not the same as truth; the theory *T* is **incomplete** (because if it is to be consistent, the G sentence cannot be proven).

Gödel's 1st incompleteness theorem

- Solution: It is possible to define a larger theory *T'* that contains the whole of *T*, **plus *G* as an additional axiom** so that G is *outside* T but within T' and no inconsistency takes place.
- In this case, *G* is indeed a theorem in *T'* (trivially so, since it is an axiom).
- This thus solves the problem!
- Or does it?
- The incompleteness theorem can now be applied to *T'*: there will be a new Gödel statement *G'* for *T'*, showing that *T'* is also incomplete.
- **Conclusion: each theory has its own Gödel statement.**

Implications?

- Gödel's 1st incompleteness theorem states that any formal system that includes enough of the theory of the natural numbers is **incomplete**: there are statements in its language that it can neither prove nor refute.
- Thus, **no formal system** that aims to characterize the natural numbers **can actually do so** as there will be true number-theoretical statements which that system cannot prove.
- This fact has severe consequences for the program of Logicism proposed by Frege and Russell, which aimed to define the natural numbers in terms of logic.
- Many logicians believe that Gödel's incompleteness theorems struck a fatal blow to Hilbert's program to produce a **finitary consistency proof for mathematics**.
- These mathematicians argue that the incompleteness theorems show that Hilbert's second problem (that the theory of natural no. be free of contradictions) **cannot be completed**. And this is the **3rd crisis**!
- Note: The second incompleteness theorem (that a system cannot demonstrate its own consistency) is often singled out as the final blow to Hilbert's program. Not all mathematicians agree with this analysis, however.

Formalism and Philosophy

- Related to Nominalism, the philosophy which claims that abstract entities have **no existence of any kind**, neither outside the human mind (realism) nor as mental constructions within the human mind (conceptualism/anti-realism).
- For nominalism, abstract entities are mere vocal utterances or written lines, <u>mere names</u>.
- So when formalists try to prove that a certain axiomatised theory *T* is free of contradictions, they do not study the abstract entities which occur in *T* but instead study that first order language *L* which was used to formalise *T*.
- Hence, this whole study of *L* is a **strictly syntactical** study since no meanings or abstract entities are associated with the sentences of *L*.
- In fact, for the strict formalist, "to do math" is "to manipulate the meaningless symbols of a first order language according to explicit, syntatical roles". It's like a **Game**.

The 3 crises – consequences

- After these 3 crises, what do we have?
- The **lack of a firm foundation** for math.
- Indeed, after 1931, with Gödel's incompleteness theorem, mathematicians on the whole threw up their hands in frustration and turned away from the philosophy of mathematics and the hope of finding a certain foundation for mathematics that is free of contradictions and complete.
- Any other alternative? Or are we 'doomed' to accept that Math, our bastion of certainty in this uncertain world, is but a fraud? Should we, perhaps, turn to coherentism and/or reliabilism to justify mathematics?

Homework

- Article J (the 3 crises)
- Article D (TOK 202-210)
- Article E (Non-European roots of Math)

- Optional – Article G