RAFFLES INSTITUTION
H2 Mathematics 9758
2023 Year 6 Term 3 Revision 12 (Summary and Tutorial)

Topic: Differential Equations

Summary for Differential Equations

Differential equations of the form $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x)$

- solved by "direct" integration, i.e. integrate $\mathrm{f}(x)$ with respect to x.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x) \Rightarrow y = \int \mathrm{f}(x)\ \mathrm{d}x$$

Example [9740/2008/01/Q4(i)(ii))]

(i) Find the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x}{x^2+1}.$$

(ii) Find the particular solution of the differential equation for which $y = 2$ when $x = 0$.

Solution:

(i) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{3x}{x^2+1}$

$y = \displaystyle\int \frac{3x}{x^2+1}\ \mathrm{d}x$ [integrate the $\mathrm{f}(x) = \dfrac{3x}{x^2+1}$ with respect to x]

$= \dfrac{3}{2}\displaystyle\int \frac{2x}{x^2+1}\ \mathrm{d}x$

$= \dfrac{3}{2}\ln(x^2+1) + c$

(ii) When $x = 0, y = 2$

$2 = \dfrac{3}{2}\ln(0^2+1) + c$

$c = 2$

Thus the particular solution is $y = \dfrac{3}{2}\ln(x^2+1) + 2$.

Differential equations of the form $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(y)$

- solved by separating the variables, i.e.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(y) \Rightarrow \int \frac{1}{\mathrm{f}(y)}\ \mathrm{d}y = \int 1\ \mathrm{d}x$$

Example [9740/2012/02/Q1(b)]
Given that u and t are related by

$$\frac{\mathrm{d}u}{\mathrm{d}t} = 16 - 9u^2$$

and that $u = 1$ when $t = 0$, find t in terms of u, simplifying your answer.

Solution:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = 16 - 9u^2$$

$$\int \frac{1}{16 - 9u^2}\ \mathrm{d}u = \int 1\ \mathrm{d}t$$

[separate the variables to obtain $\int \frac{1}{\mathrm{f}(u)}\ \mathrm{d}u = \int 1\ \mathrm{d}t$]

$$\frac{1}{3}\int \frac{3}{4^2 - (3u)^2}\ \mathrm{d}u = \int 1\ \mathrm{d}t$$

$$\frac{1}{3} \times \frac{1}{2(4)} \ln\left|\frac{4+3u}{4-3u}\right| = t + c$$

$$\frac{1}{24} \ln\left|\frac{4+3u}{4-3u}\right| = t + c$$

When $t = 0, u = 1$

$$c = \frac{1}{24}\ln 7$$

Thus $t = \dfrac{1}{24}\ln\left|\dfrac{4+3u}{4-3u}\right| - \dfrac{1}{24}\ln 7 = \dfrac{1}{24}\ln\left|\dfrac{4+3u}{7(4-3u)}\right|$

Differential equations of the form $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \mathrm{f}(x)$

- solved by integrating the right hand side twice with respect to x. Note that

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \mathrm{f}(x) \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \int \mathrm{f}(x)\ \mathrm{d}x$$

Now suppose that $\int \mathrm{f}(x)\ \mathrm{d}x = \mathrm{F}(x) + c$, then we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{F}(x) + c$$

which gives

$$y = \int \mathrm{F}(x) + c\ \mathrm{d}x$$

Example [9740/2012/02/Q1(a)]
Find the general solution of the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 16 - 9x^2$$

Giving your answer in the form $y = \mathrm{f}(x)$.

Solution:

$\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 16 - 9x^2$

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = \int 16 - 9x^2\ \mathrm{d}x$ [integrate with respect to x]

$\quad = 16x - 3x^3 + c$

$y = \int 16x - 3x^3 + c\ \mathrm{d}x$ [integrate with respect to x]

$\quad = 8x^2 - \dfrac{3}{4}x^4 + cx + d$ [note that there are two constants c and d]

Solving Differential Equations using a *given* substitution

- the substitution will be given in the question
- this substitution allows us to reduce the differential equation to familiar forms which we discussed earlier, namely $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x)$ or $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(y)$.

Example [9233/N87/02/Q14(a)]
The variables x and y are connected by the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1+x+y}{1-x-y}.$$

Show that the substitution $u = x + y$ reduces the equation to

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{2}{1-u},$$

and solve this differential equation.

Solution:

$u = x + y \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1 + \frac{\mathrm{d}y}{\mathrm{d}x}$ [differentiating both sides with respect to x]

$$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x} - 1$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1+x+y}{1-x-y}$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} - 1 = \frac{1+u}{1-u}$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1+u}{1-u} + 1$$

$$= \frac{1+u+1-u}{1-u}$$

$$= \frac{2}{1-u} \quad \text{(shown)}$$

$\int (1-u)\ \mathrm{d}u = \int 2\ \mathrm{d}x$ [separate the variables]

$$u - \frac{1}{2}u^2 = 2x + c$$

Differential Equations in Real-World Context

- this includes formulating the differential equation in the given context, solving the equation and interpreting the solution of the differential equation.

In general, one can use Rate of change $=$ Rate increase $-$ Rate decrease to formulate the differential equation for a given real-world context problem.

Example [9740/2010/01/Q7]

A bottle containing liquid is taken from a refrigerator and placed in a room where the temperature is a constant $20\,°\text{C}$. As the liquid warms up, the rate of increase of its temperature θ °C after time *t* minutes is proportional to the temperature difference $(20-\theta)$ °C. Initially the temperature of the liquid is $10\,°\text{C}$ and the rate of increase of the temperature is $1\,°\text{C}$ per minute.

By setting up and solving a differential equation, show that $\theta = 20 - 10e^{-\frac{1}{10}t}$.
Find the time it takes the liquid to reach a temperature of $15\,°\text{C}$, and state what happens to θ for large values of *t*. Sketch a graph of θ against *t*.

interpreting

Solution:

formulating

$\frac{d\theta}{dt} = k(20-\theta)$ [rate of change $= \frac{d\theta}{dt}$, rate increase $= k(20-\theta)$ where *k* is the constant of proportionality]

When $\theta = 10, \frac{d\theta}{dt} = 1$

$1 = k(20-10) \Rightarrow k = \frac{1}{10}$

$\int \frac{1}{20-\theta}\, d\theta = \int \frac{1}{10}\, dt$ [solving DE of the form $\frac{dy}{dx} = f(y)$]

$-\ln|20-\theta| = \frac{1}{10}t + c$

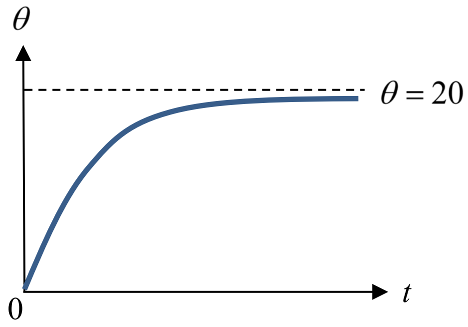
$20-\theta = \pm e^{-\frac{1}{10}t-c}$

$20-\theta = Ae^{-\frac{1}{10}t}$ where $A = \pm e^{-c}$

$\theta = 20 - Ae^{-\frac{1}{10}t}$

When $t = 0, \theta = 10$
$10 = 20 - A \Rightarrow A = 10$

$\therefore \theta = 20 - 10e^{-\frac{1}{10}t}$



When $\theta = 15$,

$15 = 20 - 10e^{-\frac{1}{10}t}$

$e^{-\frac{1}{10}t} = \frac{1}{2}$

$t = -10\ln\frac{1}{2} = 6.93$ mins

As $t \to \infty$, $e^{-\frac{1}{10}t} \to 0$ and so $\theta \to 20$.

Example [9233/N74/02/Q20]
A race called the Matrices live on an isolated island called Vector. The number of births per unit time is proportional to the population at any time and the number of deaths per unit time is proportional to the square of the population. If the population at time t is p, show that $\frac{\mathrm{d}p}{\mathrm{d}t} = ap - bp^2$, where a and b are positive constants. Solve the equation for p in terms of t, given that $p = \frac{2a}{3b}$ when $t = 0$. Show that there is a limit to the size of the population.

Solution:

Rate increase $\propto p$
$\therefore$ Rate increase $= ap$ where a is a positive constant

Rate decrease $\propto p^2$
$\therefore$ Rate decrease $= bp^2$ where b is a positive constant

Thus $\frac{\mathrm{d}p}{\mathrm{d}t} = ap - bp^2$ [formulating]

$\int \frac{1}{ap - bp^2}\,\mathrm{d}p = \int 1\,\mathrm{d}t$ [solving DE of the form $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(y)$]

…

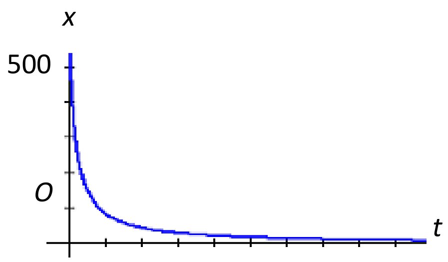
Refer to Q12 of your C9 Differential Equations tutorial for the remaining parts of the solution.

Revision Tutorial Questions

Source of Question: ACJC JC2 CT1 9758/2018/Q2

1 In a factory process, a chemical C is used up at a rate proportional to the square of the amount of chemical present, x grams, at time t seconds. At the start of the process, 500 grams of C is present. By setting up and solving a differential equation, express the solution of the differential equation in the form $x = \mathrm{f}(t)$, and sketch the part of the curve with this equation which is relevant in this context. [6]

$$[\, x = \frac{500}{1-500kt} \,]$$

1 **[6]**	$\frac{\mathrm{d}x}{\mathrm{d}t} = kx^2$ where k is a negative constant $\Rightarrow \int \frac{1}{x^2}\,\mathrm{d}x = \int k\,\mathrm{d}t$ $\Rightarrow -\frac{1}{x} = kt + c$ When $t = 0$, $x = 500$, $-\frac{1}{500} = c$ $\Rightarrow x = \frac{-1}{kt - \frac{1}{500}} = \frac{-500}{500kt - 1}$ or $x = \frac{500}{1-500kt}$ Any graph with negative value for k. 

Source of Question: DHS JC2 Prelim 9758/2018/01/Q6

2 **(a)** By using the substitution $y = zx^2$, find the general solution of the differential equation

$$x^2 \frac{\mathrm{d}y}{\mathrm{d}x} = 2xy - y^2, \quad \text{where } x \neq 0.$$

[4]

(i) Sketch the solution curve that passes through $(2, -4)$, indicating any stationary points and asymptotes clearly. [4]

(ii) State the particular solution for which y has no turning point. [1]

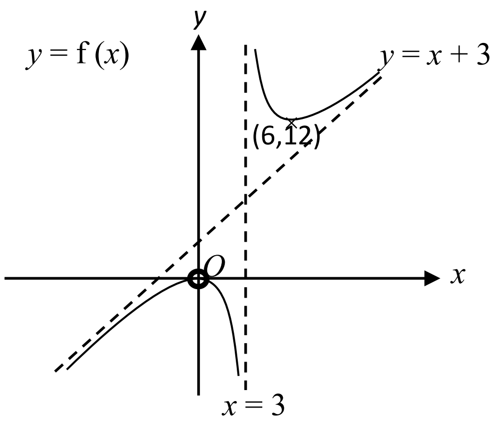
(b) A differential equation is of the form $\frac{\mathrm{d}y}{\mathrm{d}x} + y = px + q$, where p and q are constants.

Its general solution is $y = 4x - 1 + D\mathrm{e}^{-x}$, where D is an arbitrary constant.

Find the values of p and q. [2]

[(a) $y = \frac{x^2}{x - C}$ (a)(ii) $y = x$ (b) $p = 4$, $q = 3$]

2(a) **[4]**	$x^2 \frac{\mathrm{d}y}{\mathrm{d}x} = 2xy - y^2$ ...(1) Differentiating both sides of $y = zx^2$ with respect to x: $\frac{\mathrm{d}y}{\mathrm{d}x} = 2xz + x^2 \frac{\mathrm{d}z}{\mathrm{d}x}$ ...(2) Substitute (2) into (1): $x^2\left(2xz + x^2 \frac{\mathrm{d}z}{\mathrm{d}x}\right) = 2x\left(zx^2\right) - \left(zx^2\right)^2$ $x^4 \frac{\mathrm{d}z}{\mathrm{d}x} = -z^2 x^4$ $\frac{\mathrm{d}z}{\mathrm{d}x} = -z^2$ $\int \frac{1}{z^2}\, \mathrm{d}z = -\int 1 \,\mathrm{d}x$ $-\frac{1}{z} = -x + C$ $-\frac{x^2}{y} = -x + C$ $y = \frac{x^2}{x - C}$

2(a)(i) **[4]**	Given $(2,-4)$, $-4=\dfrac{2^2}{2-C}\Rightarrow C=3$ Hence, $y=\dfrac{x^2}{x-3}=x+3+\dfrac{9}{x-3}$ 
2(a)(ii) **[1]**	When $C=0$, particular solution is $y=\dfrac{x^2}{x-0}=x$ which is a straight line and has no turning point.
2(b) **[2]**	Given $y=4x-1+De^{-x}\Rightarrow \dfrac{dy}{dx}=4-De^{-x}$ $\dfrac{dy}{dx}+y=\left(4-De^{-x}\right)+\left(4x-1+De^{-x}\right)=4x+3$ $\therefore p=4,\ q=3$

Source of Question: NJC JC2 Midyear 9758/2018/Q2

3 **(i)** Show that the substitution $r^2=x^2+y^2$ reduces the differential equation

$$y\frac{dy}{dx}=4-x-x^2-y^2$$

to

$$r\frac{dr}{dx}=4-r^2.$$

Hence solve the differential equation, given that $y=2\sqrt{2}$ when $x=0$. [7]

The solution to part **(i)** is the equation of a curve C.

(ii) Explain why

(a) every point on C is more than 2 units away from the origin O. [1]

(b) C is symmetrical about the x-axis. [1]

[$x^2+y^2=4\left(1+e^{-2x}\right)$]

3(i) **[7]**	$r^2 = x^2 + y^2$ Differentiating both sides with respect to x: $\frac{\mathrm{d}}{\mathrm{d}x}\left(r^2\right) = 2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x}$ $\Rightarrow 2r\frac{\mathrm{d}r}{\mathrm{d}x} = 2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x}$ $\Rightarrow r\frac{\mathrm{d}r}{\mathrm{d}x} = x + y\frac{\mathrm{d}y}{\mathrm{d}x}$ $y\frac{\mathrm{d}y}{\mathrm{d}x} = 4 - x - x^2 - y^2$ $x + y\frac{\mathrm{d}y}{\mathrm{d}x} = 4 - \left(x^2 + y^2\right)$ $r\frac{\mathrm{d}r}{\mathrm{d}x} = 4 - r^2$ (shown) Alternatively, $r = \sqrt{x^2 + y^2} \quad \Rightarrow \quad \frac{\mathrm{d}r}{\mathrm{d}x} = \frac{1}{2\sqrt{x^2 + y^2}} \times \left(2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x}\right)$ $\Rightarrow \quad \frac{\mathrm{d}r}{\mathrm{d}x} = \frac{x + y\frac{\mathrm{d}y}{\mathrm{d}x}}{\sqrt{x^2 + y^2}}$ $y\frac{\mathrm{d}y}{\mathrm{d}x} = 4 - x - x^2 - y^2$ $x + y\frac{\mathrm{d}y}{\mathrm{d}x} = 4 - \left(x^2 + y^2\right)$ $\frac{x + y\frac{\mathrm{d}y}{\mathrm{d}x}}{\sqrt{x^2 + y^2}} = \frac{4 - \left(x^2 + y^2\right)}{\sqrt{x^2 + y^2}} = \frac{4}{\sqrt{x^2 + y^2}} - \sqrt{x^2 + y^2}$ $\frac{\mathrm{d}r}{\mathrm{d}x} = \frac{4}{r} - r$ $r\frac{\mathrm{d}r}{\mathrm{d}x} = 4 - r^2$ (shown)

	$r\frac{\mathrm{d}r}{\mathrm{d}x}=4-r^2$ $\frac{r}{4-r^2}\frac{\mathrm{d}r}{\mathrm{d}x}=1$ $-\frac{1}{2}\int\frac{-2r}{4-r^2}\,\mathrm{d}r=\int\mathrm{d}x$ $-\frac{1}{2}\left[\ln\left\|4-r^2\right\|\right]=x+c$ $\ln\left\|4-r^2\right\|=-2x+c',$ where $c'=-2c$ $\left\|4-r^2\right\|=\mathrm{e}^{-2x+c'}$ $4-r^2=\pm\mathrm{e}^{-2x+c'}$ $r^2=4-A\mathrm{e}^{-2x}$, where $A=\pm e^{c'}$ is an arbitrary constant, $A\neq 0$ $x^2+y^2=4-A\mathrm{e}^{-2x}$ When $x=0,\ y=2\sqrt{2}$: $0^2+\left(2\sqrt{2}\right)^2=4-A\mathrm{e}^{-2(0)}$ $8=4-A$ $A=-4$ So a cartesian equation of C is $x^2+y^2=4+4\mathrm{e}^{-2x}$ $x^2+y^2=4\left(1+\mathrm{e}^{-2x}\right)$ or $\ln\left\|x^2+y^2-4\right\|=-2x+\ln 4$
(ii)(a) **[1]**	$x^2+y^2=4\left(1+\mathrm{e}^{-2x}\right)$ $\sqrt{x^2+y^2}=2\sqrt{1+\mathrm{e}^{-2x}}>2$ since $\mathrm{e}^{-2x}>0$ for all $x\in\mathbb{R}$
(ii)(b) **[1]**	$x^2+y^2=4\left(1+\mathrm{e}^{-2x}\right)$ $y^2=4\left(1+\mathrm{e}^{-2x}\right)-x^2$ $(-y)^2=4\left(1+\mathrm{e}^{-2x}\right)-x^2$ If (x, y) lies on C, then $(x, -y)$ also lies on C. So S is symmetrical about the real axis.

Source of Question: PJC JC2 Prelim 9758/2018/01/Q11

4 Environmental conditions such as acidity, temperature, oxygen levels and toxins influence the rate of growth of microorganisms. A biologist investigates the change of population of a particular type of microorganism of size n thousand at time t days under different conditions. In both models I and II, the initial population of the microorganism is 3000 and the population reaches 2000 after 1 day.

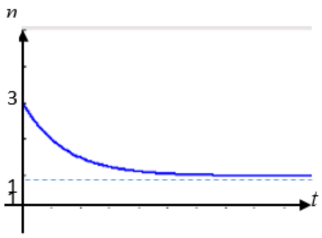
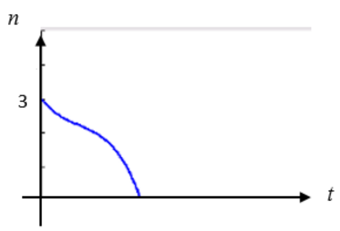
(i) Under model I, the biologist observes that the rate of growth of microorganism is a constant whereas the death rate is proportional to its population. He also observes that when the population of the microorganism is 1000, it remains at this constant value. By setting up and solving a differential equation, show that $n = 1 + 2^{1-t}$. [8]

(ii) Under model II, the biologist observes that n and t are related by the differential equation $\dfrac{d^2n}{dt^2} = 4 - 6t$. Find the particular solution of this differential equation. [3]

(iii) By sketching the graphs of n against t for both model I and II, state and explain which of the two models is more harmful for the growth of this type of microorganism. [2]

[(ii) $n = 2t^2 - t^3 - 2t + 3$]

4(i) **[8]**	Rate of change of population: $\dfrac{dn}{dt} = a - bn$ Since $\dfrac{dn}{dt} = 0$ when $n = 1$, $0 = a - b$. Hence, $a = b$ $\dfrac{dn}{dt} = a - an = a(1-n)$ $\displaystyle\int \frac{1}{1-n}\,dn = \int a\ dt$ $-\ln\lvert 1-n\rvert = at + C$ $\ln\lvert 1-n\rvert = -at - C$ $\lvert 1-n\rvert = e^{-at-C}$ $1 - n = \pm e^{-C}e^{-at}$ $n = 1 + Ae^{-at}$, where $A = \mp e^{c}$ is an arbitrary constant, $A \neq 0$ Sub $t = 0, n = 3$: $3 = 1 + Ae^{0} \Rightarrow A = 2$ Sub $t = 1, n = 2$: $2 = 1 + 2e^{-a} \Rightarrow \dfrac{1}{2} = e^{-a} \Rightarrow -a = \ln\dfrac{1}{2}$ Hence, $n = 1 + 2e^{\ln\frac{1}{2}t} = 1 + 2\left(e^{\ln\frac{1}{2}}\right)^{t} = 1 + 2\left(\dfrac{1}{2}\right)^{t} = 1 + 2^{1-t}$
(ii) **[3]**	$\dfrac{d^2n}{dt^2} = 4 - 6t$ $\dfrac{dn}{dt} = 4t - 3t^2 + C$ $n = 2t^2 - t^3 + Ct + D$

	Sub $t=0, n=3$: $D=3$ Sub $t=1, n=2$: $C=-2$ Hence, $n=2t^2-t^3-2t+3$
(iii) **[2]**	Condition I  Condition II  Based on the graphs, the population decreases but approaches a constant of 1000 for Condition I. Whereas for Condition II, the microorganism would die off quickly. Hence Conditions II is more harmful.

Source of Question: VJC JC2 Prelim 9758/2018/01/Q12

5 The logistic equation, sometimes called the *Verhulst model*, is a model of population growth. Letting N be the population size at any time, t (in years), this model is formalized by the differential equation:

$$\frac{\mathrm{d}N}{\mathrm{d}t}=rN\left(1-\frac{N}{K}\right),$$

where r and K are real positive constants.

(i) By considering the value of $\frac{\mathrm{d}N}{\mathrm{d}t}$ when N approaches K, explain, in the context of the question, the significance of K. [2]

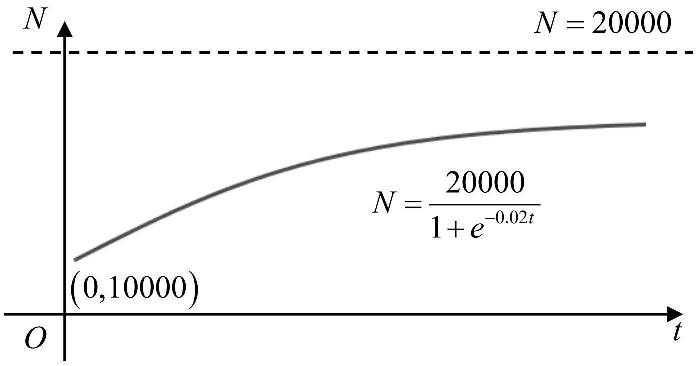
(ii) Solve the differential equation, and show that $N=\frac{K}{1+Be^{-rt}}$, where B is a constant. [5]

(iii) It is now given that the initial population in a small town is 10,000. The rate of population growth at that time is 100 per year. When the population is 15,000, the population growth rate is 75 per year. Use the *Verhulst model* to find N in terms of t. Hence, sketch the graph of N against t. [5]

[(iii) $N=\frac{20000}{1+e^{-0.02t}}$]

5(i) **[2]**	When $N\to K$, $\frac{\mathrm{d}N}{\mathrm{d}t}\to 0$. K is the population size where there is no change in the population. OR K is the population size that the population will eventually approach.

(ii) **[5]**	$\frac{\mathrm{d}N}{\mathrm{d}t}=rN\left(1-\frac{N}{K}\right)=rN\left(\frac{K-N}{K}\right)$ $\frac{K}{N(K-N)}\frac{\mathrm{d}N}{\mathrm{d}t}=r$ $\int\frac{1}{N}+\frac{1}{K-N}\,\mathrm{d}N=\int r\,\mathrm{d}t$ $\ln N-\ln\lvert K-N\rvert=rt+c$ $\frac{N}{K-N}=A\mathrm{e}^{rt}$, where $A=\pm e^{c}$ is an arbitrary constant, $A\neq 0$ $N=KA\mathrm{e}^{rt}-NA\mathrm{e}^{rt}$ $N+NA\mathrm{e}^{rt}=KA\mathrm{e}^{rt}$ $N=\frac{KA\mathrm{e}^{rt}}{1+A\mathrm{e}^{rt}}$ $=\frac{K}{\frac{1}{A}\mathrm{e}^{-rt}+1}$ Let $B=\frac{1}{A}$ $N=\frac{K}{1+B\mathrm{e}^{-rt}}$ Alternatively $\frac{1}{N(K-N)}\frac{\mathrm{d}N}{\mathrm{d}t}=\frac{r}{K}$ $\int\frac{1}{\left(K/2\right)^{2}-\left(N-K/2\right)^{2}}\,\mathrm{d}N=\int\frac{r}{K}\,\mathrm{d}t$ $\frac{1}{2\left(\frac{K}{2}\right)}\ln\left\lvert\frac{\frac{K}{2}+\left(N-\frac{K}{2}\right)}{\frac{K}{2}-\left(N-\frac{K}{2}\right)}\right\rvert=\frac{r}{K}t+C$ $\frac{1}{K}\ln\frac{N}{\lvert K-N\rvert}=\frac{r}{K}t+C$ $\ln\frac{N}{\lvert K-N\rvert}=rt+CK$ $\frac{N}{\lvert K-N\rvert}=D\mathrm{e}^{rt}$

(iii) **[5]**	When $N = 10000$, $\frac{\mathrm{d}N}{\mathrm{d}t} = 10000r\left(1 - \frac{10000}{K}\right) = 100 \Rightarrow 100r - \frac{1000000r}{K} = 1$ When $N = 15000$, $\frac{\mathrm{d}N}{\mathrm{d}t} = 15000r\left(1 - \frac{15000}{K}\right) = 75 \Rightarrow 200r - \frac{3000000r}{K} = 1$ By GC, $r = \frac{1}{50}, \frac{r}{K} = 10^{-6} \Rightarrow K = 20000$ When $t = 0$, $N = \frac{20000}{1+B} = 10000 \Rightarrow B = 1$ $N = \frac{20000}{1+e^{-0.02t}}$ 

Source of Question: EJC JC2 Midyear 9758/2018/02/Q4

6 A tank has a capacity of 100 litres. Initially, the tank contains 10 litres of water thoroughly mixed with 300 grams of salt. Salt water with a concentration of 5 g/litre is poured into the tank at a constant rate of 2 litres per minute, while the mixture flows out at a constant rate of 1 litre per minute.

Let S denote the amount of dissolved salt in the tank (in grams) at time t minutes after salt water is poured into the tank.

(i) Show that $\frac{\mathrm{d}S}{\mathrm{d}t} = 10 - \frac{S}{t+10}$, stating your assumption(s) clearly. [2]

(ii) By substituting $Q = (t+10)S$, solve the differential equation in **(i)**, and find S in terms of t. [4]

(iii) Hence**,** find the concentration of salt in the tank at the point just before it overflows. [2]

Once the volume of salt solution in the tank reaches 100 litres, the pouring stops, and the tank is allowed to drain off. The salt solution drains from the tank at a rate proportional to the volume of solution in the tank. Let V denote the volume of solution in the tank at time T minutes after the tank starts to drain off.

(iv) If the tank takes 10 minutes to drain half of its contents, find V in terms of T. [4]

[(ii) $S = \frac{5t^2 + 100t + 3000}{t+10}$ (iii) 5.25 g/litre (iv) $V = 100\mathrm{e}^{-\frac{\ln 2}{10}T}$]

6(i) **[2]**	$\frac{\mathrm{d}S}{\mathrm{d}t}$ = Rate of salt inflow − Rate of salt outflow $=5\times 2-\frac{S}{\text{Volume of salt solution in the tank}}$ $=10-\frac{S}{t+10}$ Assumption: The contents of the tank is thoroughly mixed at all times.
(ii) **[4]**	$\frac{\mathrm{d}S}{\mathrm{d}t}=10-\frac{S}{t+10}$ ---- (*) **Method 1 (Do implicit differentiation and sub in (*))** Let $Q=\left(t+10\right)S$ Differentiate both sides with respect to t: $\frac{\mathrm{d}Q}{\mathrm{d}t}=\left(t+10\right)\frac{\mathrm{d}S}{\mathrm{d}t}+S\left(1\right)$ Sub into (*) to obtain equation in terms of Q and t: $\frac{\frac{dQ}{dt}-S}{t+10}=10-\frac{S}{t+10}$ $\frac{dQ}{dt}-S=10\left(t+10\right)-S$ $\frac{dQ}{dt}=10\left(t+10\right)$ **Method 2 (Sub in and do differentiation)** $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{Q}{(t+10)}\right)=10-\frac{Q}{(t+10)^2}$ $\frac{(t+10)\frac{\mathrm{d}Q}{\mathrm{d}t}-Q(1)}{(t+10)^2}=10-\frac{Q}{(t+10)^2}$ $(t+10)\frac{\mathrm{d}Q}{\mathrm{d}t}-Q=10(t+10)^2-Q$ $\frac{\mathrm{d}Q}{\mathrm{d}t}=10(t+10)$ $Q=\int 10t+100\,\mathrm{d}t$ $=5t^2+100t+c$, where c is a constant

	$(t+10)S = 5t^2 + 100t + c$ $S = \dfrac{5t^2 + 100t + c}{t+10}$ When $t = 0$, $S = 300$, so $c = 3000$, therefore $S = \dfrac{5t^2 + 100t + 3000}{t+10}$
(iii) **[2]**	Just before the tank overflows, $t = (100-10)/1 = 90$ mins Concentration of salt $= \left(\dfrac{5(90)^2 + 100(90) + 3000}{(90)+10}\right)/100$ $= 5.25$ g/litre
(iv) **[4]**	Let V denote the volume of the salt solution in the tank. $\dfrac{dV}{dT} = -kV$ for some positive constant k $\int \dfrac{1}{V}\,\mathrm{d}V = \int -k\,\mathrm{d}T$ $\ln V = -kT + c_2$, where c_2 is a constant $V = e^{-kT+c_2} = Ae^{-kT}$, where A is a constant When $T = 0$, $V = 100$, so $A = 100$ When $T = 10$, $V = 50$, so $50 = 100e^{-k(10)}$ $k = -\dfrac{\ln 0.5}{10}$ or $\dfrac{\ln 2}{10}$ Therefore, $V = 100\,\mathrm{e}^{\frac{\ln 0.5}{10}T}$ or $V = 100\,\mathrm{e}^{-\frac{\ln 2}{10}T}$.

Source of Question: IJC JC2 Prelim 9758/2018/01/Q10

7 A curve C_k has parametric equations

$$x = 1 + k\cos\theta, \qquad y = -2 + \frac{1}{2}k\sin\theta,$$

where k is a positive constant.

(i) Find the cartesian equation of C_k and show that its gradient function is $\dfrac{1-x}{4(y+2)}$. [4]

(ii) On the same diagram, sketch the graphs of C_1 and C_4.
Label the two graphs clearly. [3]

On a map, the curves C_1, C_2, C_3 and C_4 represent the contours of a mountain. A stream flows down the mountain. Its path on the map is always at right angles to the contour it is crossing.

(iii) Explain why the path of the stream is modelled by the differential equation

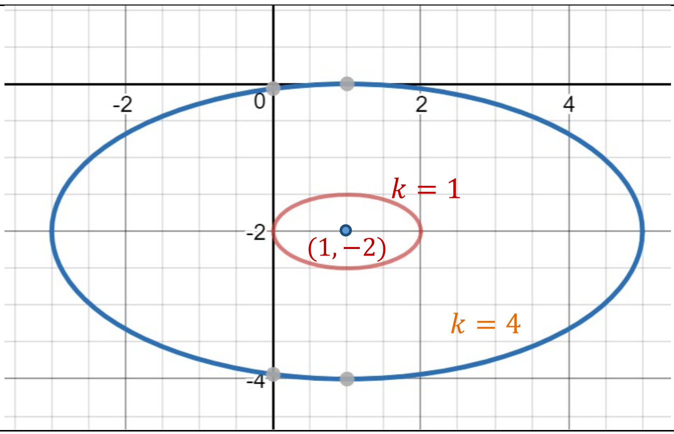
$$\left(\frac{1}{y+2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4}{x-1}.$$

By considering $\displaystyle\int \frac{1}{y+2}\,\mathrm{d}y = \int \frac{4}{x-1}\,\mathrm{d}x$, show that the path of the stream on the map is represented by the general solution $y = A(x-1)^4 - 2$, where A is an arbitrary constant. [5]

(iv) The path of the stream on the map passes through the point $(-1, -1)$. find the equation of the path. [1]

[(i) $(x-1)^2 + 4(y+2)^2 = k^2$ (iv) $y = \dfrac{1}{16}(x-1)^4 - 2$]

7(i) **[4]**	$x = 1 + k\cos\theta \quad \Rightarrow \quad \cos\theta = \dfrac{(x-1)}{k}$ $y = -2 + \dfrac{1}{2}k\sin\theta \quad \Rightarrow \quad \sin\theta = \dfrac{2(y+2)}{k}$ $\cos^2\theta + \sin^2\theta = 1 \quad \Rightarrow \dfrac{(x-1)^2}{k^2} + \dfrac{4(y+2)^2}{k^2} = 1$ Cartesian eqn of C_k is $(x-1)^2 + 4(y+2)^2 = k^2$ Applying implicit differentiation, $2(x-1) + 8(y+2)\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$ $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{(x-1)}{4(y+2)} = \dfrac{1-x}{4(y+2)}$

(ii) **[3]**	
(iii) **[5]**	The path of the stream on the map is perpendicular to the ellipses implies that (gradient of the path)×(grad of ellipse) $= -1$. Since gradient of ellipse $= -\dfrac{(x-1)}{4(y+2)}$, We have $\dfrac{dy}{dx}\left[-\dfrac{(x-1)}{4(y+2)}\right] = -1$ Thus $\dfrac{1}{y+2} \times \dfrac{dy}{dx} = \dfrac{4}{x-1}$. Using the given result, $\int \left(\dfrac{1}{y+2}\right) dy = \int \dfrac{4}{x-1}\, dx$ $\ln\lvert y+2\rvert = 4\ln\lvert x-1\rvert + C$ $\lvert y+2\rvert = e^{4\ln\lvert x-1\rvert + C}$ $= \left(e^{\ln\lvert x-1\rvert^4}\right)\left(e^C\right)$ $y+2 = \pm\left(e^C\right)(x-1)^4$ $y = A(x-1)^4 - 2$, where $A = \pm e^c$ is an arbitrary constant, $A \neq 0$
(iv) **[1]**	The path passes through the point $(-1,-1)$, $-1 = A(-1-1)^4 - 2$ $A = \dfrac{1}{16}$ Eqn of the path is $y = \dfrac{1}{16}(x-1)^4 - 2$

Source of Question: SRJC JC2 Prelim 9758/2018/01/Q11

8 A charged particle is placed in a varying magnetic field. A researcher decides to fit a mathematical model for the path of the fast-moving charged particle under the influence of the magnetic field. The particle was observed for the first 1.5 seconds. The displacement of the particle measured with respect to the origin in the horizontal and vertical directions, at time t seconds, is denoted by the variables x and y respectively. It is given that when $t=0,\ x=-\frac{1}{32}$, $y=0$ and $\frac{\mathrm{d}x}{\mathrm{d}t}=3$. The variables are related by the differential equations

$$(\cos t)\frac{\mathrm{d}y}{\mathrm{d}t}+y\sin t=4\cos^2 t-y^2 \text{ and } \frac{\mathrm{d}^2x}{\mathrm{d}t^2}=\cos 3t\cos t.$$

(i) Using the substitution $y=v\cos t$, show that $\frac{\mathrm{d}v}{\mathrm{d}t}=4-v^2$ and hence find y in terms of t. [7]

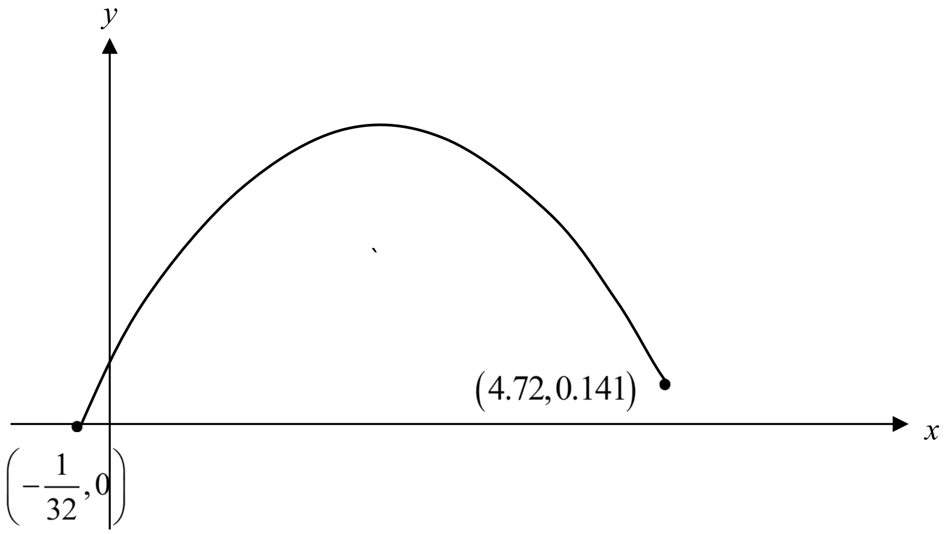
(ii) Show that $x=-\frac{1}{32}\cos 4t-\frac{1}{8}\cos 2t+3t+\frac{1}{8}$. [4]

(iii) Sketch the path travelled by the particle for the first 1.5 seconds, labelling the coordinates of the end points of the path. The evaluation of the y-intercept is not needed. [2]

[(i) $y=\dfrac{2\left(\mathrm{e}^{4t}-1\right)\cos t}{\mathrm{e}^{4t}+1}$]

8(i) **[7]**	$y=v\cos t$ Differentiate both sides with respect to t $\frac{\mathrm{d}y}{\mathrm{d}t}=-v\sin t+\frac{\mathrm{d}v}{\mathrm{d}t}\cos t$ $(\cos t)\frac{\mathrm{d}y}{\mathrm{d}t}+y\sin t=4\cos^2 t-y^2$ $(\cos t)\left(-v\sin t+\frac{\mathrm{d}v}{\mathrm{d}t}\cos t\right)+(v\cos t)\sin t=4\cos^2 t-v^2\cos^2 t$ $\frac{\mathrm{d}v}{\mathrm{d}t}\left(\cos^2 t\right)=\left(\cos^2 t\right)\left(4-v^2\right)$ $\frac{\mathrm{d}v}{\mathrm{d}t}=4-v^2$ (Shown) $\int\frac{1}{4-v^2}\,\mathrm{d}v=\int 1\,\mathrm{d}t$ $\int\frac{1}{2^2-v^2}\,\mathrm{d}v=\int 1\,\mathrm{d}t$ $\frac{1}{2(2)}\ln\left\|\frac{2+v}{2-v}\right\|=t+d$ $\left\|\frac{2+v}{2-v}\right\|=\mathrm{e}^{4t+4d}$ $\frac{2+v}{2-v}=\pm\mathrm{e}^{4t+4d}$ $\frac{2+v}{2-v}=A\mathrm{e}^{4t}$, where $A=\pm\mathrm{e}^{4d}$ is an arbitrary constant, $A\neq 0$

	$2+v=2Ae^{4t}-Ave^{4t}$ $v+Ave^{4t}=2\left(Ae^{4t}-1\right)$ $v=\dfrac{2\left(Ae^{4t}-1\right)}{Ae^{4t}+1}$ $\dfrac{y}{\cos t}=\dfrac{2\left(Ae^{4t}-1\right)}{Ae^{4t}+1}$ $y=\dfrac{2\left(Ae^{4t}-1\right)\cos t}{Ae^{4t}+1}$ When $t=0,\ y=0$ $0=\dfrac{2(A-1)}{A+1}$ $A=1$ $y=\dfrac{2\left(e^{4t}-1\right)\cos t}{e^{4t}+1}$
(ii) **[4]**	$\dfrac{d^2x}{dt^2}=\cos 3t\cos t$ $\dfrac{d^2x}{dt^2}=\dfrac{1}{2}(\cos 4t+\cos 2t)$ $\dfrac{dx}{dt}=\dfrac{1}{2}\int \cos 4t+\cos 2t\ dt$ $=\dfrac{1}{2}\left(\dfrac{1}{4}\sin 4t+\dfrac{1}{2}\sin 2t\right)+c$ $=\dfrac{1}{8}\sin 4t+\dfrac{1}{4}\sin 2t+c$ When $t=0,\ \dfrac{dx}{dt}=3$ $3=\dfrac{1}{8}\sin 4(0)+\dfrac{1}{4}\sin 2(0)+c$ $c=3$ $x=\int \dfrac{1}{8}\sin 4t+\dfrac{1}{4}\sin 2t+3\ dt$ $x=-\dfrac{1}{32}\cos 4t-\dfrac{1}{8}\cos 2t+3t+k$ When $t=0,\ x=-\dfrac{1}{32}$ $-\dfrac{1}{32}=-\dfrac{1}{32}\cos 4(0)-\dfrac{1}{8}\cos 2(0)+3(0)+k$ $k=\dfrac{1}{8}$ Hence, $x=-\dfrac{1}{32}\cos 4t-\dfrac{1}{8}\cos 2t+3t+\dfrac{1}{8}$

(iii) **[2]**	

Source of Question: EJC JC2 Prelim 9758/2018/01/Q10

9 An epidemiologist is studying the spread of a disease, dengue fever, which is spread by mosquitoes, in town A. P is defined as the number of infected people (in thousands) t years after the study begins. The epidemiologist predicts that the rate of increase of P is proportional to the product of the number of infected people and the number of uninfected people. It is known that town A has 10 thousand people of which a thousand were infected initially.

(i) Write down a differential equation that is satisfied by P. [1]

(ii) Given that the epidemiologist projects that it will take 2 years for half the town's population to be infected, solve the differential equation in **(i)** and express P in terms of t. [6]

(iii) Hence, sketch a graph of P against t [2]

A second epidemiologist proposes an alternative model for the spread of the disease with the following differential equation: $\dfrac{\mathrm{d}P}{\mathrm{d}t} = \dfrac{2\cos t}{\left(2-\sin t\right)^2}$ (*).

(iv) Using the same initial condition, solve the differential equation (*) to find an expression of P in terms of t. [3]

(v) Find the greatest and least values of P predicted by the alternative model. [2]

(vi) The government of town A deems the alternative model as a more realistic model for the spread of the disease as it more closely follows the observed pattern of the spread of the disease. What could be a possible factor contributing to this? [1]

[(i) $\dfrac{\mathrm{d}P}{\mathrm{d}t} = kP\left(10-P\right)$ (ii) $P = \dfrac{10e^{\frac{t}{2}\ln(9)}}{9+e^{\frac{t}{2}\ln(9)}}$ (iv) $P = \dfrac{2}{2-\sin t}$ (v) max P=2; min P = 2/3]

9(i) **[1]**	$\dfrac{\mathrm{d}P}{\mathrm{d}t} = kP\left(10-P\right)$
(ii) **[6]**	$\dfrac{\mathrm{d}P}{\mathrm{d}t} = kP\left(10-P\right)$

Method 1 to integrate:

$$\int \frac{1}{P(10-P)}\,\mathrm{d}P = k\int \mathrm{d}t$$

$$\frac{1}{10}\int \frac{1}{P}+\frac{1}{10-P}\,\mathrm{d}P = k\int \mathrm{d}t$$

$$\frac{1}{10}\Big[\ln|P|-\ln|(10-P)|\Big] = kt+C$$

$$\frac{1}{10}\ln\left|\frac{P}{10-P}\right| = kt+c$$

$$\frac{1}{10}\ln\left(\frac{P}{10-P}\right) = kt+C$$

$$\ln\left(\frac{P}{10-P}\right) = 10kt+C_1$$

$$\frac{P}{10-P} = e^{10kt+C_1} = Ae^{10kt}$$

Method 2 to integrate

$$\int \frac{1}{P(10-P)}\,\mathrm{d}P = k\int \mathrm{d}t$$

$$\int \frac{1}{25-(P-5)^2}\,dP = k\int \mathrm{d}t$$

$$\frac{1}{10}\ln\left|\frac{5+(P-5)}{5-(P-5)}\right| = kt+c$$

$$\frac{1}{10}\ln\left|\frac{P}{10-P}\right| = kt+c$$

From either Method 1 or 2,

since $P>0, 10-P\geq 0$

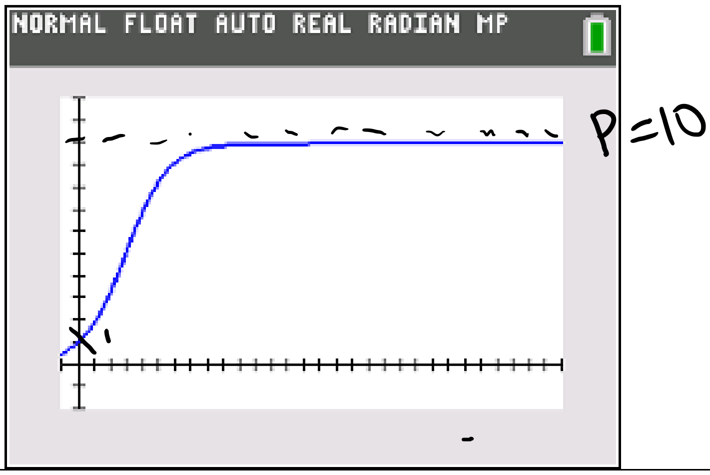
$$\frac{1}{10}\ln\left(\frac{P}{10-P}\right) = kt+C$$

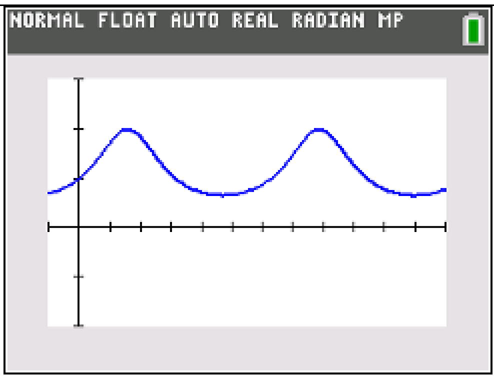
$$\ln\left(\frac{P}{10-P}\right) = 10kt+C_1$$

$$\frac{P}{10-P} = e^{10kt+C_1} = Ae^{10kt}$$

Substitute in values into solution

Sub $t=0, P=1$

	$\dfrac{P}{10-P}=e^{10kt+C_1}=Ae^{10kt}$ $\dfrac{1}{9}=Ae^{0}\Rightarrow A=\dfrac{1}{9}$ $\dfrac{P}{10-P}=\dfrac{1}{9}e^{10kt}$ Sub $t=2, P=5$ $\dfrac{5}{10-5}=\dfrac{1}{9}e^{10(2)k}$ $1=\dfrac{1}{9}e^{20k}$ $e^{20k}=9\Rightarrow k=\dfrac{1}{20}\ln(9)\approx 0.10986$ So we have, $\dfrac{P}{10-P}=\dfrac{1}{9}e^{\frac{t}{2}\ln(9)}$ $9P=(10-P)e^{\frac{t}{2}\ln(9)}$ $P\left(9+e^{\frac{t}{2}\ln(9)}\right)=10e^{\frac{t}{2}\ln(9)}$ $P=\dfrac{10e^{\frac{t}{2}\ln(9)}}{9+e^{\frac{t}{2}\ln(9)}}$
(iii) **[2]**	(iii) 
(iv) **[3]**	$\dfrac{\mathrm{d}P}{\mathrm{d}t}=\dfrac{2\cos t}{(2-\sin t)^2}=(-2)(-\cos t)(2-\sin t)^{-2}$ $P=\dfrac{-2(2-\sin t)^{-1}}{-1}=\dfrac{2}{2-\sin t}+c$ Sub $t=0, P=1$

	$1=\dfrac{2}{2-\sin 0}+c$ $c=1-1=0$ Thus $P=\dfrac{2}{2-\sin t}$
(v) **[2]**	Since $-1\le \sin t\le 1$, Largest value of P is when $\sin t=1$ Largest value of P = 2 Smallest value of P is when $\sin t=-1$ Smallest value of P = $\dfrac{2}{3}$
(vi) **[1]**	 We can use the GC to plot $P=\dfrac{2}{2-\sin t}$. The second model could be deemed more suitable, as it shows oscillating values of P, which could correspond to the population of the mosquitoes which could vary seasonally. (For example, when the season is hot and rainy, the environment is more conducive for the breeding of mosquitoes.)

Source of Question: HCI JC2 Prelim 9758/2018/01/Q12

10 An experiment is conducted at room temperature where two substances, A and B, react in a chemical reaction to form X as shown below:

$$A+B\rightarrow X.$$

The initial concentrations in mol/dm^3 of substances A and B are a and b respectively. At time t seconds, the concentration of A and B are each reduced by x, where x denotes the concentration of X at time t.

(i) State the concentrations of A and B at time t. [1]

(ii) It is known that the rate of change of concentration of X at time t is proportional to the product of concentration of A and B at time t with a constant of proportionality k.
Write down a differential equation involving x, a, b, t and k. [1]

(iii) State the maximum value of x if $a\le b$. Justify your answer. [2]

In the rest of the parts of the question, assume $a=b$.

(iv) The initial concentration of X is zero. Solve the differential equation in part **(ii)**, leave your answer in terms of x, a, t and k.

Express the solution in the form $x = \mathrm{f}(t)$ and sketch $x = \mathrm{f}(t)$ relevant in this context. Label the graph as S_1. [5]

It is known that the rate of change of concentration of *X* is doubled with every $10^{\circ}\mathrm{C}$ rise in the temperature. The experiment above is repeated but at a temperature $20^{\circ}\mathrm{C}$ above the room temperature. The concentration of *X* for this 2nd experiment at time *t* is now denoted by x_2. Let S_2 be the solution curve for the 2nd experiment.

(v) Write down a differential equation relating the rate of change of concentration of x_2 at time *t*. Without solving the differential equation, write down the solution to this differential equation, leave your answer in terms of x_2, a, t and k. [2]

(vi) On the same diagram as in part **(iv)**, sketch the solution curve for S_2. Show clearly the relative positions of S_1 and S_2 and their behaviour when $t \to \infty$. [2]

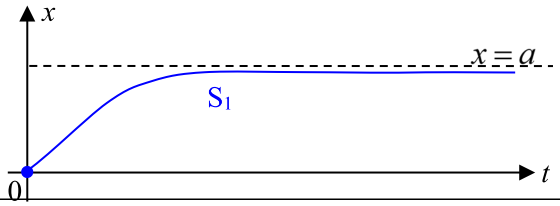
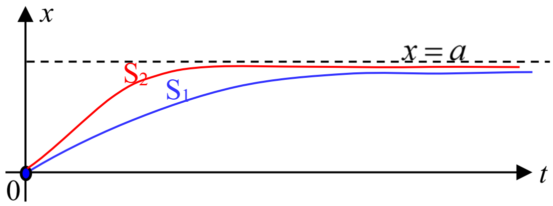
(vii) It is given that S_1 passes through the point $(1, \alpha)$ and S_2 passes through the point $(1, \beta)$. Using the concentration of *X* for the two experiments, state the inequality relating α and β in this context. [1]

[(i) the concentration of *A* and *B* at time *t* are $(a-x)$ and $(b-x)$ mol/dm^3 respectively

(ii) $\frac{\mathrm{d}x}{\mathrm{d}t} = k(a-x)(b-x),\ k \in \mathbb{R}^+$ (iii) a (iv) $x = a - \frac{a}{akt+1}$

(v) $\frac{\mathrm{d}x_2}{\mathrm{d}t} = 4k(a-x_2)^2, x_2 = a - \frac{a}{4akt+1}$ (vi) $\alpha < \beta$]

10(i) [1]	The concentration of *A* and *B* at time *t* are $(a-x)$ and $(b-x)$ mol/dm^3 respectively
(ii) [1]	$\frac{\mathrm{d}x}{\mathrm{d}t} = k(a-x)(b-x),\ k \in \mathbb{R}^+$
(iii) [2]	Max value for *x* is *a*, $\because \frac{\mathrm{d}x}{\mathrm{d}t} = 0$ and after $x = a$ there is no more concentration of substance A for reaction to continue.
(iv) [5]	$\frac{\mathrm{d}x}{\mathrm{d}t} = k(a-x)^2$ $\int (a-x)^{-2}\mathrm{d}x = kt$ $(a-x)^{-1} = kt + C$ $x = a - \frac{1}{kt+C}$ ---(1) When $t = 0, x = 0 \Rightarrow c = \frac{1}{a}$ $x = a - \frac{1}{kt + \frac{1}{a}}$ $x = a - \frac{a}{akt+1}$

	
(v) **[2]**	$\frac{\mathrm{d}x_2}{\mathrm{d}t} = 4k(a - x_2)^2$ Replacing k by $4k$ in the solution of part iv, $x_2 = a - \frac{a}{4akt + 1}$
(vi) **[2]**	
(vii) **[1]**	From the graph, $\alpha < \beta$. $\because x_2$ for $S_2 > x$ for S_1 and both curves start from the origin.