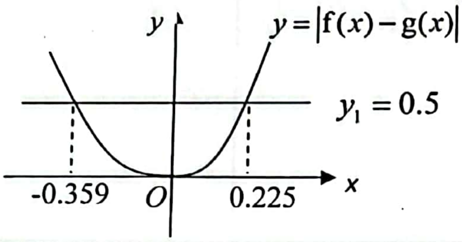
Jurong Junior College
2016 JC2 H2 Mathematics Prelim Paper 1 Solutions

Qn	Solution
1	$a+b+c=24$ $5a+2b-3c=79$ $a+b=4(c+1)$ $a=17, b=3, c=4$
2(i)	Let P_n be the statement that $u_n=\dfrac{n}{4n^2-1}$ for all $n\in\mathbb{Z}^+$ When $n=1$, LHS $=u_1=\dfrac{1}{3}$ (given) RHS $=\dfrac{1}{4\times1^2-1}=\dfrac{1}{3}$ Hence P_1 is true.
	Assume P_k is true for some $k\in\mathbb{Z}^+$, i.e. $u_k=\dfrac{k}{4k^2-1}$. We want to prove that P_{k+1} is true, i.e. $u_{k+1}=\dfrac{k+1}{4(k+1)^2-1}$ LHS $=u_{k+1}$ $=u_k-\dfrac{1}{(2k-1)(2k+3)}$ $=\dfrac{k}{(2k-1)(2k+1)}-\dfrac{1}{(2k-1)(2k+3)}$ $=\dfrac{k(2k+3)-(2k+1)}{(2k-1)(2k+1)(2k+3)}$ $=\dfrac{2k^2+k-1}{(2k-1)(2k+1)(2k+3)}$ $=\dfrac{(2k-1)(k+1)}{(2k-1)(2k+1)(2k+3)}$ $=\dfrac{k+1}{(2k+1)(2k+3)}$ $=\dfrac{k+1}{4k^2+8k+3}$ $=\dfrac{k+1}{4(k+1)^2-1}=$ RHS Hence P_k is true $\Rightarrow P_{k+1}$ is true. Since P_1 is true & P_k is true $\Rightarrow P_{k+1}$ is true, by mathematical induction, P_n is true for all $n\in\mathbb{Z}^+$.

(ii)	Sum of 1[st] n terms of $\frac{1}{5\times9}+\frac{1}{7\times11}+\frac{1}{9\times13}+\cdots\cdots+\frac{1}{(2n+3)(2n+7)}$ $=\sum_{r=3}^{n+2}\frac{1}{(2r-1)(2r+3)}$ $=\sum_{r=3}^{n+2}[u_r-u_{r+1}]$ $= u_3-u_4$ $+u_4-u_5$ $+u_5-u_6$ $+\cdots\cdots$ $+\cdots\cdots$ $+u_{n+2}-u_{n+3}$ $=u_3-u_{n+3}$ $=\frac{3}{35}-\frac{n+3}{4(n+3)^2-1}.$
(iii)	As $n\to\infty, \frac{n+3}{4(n+3)^2-1}\to 0$ Hence, the series is convergent and $\sum_{r=3}^{\infty}\frac{1}{(2r-1)(2r+3)}=\frac{3}{35}$
3 **(i)**	Required Area $=\int_{-\frac{2}{3}}^{0}\left(3-\frac{12}{(3x+2)^2+4}\right)dx$ $=\int_{-\frac{2}{3}}^{0}\left(3-\frac{12}{(3x+2)^2+2^2}\right)dx$ $=\left[3x-2\tan^{-1}\left(\frac{3x+2}{2}\right)\right]_{-\frac{2}{3}}^{0}$ $=0-2\left(\frac{\pi}{4}\right)-[-2-0]$ $=2-\frac{\pi}{2}$

(ii)	$y=\dfrac{12}{(3x+2)^2+4} \quad \Rightarrow \quad 3x=-2\pm\sqrt{\dfrac{12}{y}-4}=-2\pm\sqrt{\dfrac{12-4y}{y}}$ Since $x\geq-\dfrac{2}{3}$, $x=-\dfrac{2}{3}+\dfrac{1}{3}\sqrt{\dfrac{12-4y}{y}}$ Required volume $=\pi\int_{\frac{3}{2}}^{3}x^2\ \mathrm{d}y$ $=\pi\int_{\frac{3}{2}}^{3}\left(-\dfrac{2}{3}+\dfrac{1}{3}\sqrt{\dfrac{12-4y}{y}}\right)^2\ \mathrm{d}y$ $=0.5125$
4(i)	$y=\tan\left(2\tan^{-1}x+\dfrac{\pi}{4}\right)$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\sec^2\left(2\tan^{-1}x+\dfrac{\pi}{4}\right)\dfrac{2}{1+x^2}$ $\left(1+x^2\right)\dfrac{\mathrm{d}y}{\mathrm{d}x}=2\left(1+\tan^2\left(2\tan^{-1}x+\dfrac{\pi}{4}\right)\right)$ $\left(1+x^2\right)\dfrac{\mathrm{d}y}{\mathrm{d}x}=2\left(1+y^2\right)$
	Alternative Method $\tan^{-1}y=2\tan^{-1}x+\dfrac{\pi}{4}$ Differentiate with respect to x, $\dfrac{1}{1+y^2}\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{2}{1+x^2}$ $\Rightarrow\left(1+x^2\right)\dfrac{\mathrm{d}y}{\mathrm{d}x}=2\left(1+y^2\right)$
(ii)	Differentiate with respect to x, $\Rightarrow\left(1+x^2\right)\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+2x\dfrac{\mathrm{d}y}{\mathrm{d}x}=4y\dfrac{\mathrm{d}y}{\mathrm{d}x}$ $\Rightarrow\left(1+x^2\right)\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+(2x-4y)\dfrac{\mathrm{d}y}{\mathrm{d}x}=0$ When $x=0$, $y=\tan\dfrac{\pi}{4}=1$ $(1+0)\dfrac{\mathrm{d}y}{\mathrm{d}x}=2(1+1)\Rightarrow\dfrac{\mathrm{d}y}{\mathrm{d}x}=4$ $(1+0)\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+(0-4)(4)=0\Rightarrow\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=16$ $y=\tan\left[2\tan^{-1}x+\dfrac{\pi}{4}\right]$ $=1+4x+\dfrac{16}{2!}x^2+.....$ $=1+4x+8x^2+....$

<table>
<tr><td>(iii)</td><td>Sketch $y=|f(x)-g(x)|$ and $y_1=0.5$

</td></tr>
<tr><td></td><td>For $|f(x)-g(x)|<0.5$, $\quad -0.359<x<0.225$</td></tr>
<tr><td>5(i)</td><td>$x=2\sin 2t$, $\quad y=\cos 2t$, $\quad$ for $0\le t<\pi$.
$\frac{dx}{dt}=4\cos 2t \qquad \frac{dy}{dt}=-2\sin 2t$
$\frac{dy}{dx}=-\frac{\sin 2t}{2\cos 2t}$
Equation of normal at P, $t=\theta$:
$$y-\cos 2\theta=\frac{2\cos 2\theta}{\sin 2\theta}(x-2\sin 2\theta)$$
$(\sin 2\theta)y-\cos 2\theta\sin 2\theta=(2\cos 2\theta)x-4\cos 2\theta\sin 2\theta$
$(2\cos 2\theta)x-(\sin 2\theta)y=3\cos 2\theta\sin 2\theta$ (shown)
i.e. $m=3$</td></tr>
<tr><td>(ii)</td><td>At the x-axis, $y=0$
$(2\cos 2\theta)x=3\sin 2\theta\cos 2\theta$
$x=\frac{3}{2}\sin 2\theta \quad$ i.e. $A\left(\frac{3}{2}\sin 2\theta,\ 0\right)$
At the y-axis, $x=0$
$-(\sin 2\theta)y=3\sin 2\theta\cos 2\theta$
$y=-3\cos 2\theta \quad$ i.e. $B(0,\ -3\cos 2\theta)$
mid-point of AB: $\left(\frac{3}{4}\sin 2\theta,-\frac{3}{2}\cos 2\theta\right)$
$x=\frac{3}{4}\sin 2\theta \Rightarrow \sin 2\theta=\frac{4}{3}x$
$y=-\frac{3}{2}\cos 2\theta \Rightarrow \cos 2\theta=-\frac{2}{3}y$
Cartesian equation of the locus of the mid-point of AB:
$\sin^2 2\theta+\cos^2 2\theta=1$
$\frac{16x^2}{9}+\frac{4y^2}{9}=1 \quad$ i.e. $16x^2+4y^2=9$</td></tr>
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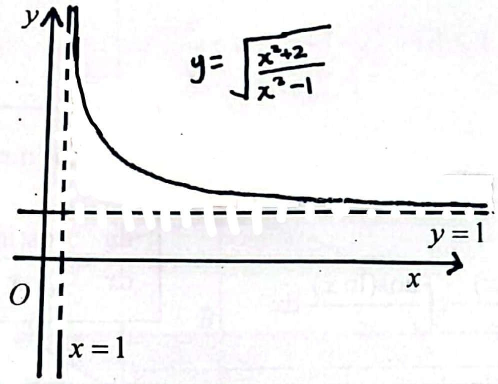
Qn	Solution
6(i)	
(ii)	Let $y=\mathrm{g}(x)=\sqrt{\frac{x^2+2}{x^2-1}},\quad x>1$ Then $y^2=\frac{x^2+2}{x^2-1}=1+\frac{3}{x^2-1}$ $y^2-1=\frac{3}{x^2-1}$ $x^2-1=\frac{3}{y^2-1}$ $x^2=1+\frac{3}{y^2-1}=\frac{y^2+2}{y^2-1}$ $x=\mathrm{g}^{-1}(y)=\sqrt{\frac{y^2+2}{y^2-1}}$ since $x>1>0$ $\Rightarrow \mathrm{g}^{-1}:x\mapsto\sqrt{\frac{x^2+2}{x^2-1}}\ ,x>1$ g is self-inverse as $\mathrm{g}(x)=\mathrm{g}^{-1}(x)$ and $\mathrm{D}_{\mathrm{g}^{-1}}=\mathrm{D}_{\mathrm{g}}$
(iii)	$\mathrm{g}^2(x)=\mathrm{gg}(x)=\mathrm{gg}^{-1}(x)=x.$ $\mathrm{g}^3(x)=\mathrm{gg}^2(x)=\mathrm{g}(x).$ (shown) It follows that $\mathrm{g}^{50}(x)=x$ and $\mathrm{g}^{51}(x)=\mathrm{g}(x),\quad x>1$ For $4-\mathrm{g}^{50}(x)=\left[\mathrm{g}^{51}(x)\right]^2$ Then $4-x=\frac{x^2+2}{x^2-1},\quad x>1$ $(4-x)(x^2-1)=x^2+2,\quad x>1$ $x^3-3x^2-x+6=0,\ x>1$ $(x-2)(x^2-x-3)=0$ $x=2$ or $x=\frac{1\pm\sqrt{1+12}}{2}$ since $x>1$, $\Rightarrow x=2$ or $x=\frac{1+\sqrt{13}}{2}$ (ans)

Qn	Solution
7 (a)	$u=\cos(\ln x)$ $\dfrac{dv}{dx}=\dfrac{1}{x^2}$ $\dfrac{du}{dx}=-\dfrac{\sin(\ln x)}{x}$ $v=-\dfrac{1}{x}$ $\int\dfrac{\cos(\ln x)}{x^2}\,dx=-\dfrac{\cos(\ln x)}{x}-\int\dfrac{\sin(\ln x)}{x^2}\,dx$ → [$u=\sin(\ln x)$, $\dfrac{dv}{dx}=\dfrac{1}{x^2}$, $\dfrac{du}{dx}=\dfrac{\cos(\ln x)}{x}$, $v=-\dfrac{1}{x}$] $=-\dfrac{\cos(\ln x)}{x}+\dfrac{\sin(\ln x)}{x}-\int\dfrac{\cos(\ln x)}{x^2}\,dx$
	$\Rightarrow\ 2\int\dfrac{\cos(\ln x)}{x^2}\,dx=\dfrac{\sin(\ln x)}{x}-\dfrac{\cos(\ln x)}{x}$ $\therefore\ \int\dfrac{\cos(\ln x)}{x^2}\,dx=\dfrac{1}{2x}\left[\sin(\ln x)-\cos(\ln x)\right]+c$
(b)	$u=\sqrt{x+3}\ \Rightarrow\ u^2=x+3$ Differentiating w.r.t. x, $2u\dfrac{du}{dx}=1$ When $x=1$, $u=2$; When $x=6$, $u=3$ $\int_1^6\dfrac{x-2}{x\sqrt{x+3}}\,dx=\int_2^3\dfrac{u^2-5}{(u^2-3)u}(2u\ du)$ $=2\int_2^3\left(1-\dfrac{2}{u^2-3}\right)du$ $=\left[2u-\dfrac{4}{2\sqrt{3}}\ln\left(\dfrac{u-\sqrt{3}}{u+\sqrt{3}}\right)\right]_2^3$ $=2(3)-\dfrac{2}{\sqrt{3}}\ln\left(\dfrac{3-\sqrt{3}}{3+\sqrt{3}}\right)-2(2)+\dfrac{2}{\sqrt{3}}\ln\left(\dfrac{2-\sqrt{3}}{2+\sqrt{3}}\right)$ $=2+\dfrac{2}{\sqrt{3}}\ln\left(\dfrac{3-\sqrt{3}}{3+\sqrt{3}}\right)$ i.e. $a=b=2$, $c=d=3$

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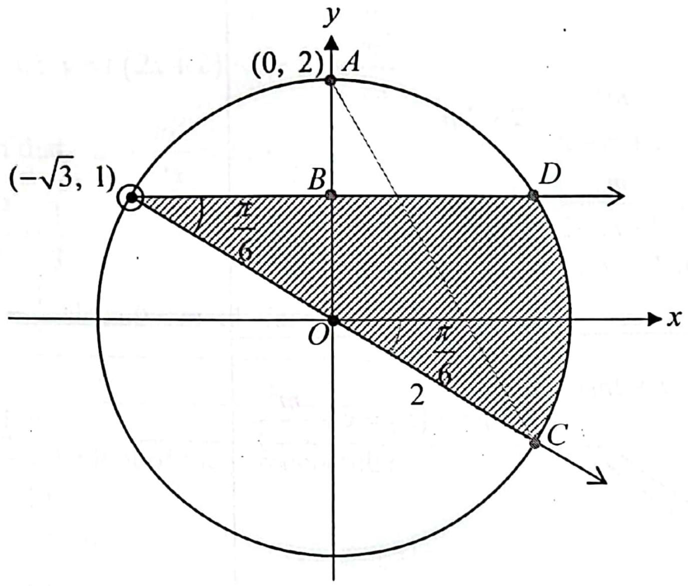
<tr><th>Qn</th><th>Solution</th></tr>
<tr><td>8
(i)</td><td>

$|z|^2 \le 4 \Rightarrow |z| \le 2$

$-\frac{\pi}{6} \le \arg(z+\sqrt{3}-\mathrm{i}) \le 0 \Rightarrow -\frac{\pi}{6} \le \arg[z-(-\sqrt{3}+\mathrm{i})] \le 0$



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<tr><td>(ii)</td><td>

Minimum value of $|z-2\mathrm{i}|$

$= AB$

$=1$

Maximum value of $|z-2\mathrm{i}|$

$= AC$

$= \sqrt{2^2+2^2-2(2)(2)\cos\frac{2\pi}{3}}$

$=2\sqrt{3}$

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<tr><td>(iii)</td><td>

$z^{100} = 2^{100} = 2^{100}\mathrm{e}^{\mathrm{i}0}$

$\Rightarrow z = 2\mathrm{e}^{\mathrm{i}\left(\frac{0+2k\pi}{100}\right)}, \ k=0, \ \pm1, \ \pm2, \ \ldots, \ \pm49, \ 50$

$\Rightarrow z = 2\mathrm{e}^{\mathrm{i}\frac{k\pi}{50}}$

Roots are found in region R (along the minor arc CD) if

$-\frac{\pi}{6} \le \frac{k\pi}{50} \le \frac{\pi}{6}$.

$\Rightarrow -8\frac{1}{3} \le k \le 8\frac{1}{3}$

$\Rightarrow k = -8, \ -7, \ -6, \ \ldots, \ 8$

$\therefore$ Number of roots found in region $R = 17$.

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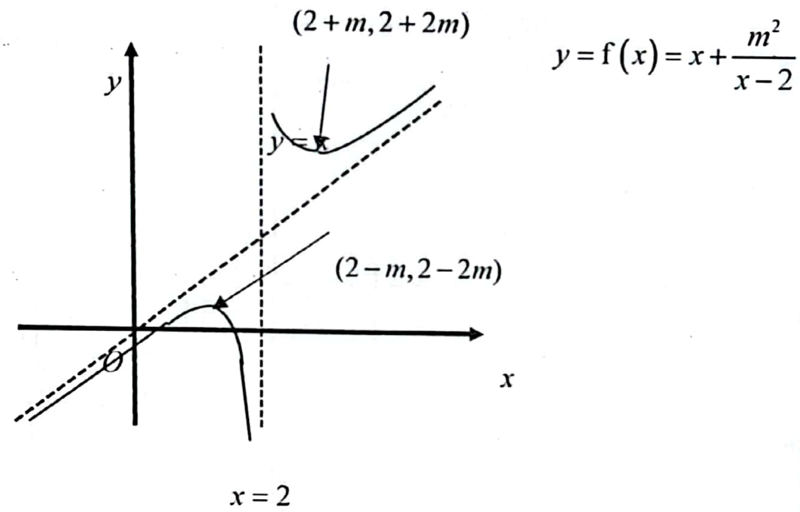
Qn	Solution
9(i)	$f(x)=x+\dfrac{m^2}{x-2}$ Let $\dfrac{df}{dx}=1-\dfrac{m^2}{(x-2)^2}=0$ $(x-2)^2-m^2=0$ $x=2\pm m$ When $x=2+m$, $f(x)=2+m+\dfrac{m^2}{2+m-2}=2+2m$ When $x=2-m$, $f(x)=2-m+\dfrac{m^2}{2-m-2}=2-2m$ The stationary points are $(2+m, 2+2m)$ and $(2-m, 2-2m)$
	
(ii)	$f(x)=\dfrac{x(x-2)+m^2}{x-2}$ when $x=0$, $f(x)=-\dfrac{m^2}{2}$ $x^2-(2+k)x+(m^2+2k)=0$ $x^2-2x+m^2=k(x-2)$ $x+\dfrac{m^2}{x-2}=k$ By inserting a horizontal line $y=k$ on the graph of C, by observation, to have two distinct positive roots, then $-\dfrac{m^2}{2}<k<2-2m$ or $k>2+2m$ (ans)

(iii)	$y=\mathrm{f}(x)=x+\dfrac{m^2}{x-2}$ After A: $y=\mathrm{f}(x+2)=x+2+\dfrac{m^2}{x}$ After B: $y=\mathrm{f}(2x+2)=2x+2+\dfrac{m^2}{2x}$ After C: $y=\mathrm{f}(2x+2)-2=2x+\dfrac{m^2}{2x}$ Given that $2x+\dfrac{m^2}{2x}=2x+\dfrac{1}{8x}$ $\Rightarrow \dfrac{m^2}{2}=\dfrac{1}{8}$ $\Rightarrow m=\dfrac{1}{2}$ since $0<m<1$ (ans)

Qn	**Solution**
10 **(i)**	Let F be the foot of the perpendicular. $\overrightarrow{AF}\cdot\begin{pmatrix}1\\1\\-1\end{pmatrix}=0$ $\Rightarrow\left[\begin{pmatrix}2\\-3\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\-1\end{pmatrix}-\begin{pmatrix}4\\-3\\0\end{pmatrix}\right]\cdot\begin{pmatrix}1\\1\\-1\end{pmatrix}=0$ $\Rightarrow -2+3\lambda-1=0$ $\Rightarrow \lambda=1$ $\therefore\ \overrightarrow{OF}=\begin{pmatrix}2\\-3\\1\end{pmatrix}+\begin{pmatrix}1\\1\\-1\end{pmatrix}=\begin{pmatrix}3\\-2\\0\end{pmatrix}.$
(ii)	Let B be $(2,\ -3,\ 1)$. A normal to π_1 is $\overrightarrow{BA}\times\begin{pmatrix}1\\1\\-1\end{pmatrix}=\begin{pmatrix}2\\0\\-1\end{pmatrix}\times\begin{pmatrix}1\\1\\-1\end{pmatrix}=\begin{pmatrix}1\\1\\2\end{pmatrix}.$ $\begin{pmatrix}4\\-3\\0\end{pmatrix}\cdot\begin{pmatrix}1\\1\\2\end{pmatrix}=1$ A cartesian equation of π_1 is $x+y+2z=1$.

(iii)	$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{pmatrix}1\\1\\2\end{pmatrix} \times \begin{pmatrix}1\\0\\7\end{pmatrix} = \begin{pmatrix}7\\-5\\-1\end{pmatrix}$ A direction vector of L is $\begin{pmatrix}-7\\5\\1\end{pmatrix}$. $\pi_1: x+y+2z=1$ $\pi_2: x+7z=c$ Let $z=0$. Then $x=c$ and $y=1-c$. A point on L is $(c,\ 1-c,\ 0)$. $\therefore$ A vector equation of L is $\mathbf{r} = \begin{pmatrix}c\\1-c\\0\end{pmatrix} + \mu\begin{pmatrix}-7\\5\\1\end{pmatrix}$, where $\mu \in \mathbb{R}$.
(iv) **(a)**	For the 3 planes to meet in the line L, $\begin{pmatrix}-7\\5\\1\end{pmatrix} \cdot \begin{pmatrix}2\\-1\\d\end{pmatrix} = 0$ and $\begin{pmatrix}c\\1-c\\0\end{pmatrix} \cdot \begin{pmatrix}2\\-1\\d\end{pmatrix} = 5$. $\Rightarrow -14-5+d=0$ and $2c-1+c=5$ $\Rightarrow d=19$ and $c=2$
(b)	For the 3 planes to have only one point in common, $\begin{pmatrix}-7\\5\\1\end{pmatrix} \cdot \begin{pmatrix}2\\-1\\d\end{pmatrix} \neq 0$. $\Rightarrow d \neq 19$

Qn	Solution		
11 (i)	Instalment	Outstanding amount at the beginning of month	Total amount with interest owed at the end of month
		50000	50000×1.002
	1	$50000\times1.002-1000$	$50000\times1.002^2-1000\times1.002$
	2	50000×1.002^2 $-1000\times1.002-1000$	$50000\times1.002^3-1000\times1.002^2$ -1000×1.002
	……..	…………………….. ……….	………………………………. ………
	Amount owed at the end of n instalments $=50000\times1.002^{n+1}-1000\times1.002^n-1000\times1.002^{n-1}-\ldots-1000\times1.002$ $=50000\times1.002^{n+1}-\dfrac{1000\times1.002\times\left[1.002^n-1\right]}{1.002-1}$ $=50000\times1.002^{n+1}-1000\times501\times(1.002^n-1)$ ------------(*) $=50000\times1.002^{n+1}-501000\times(1.002^n-1)$ (shown)		
(ii)	$50000\times1.002^{n+1}-501000\times(1.002^n-1)\le0$ By using G.C., least integer $n=53$ i.e. no of instalments required = 53		
(iii)	Amount paid at the 53th instalment = Amount owed at the end of 52 instalments $=50000\times1.002^{53}-501000\times\left(1.002^{52}-1\right)$ =733.12 (to 2 d.p.)		
(iv)	Let the amount need to be paid for each instalment be k. Then from (*) in (i) $50000\times1.002^{20}-k\times501\left[1.002^{19}-1\right]\le0$ $k\ge\dfrac{50000\times1.002^{20}}{501\left[1.002^{19}-1\right]}=2684.53$ Least value of $k=2685$ (to the nearest dollars)		