Class Adm No

Candidate Name: ________________________________

Class	Adm No

H

millennia institute

2015 Preliminary Examination II
Pre-University 3

MATHEMATICS **9740/01**

Paper 1 **16 September 2015**

3 hours

Additional Materials: Answer Paper
List of Formulae (MF 15)

READ THESE INSTRUCTIONS FIRST

Write your name and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, arrange your answers in NUMERICAL ORDER and fasten all your work securely together.

The number of marks is given in brackets [ ] at the end of each question or part question.

Answer all the questions [100 marks]

1. It is given that $\mathrm{f}(x) = ax^3 + bx^2 + cx + d$, where $a,\ b,\ c$ and d are constants. The curve $y = \mathrm{f}(x)$ passes through the points $(-1,\ -3)$, $(3,\ 13)$ and has a maximum point at $\left(\dfrac{1}{3},\ \dfrac{31}{27}\right)$. Find $\mathrm{f}(x)$. **[4]**

2. Find

(i) $\displaystyle\int \frac{\sin x}{1+2\cos x}\,\mathrm{d}x$, **[2]**

(ii) $\displaystyle\int_0^{\frac{\pi}{2}} e^x \cos 2x\,\mathrm{d}x$. **[4]**

3. **(i)** Expand $\dfrac{\sqrt[4]{1+3x^2}}{2+x}$ in ascending powers of x, up to and including the term in x^2. **[3]**

(ii) Find the set of values of x for which the expansion in part (i) is valid. **[2]**

(iii) By substituting $x = -\dfrac{1}{4}$ and using your result in part (i), show that $\sqrt[4]{19} \approx \dfrac{p}{q}$, where p and q are integers to be determined. **[2]**

4. **(i)** The n^{th} term of a sequence is $\mathrm{T}_n = \ln 3x^{n-1}$ where x is a constant. Show that the sequence is an arithmetic progression for all positive integers n. **[2]**

(ii) When $\ln 3$ is subtracted from the 19^{th}, 7^{th} and 3^{rd} terms of the arithmetic progression in part **(i)**, these terms become the first three terms of a geometric progression.

(a) Find the common ratio of the geometric progression. **[2]**

(b) Find the range of values of x for which the sum of the first 20 terms of the arithmetic progression exceeds the sum to infinity of the geometric progression. **[3]**

5. **(i)** Find the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x}=1-y^2. \qquad \textbf{[3]}$$

(ii) Find the particular solution of the differential equation for which $y=\frac{1}{3}$ when $x=0$. **[1]**

(iii) What can you say about the gradient of every solution curve as $x\to\pm\infty$? **[1]**

(iv) Sketch, on a **single** diagram, the graph of the solution found in part (ii), together with 2 other members of the family of solution curves. **[3]**

6. It is given that $y=e^{\tan^{-1}x}$. Show that

(i) $\left(1+x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x}=y$, **[2]**

(ii) the value of $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}$ when $x=0$ is -1. **[2]**

Write down the Maclaurin series for $e^{\tan^{-1}x}$ up to and including the term in x^3. **[2]**

Hence find the Maclaurin series for $\frac{e^{\tan^{-1}x}}{1+x^2}$ up to and including the term in x^2. **[2]**

7. A line *l* passes through the points *A* and *B* with coordinates $(0,-2,\ 2)$ and $(1,\ 0,\ 1)$ respectively.

(i) Find the acute angle between $\overrightarrow{OA}$ and the line *l*, where *O* is the origin. **[2]**

(ii) Hence, find the shortest distance from *O* to the line *l*, leaving your answer in exact form. **[1]**

A plane π_1 has equation $\mathbf{r}\cdot(2\mathbf{i}-\mathbf{j})=2$

(iii) Show that the line *l* lies in the plane π_1. **[2]**

A second plane π_2 contains the line *l* and is perpendicular to the plane π_1.

(iv) Find the cartesian equation of the plane π_2. **[2]**

(v) A third plane π_3 is perpendicular to both π_1 and π_2, and is at a perpendicular distance of $\sqrt{6}$ units from *O*. Find the possible vector equations of π_3, expressing your answer in the form $\mathbf{r}\cdot\mathbf{n}=d$. **[3]**

[Turn over

8. The function f is defined as follows.

$$\mathrm{f} : x \mapsto |x-\lambda| + |x+\lambda| \text{ for } x \in \mathbb{R},$$

where λ is a positive constant.

(i) By giving a sketch of f, give a reason why f does not have an inverse. **[2]**

(ii) The function f has an inverse if its domain is restricted to $x \geq a$ and also has an inverse if its domain is restricted to $x \leq b$. State minimum value of *a* and maximum value of *b* in terms of λ. **[2]**

(iii) By rewriting $\mathrm{f}(x) = c(x-\lambda) + d(x+\lambda)$ where *c* and *d* are constants to be determined, find an expression for $\mathrm{f}^{-1}(x)$ corresponding to $x \leq b$ for f. **[2]**

In the rest of the question, let $\lambda = 1$.

The function g is defined as follows.

$$\mathrm{g} : x \mapsto \ln(x-1) \text{ for } x \in \mathbb{R},\ x > 1.$$

(iv) Show that the composite function gf exists. **[2]**

(v) Find the range of gf. **[2]**

9. The curve *C* has equation $y = \mathrm{f}(x)$, where $\mathrm{f}(x) = \dfrac{x}{1+x^2}$.

(i) Sketch the curve *C*, stating the equation of the asymptote. **[2]**

(ii) Find the exact area of the region bounded by the curve, the lines $x = \frac{1}{2}$, $x = \frac{3}{2}$ and the *x*-axis. **[3]**

(iii) Find the exact value of $\int_{-\frac{1}{2}}^{\frac{3}{2}} |\mathrm{f}(x)| \, \mathrm{d}x$. **[3]**

(iv) Use the substitution $x = \tan\theta$ to find the exact volume of revolution when the region bounded by the curve, the lines $x = 0$, $x = 1$ and the *x*-axis is rotated completely about the *x*-axis. **[4]**

10. **(i)** Show that $\dfrac{2\left(3^r\right)}{3\left(1+3^{r-1}\right)\left(1+3^r\right)}=\dfrac{1}{1+3^{r-1}}-\dfrac{1}{1+3^r}$. **[1]**

(ii) Use the method of differences to show that

$$\sum_{r=1}^{n}\frac{3^r}{\left(1+3^{r-1}\right)\left(1+3^r\right)}=\frac{3}{2}\left(\frac{1}{2}-\frac{1}{1+3^n}\right).$$ **[4]**

(iii) Use the method of mathematical induction to prove the result in part (ii). **[5]**

(iv) Hence find $\displaystyle\sum_{r=1}^{\infty}\frac{3^{r+1}}{\left(1+3^{r-1}\right)\left(1+3^r\right)}$. **[3]**

11. **(a)** **(i)** A student wrote the statement "if $z=x+y\mathrm{i}$, where x and y are real, is a root of the equation $\mathrm{P}(z)=a_n z^n+a_{n-1}z^{n-1}+\ldots+a_1 z+a_0=0,\ n\in\square^+$, then $z=x-iy$ is also a root" in his notes. Explain whether the statement is always true. **[1]**

(ii) Given that $2+\mathrm{i}$ is a root of the equation $4z^3-11z^2+25=0$, without the use of a calculator, find the other roots of the equation. **[4]**

(b) **(i)** Solve the equation

$$z^4+1+\sqrt{3}\mathrm{i}=0,$$

giving the roots in the form $re^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\leq\pi$. **[4]**

(ii) Show the roots on an Argand diagram. **[2]**

(c) Given that $\left|z^2\right|=2,\ |wz|=2\sqrt{2},\ \arg(-\mathrm{i}z)=\dfrac{\pi}{4}$ and $\arg\left(\dfrac{z^2}{w}\right)=-\dfrac{5\pi}{6}$, find w in the form $a+b\mathrm{i}$, where a and b are real coefficients to be determined. **[4]**

- End of paper-