Answer all the questions [100 marks]

1 The position vectors **a** and **b** are given by

$$\mathbf{a} = 3\mathbf{i} - 2\mathbf{j} + 6\mathbf{k} \quad \text{and} \quad \mathbf{b} = 4p\mathbf{i} + 7p\mathbf{j} - 4p\mathbf{k},$$

where $p > 0$. Given **b** is a unit vector,

(i) find the exact value of p, [2]

(ii) give a geometrical interpretation of |**a**□**b**|, [1]

(iii) evaluate $\mathbf{a} \times \mathbf{b}$. [2]

2 Without using a calculator, solve the inequality

$$\frac{2x^2 + 2x + 1}{x^2 - 2x + 1} \geq 1 \ ,$$ [4]

Hence solve

$$\frac{2(\ln x)^2 + \ln x^2 + 1}{(\ln x)^2 - \ln x^2 + 1} \geq 1 \ .$$ [3]

3 It is given that $f(x) = ax^2 + bx + c$, where a, b and c are constants.

(i) Given that the curve with equation $y = f(x)$ passes through the points with coordinates $(-1, 6)$, $(1.1, 0.12)$ and $(1.5, 1)$, find the values of a, b and c. [3]

(ii) Find the set of values of x for which $y = f(x)$ is an increasing function. [2]

(iii) Sketch the graph of $y = f'(x)$. [2]

4 Find the first three terms in the expansion of $\sqrt{x+4}$, in ascending powers of x. State the set of values of x for which the expansion is valid. [4]

Hence, by substituting with a suitable value of x, find an approximate value for $\sqrt{8}$ as a fraction. [3]

5(a) An educational fund is started at \$2000 and the bank offers a compound interest at 2% per annum. If withdrawals of \$50 are made at the beginning of each of the subsequent years, show that the amount in the fund at the beginning of the (n+1)th year is $\$500\left(5-1.02^{n}\right)$. [3]

Find the amount in the fund to the nearest dollar after the 30th withdrawal. [1]

Calculate the number of years this educational fund can last. [2]

(b) The 9th term of an arithmetic progression is 50 and the sum of the first 15 terms is 570. It is given that the sum of the first *n* terms is greater than 500. Find the least possible value of *n*. [4]

6

(i) Given that $x^2 - y^2 = 2xy - 1$, find $\frac{dy}{dx}$ in terms of *x* and *y*. [4]

(ii) For the curve $y^2 - x^2 + 2xy - 1 = 0$, find the coordinates of each point at which the tangent is parallel to the *x*-axis. [4]

7

Given *y* = 1 when *x* = 0 and $\frac{dy}{dx} = \frac{1}{2}(x+y)^2$, show that

$$\left(\frac{d^3y}{dx^3}\right) - (x+y)\frac{d^2y}{dx^2} - \left(1+\frac{dy}{dx}\right)^2 = 0 \ .$$ [3]

Hence, find the first four terms of the Maclaurin series for *y*. [4]

8

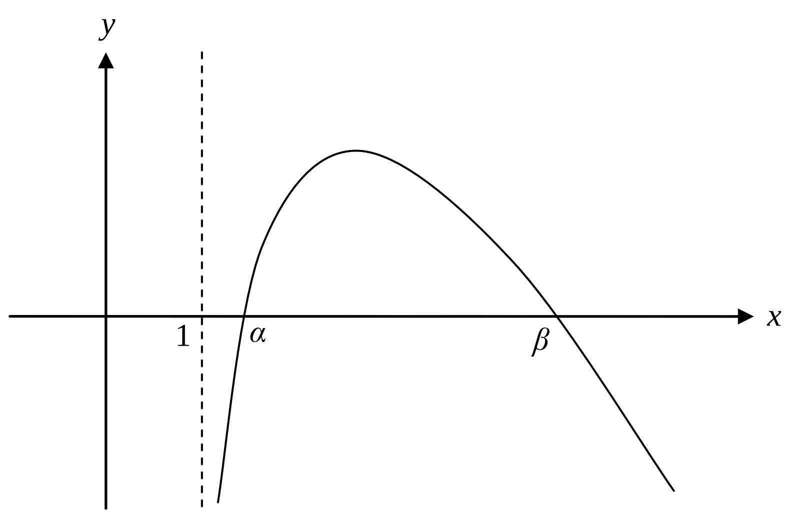
The region bounded by the curve $y = \tan x$, the *x*-axis and the line $x = \frac{\pi}{3}$ is rotated through 2π about the *x*-axis. Find the exact value of the volume of the solid formed. [4]

9(a) Using De Moivre's Theorem, show that

$$\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)^5\left(\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}\right)^9=-1 \text{ .}$$ [3]

(b) The roots of the equation $z^3=27i$ are z_1, z_2 and z_3. Without using a calculator, find z_1, z_2 and z_3 in the Cartesian form $x+iy$, showing your working. [4]

10 The diagram below shows that graph of $y=\ln(x-1)-x+4$.
The roots of the equation $\ln(x-1)-x+4=0$ are denoted by α and β, where $\alpha<\beta$.



(i) Find the values of α and β, each correct to 3 decimal places. [2]

A sequence of real numbers $x_1, x_2, x_3, ...$ satisfies the recurrence relation $x_{n+1}=\ln(x_n-1)+4$ for $n\geq 1$.

(ii) Prove algebraically that, if the sequence converges, then it converges to either α or β. [3]

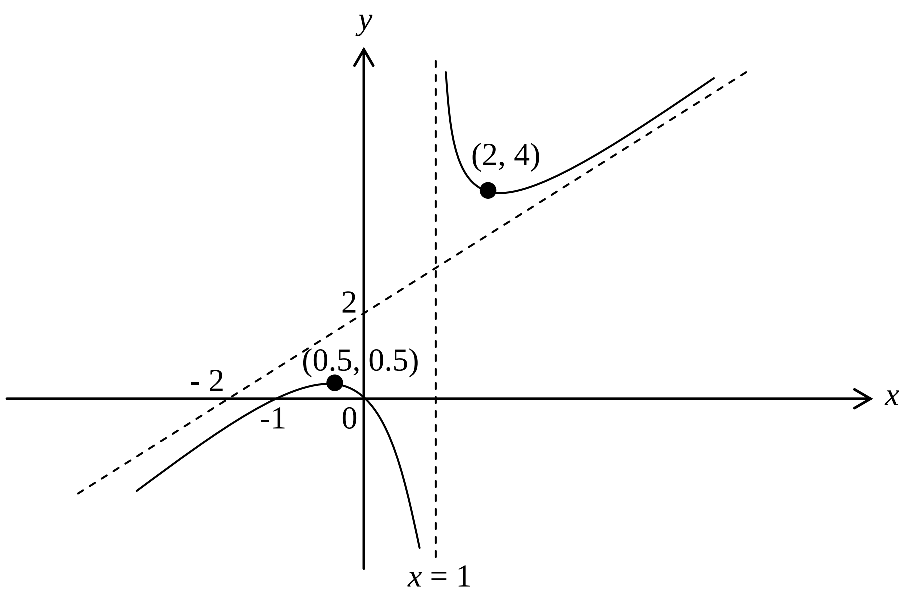
(iii) By considering the maximum point of $y=\ln(x-1)-x+4$, show that $x_{n+1}\leq x_n+2$. [3]

11 Water is flowing into a rectangular tank, with a horizontal base, at a constant rate. Water is flowing out of the tank at a rate which is proportional to the depth of water in the tank. At time t seconds, the depth of water in the tank is x metres. The depth of the water remains constant when it is 0.5 m. Show that

$$\frac{dx}{dt} = -k(2x-1) \text{, where k is a positive constant.}$$ [3]

Initially, the depth of water in the tank is 1.5 m and is decreasing at a rate of 0.02m/s. Find the time at which the depth of water is 1.01 m. [5]

12 The diagram shows the graph of $y = f(x)$.



On separate diagrams, sketch the graphs of

(i) $y = f(|x|)$, [3]

(ii) $y^2 = f(x)$, [3]

In each case, state the equation of any asymptote(s), the coordinates of any intercept(s) with the axes and any stationary point(s).

13 The plane p passes through the points with coordinates $(0,-2,1)$, $(-2,-3,2)$ and $(-4,5,-2)$.

(i) Find the vector equation of the plane p, in scalar product form. [4]

The line l_1 has equation $\frac{x-1}{2}=\frac{y+4}{1}=\frac{z+1}{4}$ and the line l_2 has equation $\frac{x}{k}=\frac{y-1}{2}=\frac{z+1}{3}$, where k is a constant. It is given that l_1 and l_2 intersect.

(ii) Find the value of k. [5]

(iii) Explain why l_1 does not lie in p. [2]

(iv) Find the coordinates of the point at which l_1 intersects p. [3]

(v) Find the acute angle between l_2 and p. [2]