ANGLO-CHINESE JUNIOR COLLEGE
JC2 PRELIMINARY EXAMINATION

H

Higher 2

/100

CANDIDATE NAME

TUTORIAL/ FORM CLASS

INDEX NUMBER

MATHEMATICS 9758/02

Paper 2 **28 August 2020**

3 hours

Candidates answer on the Question Paper.
Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your index number, class and name on all the work you hand in.
Write in dark blue or black pen.

You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing calculator is expected, where appropriate.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

Question	Marks
1	/9
2	/9
3	/10
4	/12
5	/5
6	/6
7	/7
8	/7
9	/9
10	/12
11	/14

This document consists of __ printed pages.

Section A: Pure Mathematics [40 marks]

1 **(a)** If $|\mathbf{a}| = 2$, $|\mathbf{a} - \mathbf{b}| = \sqrt{3}$ and $\mathbf{a}.\mathbf{b} = \frac{5}{2}$, use the cosine rule for triangles to find $|\mathbf{b}|$. [3]

(b) It is given that $\mathbf{r} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ where *a, b* and *c* are constants.

Give the geometrical interpretations of

(i) $|\mathbf{r}.\mathbf{k}|$, [1]

(ii) $|\mathbf{r} \times \mathbf{k}|$. [1]

(iii) Given that $\mathbf{k} \times \mathbf{r} = \mathbf{p}$ and $\mathbf{r} \times \mathbf{p} = \mathbf{k}$, in either order, show that $a^2 + b^2 = 1$ and find the value of *c*. [4]

2 A scientist and his student are investigating the growth of bacteria in a petri dish. At time *t* minutes after the bacteria is first introduced into the dish, the number of bacteria in the dish is denoted by *x*, in hundreds. Initially, 100 bacteria were introduced into the dish. In 10 minutes, the number of bacteria became 400.

(i) The student claims that the growth rate of the bacteria in the dish is inversely proportional to the square root of the number of bacteria in the dish. Write down a differential equation for this situation and solve it to get *x* as a function of *t*. [4]

(ii) The scientist claims that

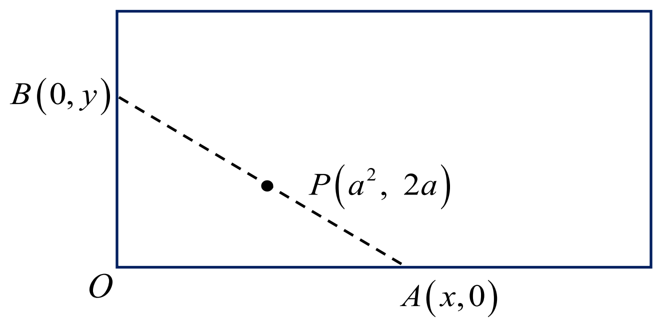
$$\frac{\mathrm{d}x}{\mathrm{d}t} = m\left(100x - x^2\right),$$

where *m* is a constant.

Show that $x = \dfrac{H}{1 + A\mathrm{e}^{-Hmt}}$, where *H* is a whole number to be determined. (You need not find the values of *A* and *m*.) [4]

(iii) Whose model do you think is better, the student's or the scientist's? Justify your answer. [1]

3 **(a)** The diagram below shows a large rectangular field, with a corner of the field being fenced off. The dotted line *AB*, represents the fence. *O* is the origin and the points *A* and *B* have coordinates $(x, 0)$ and $(0, y)$ respectively. The point *P* on line *AB*, is a permanent sprinkler that is used as a support for the fencing. Point *P* has coordinates $(a^2, 2a)$, where *a* is a positive constant.



(i) Show that *S*, the area of the corner *AOB*, is given by $S = \dfrac{ax^2}{x - a^2}$ unit^2. [2]

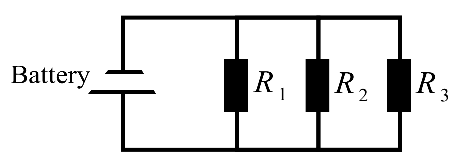
(ii) Use differentiation to find, in terms of *a*, the minimum value of *S*, proving that it is a minimum. [4]

(b) In a parallel electrical circuit, the **equivalent resistance**, R_P, is given by

$$\frac{1}{R_P} = \sum_{i=1}^{n} \frac{1}{R_i},$$

where R_i, $i = 1,\ 2,\ \ldots,\ n$, is the resistance of the *i*-th resistor connected to a battery.

The diagram below shows a parallel electrical circuit, with three resistors of resistance, R_1, R_2 and R_3 respectively, connected to a battery.

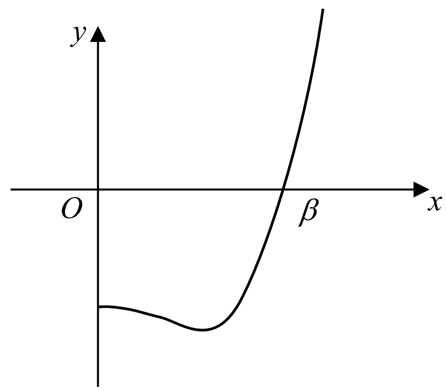


It is known that R_1 changes with time at a rate *r*, R_2 changes with time at a rate 2*r*, and R_3 remains unchanged. Find the rate of change of the **equivalent resistance**, $\dfrac{dR_P}{dt}$, when $R_2 = R_1$, and $R_3 = 2R_1$, leaving your answer in terms of *r*. [4]

4 **(a)** It is given that $\mathrm{f}(x) = x^6 - ax^4 - x^2 - b$, where a and b are real numbers.

(i) Show that $\mathrm{f}(x) = \mathrm{f}(-x)$. [1]

The diagram shows the curve with equation $y = \mathrm{f}(x)$ for $x \geq 0$. The curve crosses the positive x-axis at $x = \beta$.



(ii) Determine the number of non-real roots of the equation $\mathrm{f}(x) = 0$. Justify your answer. [2]

(iii) Given that $z_1 = r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $0 < \theta < \dfrac{\pi}{2}$, is one of the roots of the equation $\mathrm{f}(x) = 0$, express all the roots of $\mathrm{f}(x) = 0$ in the modulus-argument form. [3]

(iv) Hence, show that $\mathrm{f}(x)$ can be expressed as a product of three quadratic factors of the form

$$\left(x^2 - A\right)\left(x^2 - Brx\cos\theta + r^2\right)\left(x^2 - Crx\cos\theta + r^2\right),$$

where A, B and C are constants. [3]

(b) Find the modulus of the complex number

$$\frac{\mathrm{i}\left(2\mathrm{i}z + z^2\right)^*}{z\left(2z^* - 4\mathrm{i}\right)},$$

where z is a complex number. [3]

Section B: Probability and Statistics [60 marks]

5 For events A and B, it is given that $\mathrm{P}(A \cup B) = 0.8$ and $\mathrm{P}(A \mid B) = 0.5$. It is also given that events A and B are independent.

(i) Find $\mathrm{P}(B)$. [2]

A third event C is such that events A and C are independent and events B and C are independent.

(ii) Given also that $\mathrm{P}(C) = 0.5$, find exactly the maximum and minimum possible values of $\mathrm{P}(A \cap B \cap C)$. [3]

6 Drew has a sock drawer which contains 3 pairs of red socks, 2 pairs of white socks and 1 pair of black socks. In the mornings, Drew randomly pulls out socks from the drawer, one sock at a time, until he has a pair of socks which matched in colour. The random variable S denote the total number of socks Drew pulls out in a morning.

(i) Determine the probability distribution of S. [3]

(ii) Find $\mathrm{E}(S)$ and $\mathrm{Var}(S)$. [2]

(iii) Find the probability that Drew took more than 2 pulls to obtain a pair of matching socks for 7 consecutive mornings. [1]

7 Peter and Calvin are brothers. They went to a birthday party with 6 other friends.

(i) The 8 friends started off the party with dinner, sitting down on 8 chairs around a circular dining table. Given that there was at least one other person seated between the two brothers, find the number of different arrangements of the 8 friends at the table. [2]

(ii) The 8 friends then decided to play a game of musical chairs. 5 chairs were placed in one row. When the music stopped, Peter could not get a seat but Calvin managed to sit on the chair in the middle of the row. Find the number of ways that the 5 chairs could be occupied. [1]

(iii) The 8 friends decided to watch a movie to end off their night. There are 4 movies for selection. Each person will vote for a movie to watch, and the movie with the highest number of votes will be played. Find the total number of ways the 8 friends could have voted such that there were movies tied for the highest number of votes. [4]

8 Wally is looking for an investment plan that generates a monthly income of more than \$200. He shortlisted an investment plan whose prospectus promises investors a variable monthly income with a mean of \$200 per month. Wally decides to research further by collecting a random sample of 36 of the plan's past monthly income returns. The monthly income returns, \$$x$, are summarised as follows.

$$\sum(x-200)=662.4 \qquad \sum(x-200)^2=307141.56$$

(i) Test, at the 10% level of significance, whether Wally should invest in this plan. You should state your hypotheses and define any symbols you use. [6]

(ii) Explain what is meant by the phrase '10% significance level' in the context of this question. [1]

9 The Bacterial Filtration Efficiency (BFE), in percentages, of Brand *A* surgical masks against the *E. Coli* bacteria is assumed to have the distribution $\mathrm{N}(\mu,\sigma^2)$. It is given that the proportion of surgical masks having a BFE less than 95.70% is the same as the proportion of the surgical masks having a BFE more than 95.78%.

(i) The probability that the BFE of a Brand *A* surgical mask is more than 95.78% is 0.0912. Find the values of μ and σ. Leave your answers to 2 decimal places. [3]

The BFE of Brand *A* surgical masks against the *S. Aureus* bacteria have the distribution $\mathrm{N}(91.09, 0.08^2)$ while the BFE of Brand *B* surgical masks against the *S. Aureus* bacteria have the distribution $\mathrm{N}(92.19, 0.03^2)$.

(ii) Given that the probability that the BFE of a randomly chosen Brand *B* surgical mask differs from the sample mean BFE of n randomly chosen Brand *A* surgical masks by at most 1.15% is at least 0.9405, find the least value of n. [4]

The masses in grams of a box of Brand *A* surgical masks have the distribution $\mathrm{N}(203, \sigma_1^2)$ while the masses in grams of a box of Brand *B* surgical masks have the distribution $\mathrm{N}(203, \sigma_2^2)$.

(iii) Find the probability that three times the mass of a randomly chosen box of Brand *A* surgical masks exceeds the total mass of 3 randomly chosen boxes of Brand *B* surgical masks. You should state the parameters of any distributions that you use. [2]

10 A popular investment manager claims that the mean dividend yield of his clients is 12%. A research firm investigates further by collecting a random sample of 40 of the manager's clients. The sample was found to have a mean dividend yield of 10.9% and a standard deviation of 2.5%.

(i) State, giving a reason, whether there is a need to make any assumption about the distribution of the dividend yield earned by the manager's clients in order for a hypothesis test to be valid. [1]

(ii) Test, at the 5% level of significance, whether the manager overstated his claim. [5]

It was later known that the standard deviation of the dividend yield of the popular investment manager's clients is 2.3%. A potential investor collected the dividend yield data of a small sample size, n, of the manager's clients. The sample was found to have a mean dividend yield of 10.2%.

(iii) State, in context, two assumptions that are needed for you to carry out a test to examine the manager's claim that the mean dividend yield of his clients is 12%. [2]

(iv) Hence find the least value of n such that the manager's claim, at the 5% level of significance, will be rejected. You should state your hypotheses clearly. [4]

11 In an epidemic outbreak, an infectious disease spread through a population. It is estimated that 5% of the population are infected. A quick test for the presence of the disease was carried out on the population. An infected person tested positive 90% of the time, while a non-infected person tested positive 6% of the time.

(i) Draw a probability tree diagram to represent the above scenario and find the probability that the quick test is accurate. [2]

(ii) Find the probability that a person is infected given that he tested positive. [2]

(iii) To increase the accuracy of identifying infected or non-infected people, a more comprehensive but more costly laboratory confirmation test can be carried out.

Give **one** reason why you would recommend that people who tested positive on the quick test be sent for the laboratory confirmation test. Justify your answer. [1]

State also, **one** drawback of selecting only the people who tested positive on the quick test to be sent for the laboratory confirmation test. [1]

In another population, 5% of the population were confirmed to be infected. A sample of 20 people were randomly chosen from this population.

(iv) State one assumption needed for the number of infected people out of these 20 people to be well-modelled by a binomial distribution, and hence find the probability that there were fewer than 2 infected people out of these 20 people. [2]

(v) Find the probability that, for the 20 people chosen, there were more than 2 infected people and the first infected person was the tenth person chosen for the sample. [3]

(vi) A total of n random samples of 20 people were chosen from this population. Given that the probability of at most 5 of the samples having fewer than 2 infected people is greater than 0.1, find the largest possible value of n. [3]