SECONDARY 4
PRELIMINARY EXAMINATION
ADDITIONAL MATHEMATICS
Paper 2 4047/2

16 September 2020 (Wednesday) **2 hours 30 minutes**

CANDIDATE NAME	**Solutions**
CLASS	
INDEX NUMBER	

READ THESE INSTRUCTIONS FIRST
Do not turn over the page until you are told to do so.
Write your name, class and index number in the spaces above.
Write in dark blue or black pen in the writing papers provided.

You may use a pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

INFORMATION FOR CANDIDATES
Answer **all** the questions in the space provided.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

The use of an approved scientific calculator is expected, where appropriate.

You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your answer scripts securely together.

The number of marks is given in brackets [ ] at the end of each question or part question.

The total number of marks for this paper is **100.**

For Examiner's Use		
Q1	5	
Q2	5	
Q3	6	
Q4	5	
Q5	8	
Q6	8	
Q7	9	
Q8	10	
Q9	10	
Q10	10	
Q11	11	
Q12	13	
Total		/100

Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Binomial Expansion

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \ldots + \binom{n}{r} a^{n-r} b^r + \ldots + b^n$$

where n *is a positive integer and* $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\ldots(n-r+1)}{r!}$

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A$$

$$\mathrm{cos}ec^2 A = 1 + \cot^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Formulae for $\Delta\ ABC$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\Delta = \frac{1}{2} ab \sin C$$

1 **(i)** Differentiate $x^2 \ln 3x$ with respect to x. [2]

$$\frac{dy}{dx} = x^2\left(\frac{3}{3x}\right) + \ln 3x(2x)$$
$$= x + 2x \ln 3x$$

– apply product rule

(ii) Hence find $\int x \ln 3x \, dx$. [3]

$$\int x + 2x \ln 3x dx = x^2 \ln 3x + C$$
$$\int x dx + \int 2x \ln 3x dx = x^2 \ln 3x + C$$
$$\int 2x \ln 3x dx = x^2 \ln 3x - \frac{x^2}{2} + C$$
$$\int x \ln 3x dx = \frac{1}{2}x^2 \ln 3x - \frac{x^2}{4} + D$$

– apply anti-differentiation

– separating terms to integrate and integration of *x*

2 Express $\dfrac{x^3+1}{(x^2+1)(x-2)}$ in partial fractions. [5]

$$\frac{x^3+1}{x^3-2x^2+x-2}=1+\frac{2x^2-x+3}{(x^2+1)(x-2)}$$

– Use long division to convert improper to proper rational function

Let

$$\frac{2x^2-x+3}{(x^2+1)(x-2)}=\frac{Ax+B}{x^2+1}+\frac{C}{x-2}$$

– Decomposition of partial fractions into its correct form

$$2x^2-x+3=(Ax+B)(x-2)+C(x^2+1)$$

By Substitution:

At $x=2$,

$$2(2)^2-2+3=C(5)$$

$$C=\frac{9}{5}$$

At $x=0$,

$$3=-2B+C$$

$$2B=\frac{9}{5}-3$$

$$B=-\frac{3}{5}$$

By comparing coefficient of x^2,

$$A+C=2$$

$$A=2-\frac{9}{5}=\frac{1}{5}$$

$$\frac{x^3+1}{(x^2+1)(x-2)}=1+\frac{x-3}{5(x^2+1)}+\frac{9}{5(x-2)}$$

3 The quadratic equation $3x^2+2x+1=0$ has roots α and β.

(i) Show that the value of $\alpha^3+\beta^3$ is $\frac{10}{27}$. [3]

$$3x^2+2x+1=0$$

$$x^2+\frac{2}{3}x+\frac{1}{3}=0$$

$$\alpha+\beta=-\frac{2}{3}$$

$$\alpha\beta=\frac{1}{3}$$

– for obtaining sum and product of roots α and β.

$$\begin{aligned}&\alpha^3+\beta^3\\&=(\alpha+\beta)(\alpha^2-\alpha\beta+\beta^2)\\&=\frac{-2}{3}\left[(\alpha+\beta)^2-3\alpha\beta\right]\\&=\frac{-2}{3}\left[\left(\frac{-2}{3}\right)^2-1\right]\\&=\frac{-2}{3}\left(\frac{-5}{9}\right)\\&=\frac{10}{27}\end{aligned}$$

– for re-expressing $\alpha^3+\beta^3$ into $\alpha+\beta$ and $\alpha\beta$. Acceptable too:

$$\begin{aligned}&(\alpha+\beta)^3\\&=\alpha^3+3\alpha^2\beta+2\alpha\beta^2+\beta^3\\&=(\alpha+\beta)^3+3\alpha\beta(\alpha+\beta)\end{aligned}$$

(ii) Find a quadratic equation whose roots are $\frac{1}{\alpha^3}$ and $\frac{1}{\beta^3}$. [3]

$$\begin{aligned}&\frac{1}{\alpha^3}+\frac{1}{\beta^3}\\&=\frac{\beta^3+\alpha^3}{\alpha^3\beta^3}\\&=\frac{\frac{10}{27}}{\frac{1}{27}}\\&=10\end{aligned}$$

– for finding sum of roots

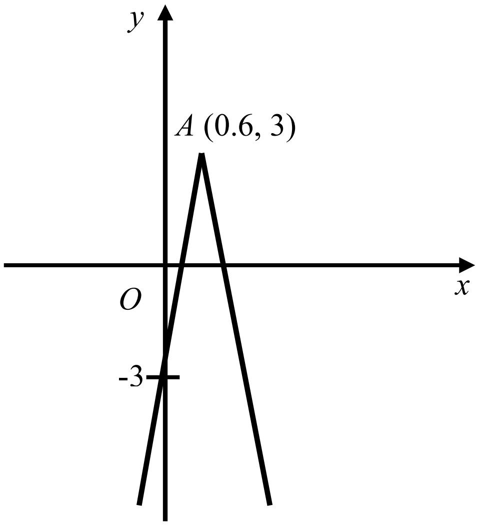
$$\frac{1}{\alpha^3}\left(\frac{1}{\beta^3}\right)$$
$$=\frac{1}{\alpha^3\beta^3}$$
$$=\frac{1}{\left(\frac{1}{3}\right)^3}$$

– for finding product of roots

$$=27$$

$$x^2-10x+27=0$$

4



The diagram shows part of the graph $y=r-q|3-px|$, where p, q and r are positive constants. The graph has a vertex at A (0.6, 3) and y-intercept of -3.

(i) Determine the values of p, q and r. [3]

At vertex,

$3-px=0$

$px=3$

$x=\frac{3}{p}$

$\frac{3}{p}=0.6$

$p=5$

$r=3$

At the y-intercept of -3, $x = 0$,

$-3 = 3 - q|3|$

$3q = 6$

$q = 2$

(ii) State the value or range of values of k such that $k = r - q|3 - px|$ has

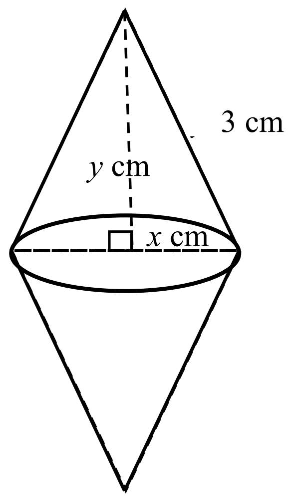
(a) 1 solution, [1]

$k = 3$

(b) 2 solutions. [1]

$k < 3$

5 A buoy is formed by two identical right circular cones of sheet iron joined by its bases with a radius of x cm. The buoy has a vertical height of y cm and a slant height of 3 cm.



(i) Express y in terms of x. [1]

$y^2 + x^2 = 3^2$

$y = \pm\sqrt{9 - x^2}$ (-ve rejected as vertical height is positive)

$y = \sqrt{9 - x^2}$

– applying Pythagoras' Theorem to obtain y in terms of x

(ii) Given that x can vary, find the exact value of x for which the volume, V, of the buoy is stationary. [4]

$$V=\frac{2}{3}\pi x^2 y$$

$$=\frac{2}{3}\pi x^2\sqrt{9-x^2}$$

-expressing volume of buoy in terms of x

$$\frac{dV}{dx}=\frac{2}{3}\pi x^2\left[\frac{1}{2}\left(9-x^2\right)^{\frac{-1}{2}}\left(-2x\right)\right]+\sqrt{9-x^2}\left(\frac{4\pi}{3}x\right)$$

-applying product rule to find dV/dx

$$=\frac{-2\pi x^3}{3\sqrt{9-x^2}}+\sqrt{9-x^2}\left(\frac{4\pi}{3}x\right)$$

$$=\frac{1}{3\sqrt{9-x^2}}\left[-2\pi x^3+4\pi x(9-x^2)\right]$$

$$=\frac{1}{3\sqrt{9-x^2}}\left[36\pi x-6\pi x^3\right]$$

At stationary point,

$$\frac{dV}{dx}=0$$

-identifying that dV/dx = 0 at stationary point

$$\frac{1}{3\sqrt{9-x^2}}\left[36\pi x-6\pi x^3\right]=0$$

$$\left[36\pi x-6\pi x^3\right]=0$$

$$x(6-x^2)=0$$

$$x=0, x=\pm\sqrt{6}$$

x=0 rejected;
-ve Rejected as radius x is positive

(iii) Determine with reasons whether this value of V is a maximum or minimum. [2]

Using first derivative test,

x	$\sqrt{6}^-$	$\sqrt{6}$	$\sqrt{6}^+$
$\frac{dV}{dx}$	>0	0	<0

Volume is maximum.

Using second derivative test,

$$\frac{d^2V}{dx^2}=\frac{2\pi}{3}\left[\frac{(18-9x^2)\sqrt{9-x^2}-(18x-3x^3)\frac{1}{2}\left(9-x^2\right)^{-\frac{1}{2}}(-2x)}{9-x^2}\right]$$

When

$$x=\sqrt{6}$$

$$\frac{d^2V}{dx^2}=-43.5<0$$

Volume is maximum

-either first or second derivative test – conclusion that volume is maximum

(iv) Find the exact surface area of the buoy when V is stationary, leaving your answer in terms of π. [1]

Surface area

$$= 2\pi(x)(3)$$
$$= 6\sqrt{6}\pi cm^2$$

6 The equation of a polynomial is given by $\mathrm{p}(x) = 2x^3 + 2ax^2 - x^2 + ax - a$ where a is a constant.

(i) Find the remainder when $\mathrm{p}(x)$ is divided by $(x + 1)$. [1]

$$p(-1)$$
$$= 2(-1)^3 + 2a(-1)^2 - (-1)^2 + a(-1) - a$$
$$= -2 + 2a - 1 - a - a$$
$$= -3$$

Remainder = -3

(ii) Show that $(2x – 1)$ is a factor of $\mathrm{p}(x)$. [2]

$$p\left(\frac{1}{2}\right)$$
$$= 2\left(\frac{1}{2}\right)^3 + 2a\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + a\left(\frac{1}{2}\right) - a$$
$$= \frac{1}{4} + \frac{a}{2} - \frac{1}{4} + \frac{a}{2} - a = 0$$

– for finding p(1/2) or using long division/synthetic division

Since remainder = 0, $(2x – 1)$ is a factor of $\mathrm{p}(x)$.

-conclusion stating remainder = 0, therefore (2x-1) is a factor of p(x).

(iii) By showing clearly your working, factorise $\mathrm{p}(x)$. [2]

	2	2a – 1	a	-a
0.5		1	a	a
	2	2a	2a	0

– Using synthetic division/long division

Alternatively, by long division

$$
\begin{array}{r}
x^2 + ax + a \\
2x-1 \overline{\smash{\big)}\, 2x^3 + (2a-1)x^2 + ax + a} \\
2x^3 - \quad x^2 \qquad\qquad \\
\hline
2ax^2 + ax \qquad \\
2ax^2 - ax \qquad \\
\hline
2ax + a \\
2ax - a
\end{array}
$$

$2x^3 + 2ax^2 - x^2 + ax - a = (2x-1)(x^2 + ax + a)$

(iv) Find the range of values of a for which the equation $\mathrm{p}(x) = 0$ has only one real root. [3]

$(2x-1)(x^2 + ax + a) = 0$

For p(x) = 0 to have only one real root,

$(a)^2 - 4(1)(a) < 0$

-apply discriminant < 0

$a^2 - 4a < 0$

– solve inequality by factorising

$a(a-4) < 0$

$0 < a < 4$

7 The table below shows the data obtained from an experiment on the vertical motion based on the oscillation of a spring with different masses attached to it.

Mass, x kg	0.02	0.03	0.04	0.05	0.15
Frequency of oscillations, y	16	13	11.4	10	6

It is known that the mass, x kg, and the frequency of oscillations per second, y, are related by the equation $xy^2 = k$, where k is a constant.

(a) Plot y^2 against $\frac{1}{x}$ and draw a straight line graph. [3]

$y^2 = \frac{k}{x}$

$\frac{1}{x}$	50	33.3	25	20	6.67
y^2	256	169	129.96	100	36

– table of values
– 5 plotted points
– straight line that passes through the points

(b) Use your graph to estimate

(i) the frequency of oscillations when a mass of 0.08 kg is attached to the spring, [1]

$x = 0.08$

$\frac{1}{x} = 12.5$

From graph, read off corresponding value of y^2
Acceptable range $60 < y^2 < 70$

$y^2 = 65$

$y = 8.06$

– Acceptable range $7.75 < y < 8.37$

(ii) the mass which produces 15 oscillations per second, [1]

$y = 15$

$y^2 = 225$

$\frac{1}{x} = 44$

From graph, read off corresponding value of $1/x$
Acceptable range $43 < 1/x < 45$

$x = 0.0227$

- Acceptable range $0.0233 < x < 0.0222$

(iii) the value of k. [1]

$k = \frac{230-0}{45-0} = 5.11$

– Acceptable range
$5 < k < 5.22$

(c) When the spring is replaced by a second spring, the relation between y and x is represented by $y^2 = \frac{2}{x} + 80$.

(i) On the same diagram, draw the line representing the second spring. [1]

[B1] – drawing the line, needs to pass through vertical intercept

y^2	80	130	180
1/x	0	25	50

(ii) Hence, explain how to find the mass which produces the same frequency of oscillations by both springs. [2]

Both the lines intersect at (25, 130).
Acceptable range of $1/x$-coord $24<1/x<26$; Acceptable range of y^2-coord $125 < y^2 < 135$

The intersection point indicates the mass that produces the same frequency of oscillations by both springs = 1/25 = 0.0400g

– Acceptable range
1/24 < mass < 1/26
0.0417g < mass < 0.0385g

8 **(i)** Prove that $\dfrac{\sec A+\tan A-1}{1-\sec A+\tan A}\equiv\dfrac{1+\sin A}{\cos A}$. [5]

LHS

$=\dfrac{\sec A+\tan A-(\sec^2 A-\tan^2 A)}{1-\sec A+\tan A}$ -using trigo identity $\sec^2 A-\tan^2 A=1$

$=\dfrac{\sec A+\tan A-(\sec A+\tan A)(\sec A-\tan A)}{1-\sec A+\tan A}$ -applying $a^2-b^2=(a+b)(a-b)$

$=\dfrac{(\sec A+\tan A)[1-(\sec A-\tan A)]}{1-\sec A+\tan A}$ -factoring out sec A + tan A

$=\sec A+\tan A$ -division of 1- sec A + tan A

$=\dfrac{1}{\cos A}+\dfrac{\sin A}{\cos A}$ -conversion of sec A & tan A in terms of sin A & cos A

$=\dfrac{1+\sin A}{\cos A}$

(ii) Hence solve the equation $\dfrac{\sec A+\tan A-1}{1-\sec A+\tan A}=3\cos A$ for $0<A<2\pi$. [5]

$\dfrac{1+\sin A}{\cos A}=3\cos A$ -use result from trigo identity in 8(i)

$1+\sin A=3\cos^2 A$

$1+\sin A=3(1-\sin^2 A)$ -use of identity $\sin^2 A+\cos^2 A=1$ to form quadratic equation in trigo function

$1+\sin A=3-3\sin^2 A$

$3\sin^2 A+\sin A-2=0$

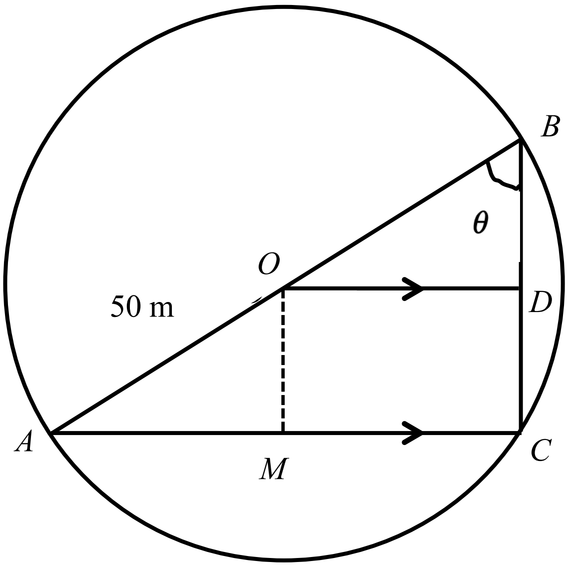
$(3\sin A-2)(\sin A+1)=0$

$\sin A=\dfrac{2}{3},\quad \sin A=-1$ -solve to obtain values for sin *A*

Basic angle = 0.72973

$A=0.730, 2.41\quad A=\dfrac{3\pi}{2}$

9



The diagram shows a circular field with centre O and radius 50 m.
A, B and C are points on the circumference of the field and angle $ABC = \theta$. D is a point on BC such that OD is parallel to AC.
The trapezium $AODC$ is the jogging path of a man.

(i) Explain why $BD = DC = 50\cos\theta$. [2]

Method 1

Angle BDO = Angle OMA = 90°
Angle OBD = Angle AOM = θ
Triangle OBD is similar to Triangle AOM.

– Identify similar triangles to use corresponding lengths

$$\frac{OA}{BO} = \frac{OM}{BD} = 1$$

$$OM = BD$$

Since $OM = DC$, $DC = BD$.
Using triangle OBD,

$$\cos\theta = \frac{BD}{50}$$

$$BD = 50\cos\theta$$

– use trigo ratios to find BD/BC

Method 2

Angle ACB = 90° (right angle in a semicircle)

$$\cos\theta = \frac{BC}{100}$$

$$BC = 100\cos\theta$$

-identifying right angle and applying trigo ratio to find BC

Angle ODB = 90° (corresponding angles since OD is parallel to AC)

$$\cos\theta = \frac{BD}{50}$$

-identifying corresponding right angle and applying trigo ratio to find BD

$$BD = 50\cos\theta$$
$$DC = 100\cos\theta - 50\cos\theta = 50\cos\theta$$
$$\therefore BD = DC$$

(ii) Show that the perimeter, L m, of the jogging path $AODC$ can be expressed in the form $p + q\cos\theta + r\sin\theta$, where p, q and r are constants to be found. [3]

$$\sin\theta = \frac{AC}{100}$$
$$AC = 100\sin\theta$$
$$\sin\theta = \frac{OD}{50}$$
$$OD = 50\sin\theta$$

– find AC using trigo ratio

– find OD using trigo ratio

$$L = 50 + 50\cos\theta + 100\sin\theta + 50\sin\theta$$
$$= 50 + 50\cos\theta + 150\sin\theta$$

(iii) Express L in the form of $p + R\cos(\theta - \alpha)$ where $R > 0$ and $0^\circ < \alpha < 90^\circ$. [3]

$$L = 50 + 50\cos\theta + 150\sin\theta$$
$$= 50 + \sqrt{50^2 + 150^2}\cos(\theta - \alpha)$$
$$= 50 + \sqrt{25000}\cos(\theta - 71.6^\circ)$$

$$\tan\alpha = \frac{150}{50} = 3$$
$$\alpha = 71.6^\circ$$

– Obtaining $\tan\alpha$
– Finding α

– Obtaining /158 (to 3sf)

(iv) Hence state the maximum perimeter, L m, of the jogging path $OACD$. Find the value of θ at which this occurs. [2]

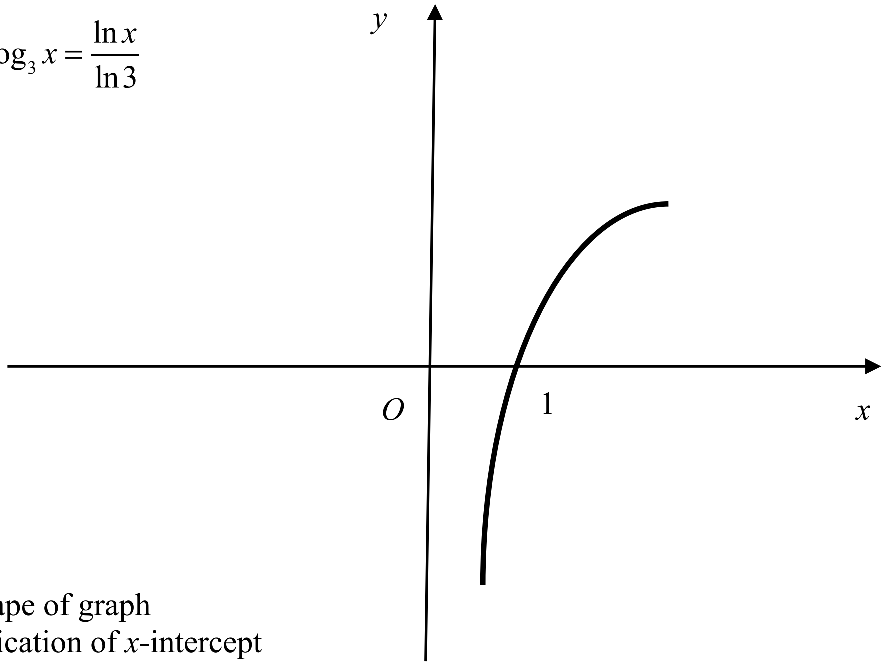
Maximum $L = 50 + \sqrt{25000} = 208$
Occurs when

$$\cos(\theta - 71.6^\circ) = 1$$
$$\theta - 71.6^\circ = 0^\circ$$
$$\theta = 71.6^\circ$$

-equating max of $\cos(\theta - 71.6^\circ) = 1$ to find θ.

10 **(a)** Sketch the graph of $y = \log_3 x$. [2]

$y = \log_3 x = \dfrac{\ln x}{\ln 3}$



– shape of graph
– indication of x-intercept

(b) Express $\log_9 4 + \log_3(x-4) = 2\log_3 x$ as a quadratic equation in x and explain why there are no real solutions. [4]

$\dfrac{\log_3 4}{\log_3 9} + \log_3(x-4) = \log_3 x^2$

– apply change of base and power law

$\dfrac{\log_3 4}{2} + \log_3(x-4) = \log_3 x^2$

$\log_3 4^{\frac{1}{2}} + \log_3(x-4) = \log_3 x^2$

$4^{\frac{1}{2}}\left(x-4\right) = x^2$

– apply product law

$2x - 8 = x^2$

$x^2 - 2x + 8 = 0$

– obtain quadratic equation

Discriminant

$= \left(-2\right)^2 - 4(1)(8)$

$= -28 < 0$

– explain why no real solutions using discriminant

There are no real solutions.

(c) Given that $\log_b(x^2y) = m$ and $\log_b(x^3y) = n$, express, $\log_b x$ and $\log_b y$ in terms of m and n. [4]

$\log_b\left(x^2y\right) = m$

$\log_b x^2 + \log_b y = m$

$2\log_b x + \log_b y = m - - - (1)$

- apply product and power laws to obtain equation (1)

$\log_b\left(x^3y\right) = n$

$\log_b x^3 + \log_b y = n$

$3\log_b x + \log_b y = n - - - (2)$

– apply product and power laws to obtain equation (2)

$(2) - (1)$

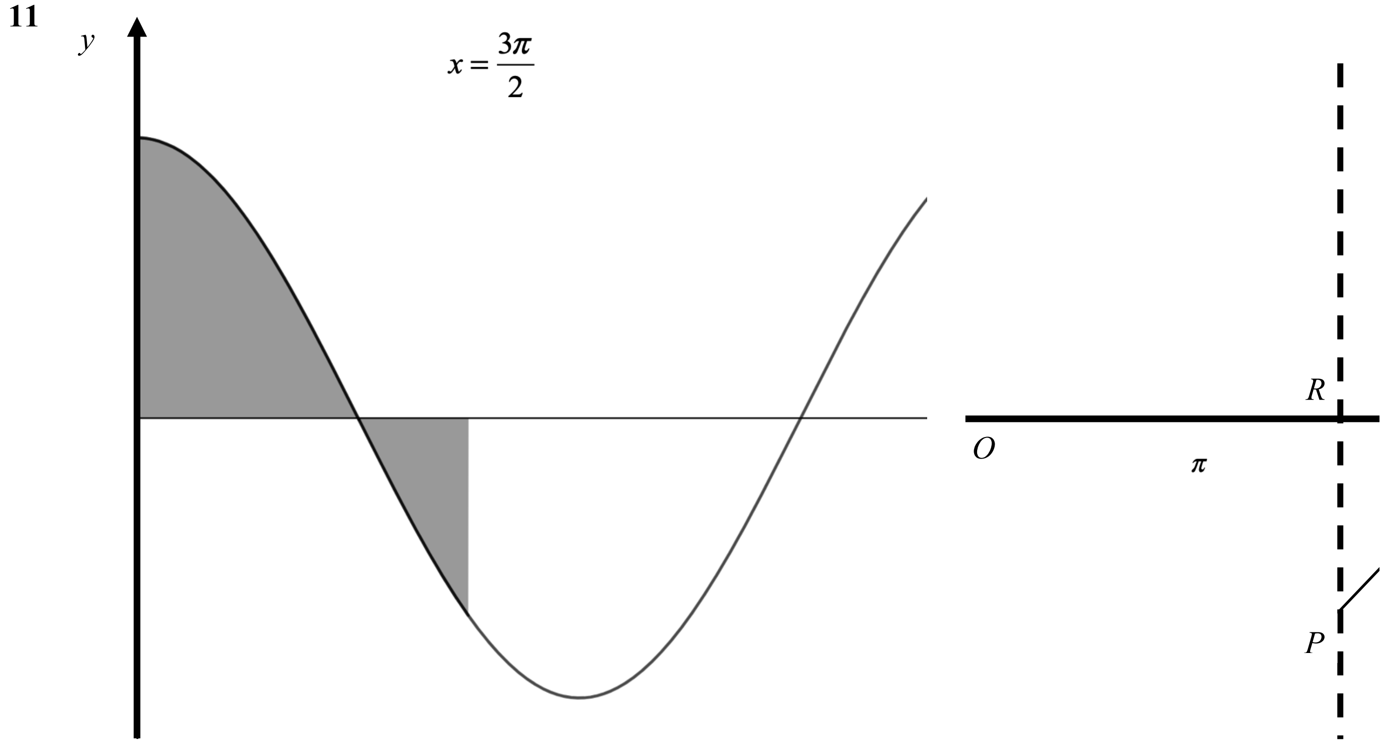
$\log_b x = n - m$

$\log_b y$

$= m - 2(n - m)$

$= 3m - 2n$

11



The diagram shows part of the curve $y = 4\cos\left(\frac{x}{2}\right)$ that meets the x-axis at $x = \pi$ and $x = 3\pi$. The line $x = \frac{3\pi}{2}$ meets the x-axis at R and the curve at P. The normal to the curve at P meets the x-axis at Q.

(i) Find the equation of the normal at P, expressing your answer in exact form. [4]

$$y = 4\cos\left(\frac{x}{2}\right)$$

$$\frac{dy}{dx} = -4\sin\left(\frac{x}{2}\right)\left(\frac{1}{2}\right) = -2\sin\left(\frac{x}{2}\right)$$

-Differentiating to find gradient of tangent

At $x = \frac{3\pi}{2}$,

gradient of tangent

$$= -2\sin\left(\frac{3\pi}{4}\right)$$

$$= -2\left(\frac{1}{\sqrt{2}}\right)$$

$$= -\sqrt{2}$$

Gradient of normal $= \frac{1}{\sqrt{2}}$

– evaluation of gradient of normal

Coordinate of P:

$$y = 4\cos\left(\frac{x}{2}\right)$$

$$= 4\cos\left(\frac{3\pi}{4}\right)$$

$$= 4\left(\frac{-\sqrt{2}}{2}\right)$$

$$= -2\sqrt{2}$$

$$P\left(\frac{3\pi}{2}, -2\sqrt{2}\right)$$

– finding coordinate of P

Equation of normal:

$$y - \left(-2\sqrt{2}\right) = \frac{1}{\sqrt{2}}\left(x - \frac{3\pi}{2}\right)$$

$$y = \frac{x}{\sqrt{2}} - \frac{3\pi}{2\sqrt{2}} - 2\sqrt{2}$$

– finding equation of normal

OR

$y = \frac{\sqrt{2}}{2}x - \frac{3\sqrt{2}}{4}\pi - \frac{4}{\sqrt{2}}$ (equivalent forms are acceptable)

(ii) Find the exact coordinates of Q. [2]

At $y = 0$,

$$\frac{x}{\sqrt{2}} - \frac{3\pi}{2\sqrt{2}} - 2\sqrt{2} = 0$$

$$\frac{x}{\sqrt{2}} = 2\sqrt{2} + \frac{3\pi}{2\sqrt{2}}$$

$$x = 4 + \frac{3\pi}{2}$$

$$Q(4 + \frac{3\pi}{2}, 0)$$

(iii) Find the exact area of the shaded region. [5]

Area

$$= \int_0^{\pi} 4\cos\left(\frac{x}{2}\right)dx + \left|\int_{\pi}^{\frac{3\pi}{2}} 4\cos\left(\frac{x}{2}\right)dx\right|$$

– Expression or function to represent area for $0 \le x \le \pi$
– Expression or function to represent area below x-axis

$$= \left[\frac{4\sin\left(\frac{x}{2}\right)}{\frac{1}{2}}\right]_0^{\pi} + \left|\left[\frac{4\sin\left(\frac{x}{2}\right)}{\frac{1}{2}}\right]_{\pi}^{\frac{3\pi}{2}}\right|$$

-Integration of trig function

$$= 8\sin\frac{\pi}{2} + \left|8\sin\frac{3\pi}{4} - 8\sin\frac{\pi}{2}\right|$$

– evaluation of limits

$$= 8 + \left|8\left(\frac{\sqrt{2}}{2}\right) - 8\right|$$

$$= 8 + \left|4\sqrt{2} - 8\right|$$

$$= 16 - 4\sqrt{2}units^2$$

– area in exact form

12 A circle, C_1 has equation $x^2 + y^2 + 8x - 12y + 16 = 0$.

(i) Find the radius and the coordinates of the centre of C_1. [3]

$$x^2 + 8x + y^2 - 12y + 16 = 0$$

$$(x+4)^2 - 16 + (y-6)^2 - 36 + 16 = 0$$

- completing the square

$$(x+4)^2 + (y-6)^2 = 6^2$$

Radius = 6
Centre = (-4, 6)

(ii) The lowest point on the circle is *A*. Explain why *A* lies on the *x*-axis. [1]
Since the circle has centre at (-4, 6) with radius 6, the lowest point on the circle is (-4, 0). Therefore, *A* lies on the *x*-axis.

A second circle, C_2 , has a diameter *PQ*. The point *P* has coordinates (-1, 3) and the equation of the tangent to C_2 at *Q* is $2y = x - 18$.

(iii) Find the equation of the diameter *PQ* and hence the coordinates of *Q*. [4]

$$2y = x - 18$$

$$y = \frac{x}{2} - 9$$

Gradient of tangent = ½
Gradient of diameter = -2

– find gradient of diameter PQ

Equation of diameter *PQ*:

$$y - 3 = -2(x+1)$$

$$y = -2x + 1$$

To find coord of *Q*, find intersection between equation of tangent and equation of diameter

$$-2x + 1 = \frac{x}{2} - 9$$

– find intersection between equation of tangent and equation of diameter

$$2.5x = 10$$

$$x = 4$$

$$y = -7$$

$$Q(4, -7)$$

(iv) Find the equation of the circle, C_2. [3]

Length of PQ

$$= \sqrt{\left(-1-4\right)^2 + (3+7)^2}$$

$$= \sqrt{25+100}$$

$$= \sqrt{125}$$

Radius $= \frac{\sqrt{125}}{2}$

– finding length of diameter to obtain radius

Midpoint of PQ

$$= \left(\frac{-1+4}{2}, \frac{3-7}{2}\right)$$

$$= \left(\frac{3}{2}, -2\right)$$

– finding center of circle

Equation of the circle, C_2,

$$\left(x-\frac{3}{2}\right)^2 + \left(y+2\right)^2 = \left(\frac{\sqrt{125}}{2}\right)^2$$

$$\left(x-\frac{3}{2}\right)^2 + \left(y+2\right)^2 = \frac{125}{4}$$

(v) Determine whether the circles C_1 and C_2 intersect each other. [2]

Distance between centre of circles C_1 and C_2

$$= \sqrt{\left(-4-\frac{3}{2}\right)^2 + \left(6+2\right)^2}$$

$$= 9.71$$

– computing distance between centre of circles and sum of radii

Sum of radii of circles C_1 and C_2

$$= \frac{\sqrt{125}}{2} + 6$$

$$= 11.6$$

Since distance between centre of circles C_1 and C_2 < sum of radii, both circles C_1 and C_2 intersect each other.

END OF PAPER