Solutions to Tutorial 6A: Integration Techniques

Basic Mastery Questions

Part 1: Standard Integrals

1a) $\int \frac{1}{2}x - \frac{2}{\sqrt{x}} - 1 \, dx = \frac{1}{2}\cdot\frac{x^2}{2} - \frac{2x^{\frac{1}{2}}}{\frac{1}{2}} - x + c$

$= \frac{x^2}{4} - 4\sqrt{x} - x + c$ #

1b) $\int \frac{1}{(1-3x)^4} \, dx = \frac{(1-3x)^{-3}}{9} + c$

$= \frac{1}{9(1-3x)^3} + C$ #

1c) $\int (2x^3+x)^2 \, dx = \int 4x^6 + 4x^4 + x^2 \, dx$

$= \frac{4}{7}x^7 + \frac{4}{5}x^5 + \frac{1}{3}x^3 + C$ #

1d) $\int 2\sec^2 x \, dx = 2\tan x + C$ #

1e) $\int \frac{3}{(2x+1)^3} + \sqrt{1-2x} \, dx = \frac{3(2x+1)^{-2}}{-4} + \frac{(1-2x)^{3/2}}{-3} + c$

$= -\frac{3}{4(2x+1)^2} - \frac{1}{3}(1-2x)^{3/2} + C$ #

1f) $\int 2\operatorname{cosec} 3x \cot 3x \, dx = 2\left(-\frac{\operatorname{cosec} 3x}{3}\right) + C$

$= -\frac{2}{3}\operatorname{cosec} 3x + C$ #

1g) $\int 5\sec\left(\frac{\pi}{4}-x\right)\tan\left(\frac{\pi}{4}-x\right) dx = \frac{5\sec\left(\frac{\pi}{4}-x\right)}{-1} + C$

$= -5\sec\left(\frac{\pi}{4}-x\right) + C$ #

1h) $\int e^{4-5x} \, dx = \frac{e^{4-5x}}{-5} + c$

$= -\frac{1}{5}e^{4-5x} + C$ #

Part 2 Integration by Substitution

2 a) $\displaystyle\int \frac{x}{(2x-1)^4}\,dx$, $u = 2x-1 \Rightarrow x = \frac{u+1}{2}$

$\frac{du}{dx} = 2$

$$= \int \frac{\frac{u+1}{2}}{u^4}\cdot\frac{1}{2}\,du$$

$$= \int \frac{1}{4}\left(\frac{u+1}{u^4}\right) du$$

$$= \frac{1}{4}\int \frac{1}{u^3} + \frac{1}{u^4}\,du$$

$$= \frac{1}{4}\left[\frac{u^{-2}}{-2} + \frac{u^{-3}}{-3}\right] + C$$

$$= -\frac{1}{8(2x-1)^2} - \frac{1}{12(2x-1)^3} + C \;\#$$

b) $\displaystyle\int \frac{1}{e^x+e^{-x}}\,dx$, $u = e^x$

$\frac{du}{dx} = e^x = u$

$$= \int \frac{1}{u+\frac{1}{u}}\cdot\frac{1}{u}\,du$$

$$= \int \frac{1}{u^2+1}\,du$$

$$= \tan^{-1}(u) + C$$

$$= \tan^{-1}(e^x) + C \;\#$$

Concept

For **part (c),** note that upon completion of integration wrt θ, we have answer in terms of $\theta, \sin\theta, \cos\theta$.
However, we should apply "right angled triangle" method to express them "back" **in terms of *x*** for the final answer.

c) $\displaystyle\int \sqrt{16-x^2}\,dx$, $x = 4\sin\theta$

$\frac{dx}{d\theta} = 4\cos\theta$

$$= \int \sqrt{16-16\sin^2\theta}\cdot 4\cos\theta\,d\theta$$

$$= \int 4\cos\theta\cdot 4\cos\theta\,d\theta$$

$$= 16\int \cos^2\theta\,d\theta$$

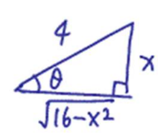
$*$ $\cos 2\theta = 2\cos^2\theta - 1$

$\cos^2\theta = \frac{\cos 2\theta + 1}{2}$

$$= 16\int \frac{\cos 2\theta + 1}{2}\,d\theta$$

$$= 8\left[\frac{\sin 2\theta}{2} + \theta\right] + C$$

$\sin\theta = \frac{x}{4}$



$$= 4\cdot 2\sin\theta\cos\theta + 8\theta + C$$

$$= 8\cdot\frac{x}{4}\cdot\frac{\sqrt{16-x^2}}{4} + 8\sin^{-1}\left(\frac{x}{4}\right) + C = \frac{x\sqrt{16-x^2}}{2} + 8\sin^{-1}\left(\frac{x}{4}\right) + C \;\#$$

Part 3 Integration by Parts

3a) $\int x^{\frac{1}{2}} \ln x \, dx$

$= \frac{2}{3}x^{\frac{3}{2}}\ln x - \frac{2}{3}\int x^{\frac{1}{2}} dx$

$= \frac{2}{3}x^{\frac{3}{2}}\ln x - \frac{2}{3}\left[\frac{2}{3}x^{\frac{3}{2}}\right] + C$

$= \frac{2}{3}x^{\frac{3}{2}}\ln x - \frac{4}{9}x^{\frac{3}{2}} + C$ #

$u = \ln x$ $\quad \frac{dv}{dx} = x^{\frac{1}{2}}$

$\frac{du}{dx} = \frac{1}{x}$ $\quad v = \frac{2}{3}x^{3/2}$

b) $\int x \cos\frac{x}{2} \, dx$

$= 2x\sin\frac{x}{2} - 2\int \sin\frac{x}{2} \, dx$

$= 2x\sin\frac{x}{2} - 2\left[\frac{-\cos x/2}{1/2}\right] + C$

$= 2x\sin\frac{x}{2} + 4\cos\frac{x}{2} + C$ #

$u = x$ $\quad \frac{dv}{dx} = \cos\frac{x}{2}$

$\frac{du}{dx} = 1$ $\quad v = 2\sin\frac{x}{2}$

c) $\int e^{-x}\sin x \, dx$

$= \int e^{-x}\cos x \, dx - e^{-x}\sin x$

$= \left[-e^{-x}\cos x - \int e^{-x}\sin x \, dx\right] - e^{-x}\sin x$

$\therefore$

$2\int e^{-x}\sin x \, dx = -e^{-x}\sin x - e^{-x}\cos x$

$\therefore \int e^{-x}\sin x \, dx = -\frac{1}{2}e^{-x}(\sin x + \cos x) + C$ #

$u = \sin x$ $\quad \frac{dv}{dx} = e^{-x}$

$\frac{du}{dx} = \cos x$ $\quad v = -e^{-x}$

$u = \cos x$ $\quad \frac{dv}{dx} = e^{-x}$

$\frac{du}{dx} = -\sin x$ $\quad v = -e^{-x}$

d) $\int \ln(x+2) \, dx$

$= x\ln(x+2) - \int \frac{x}{x+2} \, dx$

$= x\ln(x+2) - \int 1 - \frac{2}{x+2} \, dx$

$= x\ln(x+2) - \left[x - 2\ln(x+2)\right] + C$

$= x\ln(x+2) - x + 2\ln(x+2) + C$

$= (x+2)\ln(x+2) - x + C$ #

$u = \ln(x+2)$ $\quad \frac{dv}{dx} = 1$

$\frac{du}{dx} = \frac{1}{x+2}$ $\quad v = x$

$x + 2 > 0$

Concept

For integrating by parts, we may use the code **"LIATE"** as a guide in determining the order of function to differentiate:
L-Logarithmic, I-Inverse trigo,
A-Algebraic, T-Trigo, E-Exponential