Chapter 11: Maclaurin's Series & Binomial Expansions

Content Outline

- standard series expansion of $(1+x)^n$ for any rational n, e^x, $\sin x$, $\cos x$ and $\ln(1+x)$
- derivation of the first few terms of the Maclaurin's series by
  - repeated differentiation, e.g. $\sec x$
  - repeated implicit differentiation, e.g. $y^3 + y^2 + y = x^2 - 2x$
  - using standard series, e.g. $\mathrm{e}^x \cos 2x$, $\ln\left(\dfrac{1+x}{1-x}\right)$
- range of values of x for which a standard series converges
- concept of "approximation"
- small angle approximations: $\sin x \approx x$, $\cos x \approx 1 - \dfrac{1}{2}x^2$, $\tan x \approx x$

Exclude derivation of the general term of the series.

References

- **Websites**

  https://www.h2maths.site/

- **Books**

  (1) Ho Soo Thong, Tay Yong Chiang & Koh Khee Meng, "College Mathematics Syllabus C Volume 1", Pan Pacific Publications, Call Number: HO510

  (2) Pure Mathematics by Alan Sherlock, Elizabeth Roebuck, Timothy Heneage, Shirley Beck: Chapter 16

  (3) Pure Mathematics 2 by L Bostock, S Chandler: Chapter 5

Pre-requisites:

(1) Differentiation techniques including implicit differentiation (Chapter 6)

(2) O level knowledge: Binomial Theorem, Sine Rule, Cosine Rule, Inequalities

1. Introduction

A **power series** in x is a series of ascending powers of x of the form

$$\mathrm{f}(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n + \ldots$$

where $a_0, a_1, a_2, a_3, \ldots, a_n, \ldots$ are constants.

Note that a polynomial is a subset of a power series, as a polynomial has only **finite** number of terms with positive integer powers.

Examples of Power Series:

$2-4x$, $\quad 5+4x-x^2$, $\quad 3-x-5x^3$,

$\frac{1}{2}x^4+x^3+x^2-5x-10$, $\quad (1+x)^{-1}=1-x+x^2-x^3+\ldots$, $|x|<1$

Example of Non-power Series: $x^2-\frac{4}{x}+7x^{\frac{3}{2}}$

Reason: The second term contains a ______________________ of x, and

the third term contains a _______________________ for x.

In this chapter, we shall learn to approximate a function by using polynomials.

2. Maclaurin's Series

Key Result: $$\boxed{\mathrm{f}(x) = \mathrm{f}(0) + \mathrm{f}'(0)x + \frac{\mathrm{f}''(0)}{2!}x^2 + \frac{\mathrm{f}'''(0)}{3!}x^3 + \ldots + \frac{\mathrm{f}^{(n)}(0)}{n!}x^n + \ldots}$$

Note that $r! = 1\times2\times3\times\ldots\times(r-1)\times r$

Remarks: (1) The above formula is given in the formula list **MF26** as

$$\mathrm{f}(x) = \mathrm{f}(0) + x\,\mathrm{f}'(0) + \frac{x^2}{2!}\mathrm{f}''(0) + \ldots + \frac{x^n}{n!}\mathrm{f}^{(n)}(0) + \ldots$$

(2) The proof is in the Appendix.

Example 1

Find the Maclaurin's expansion for $\mathrm{f}(x)=\mathrm{e}^x$ up to and including the term in x^4.

Solution:

$\mathrm{f}(x) = \mathrm{e}^x$ $\quad \mathrm{f}'(x) =$ $\quad \mathrm{f}''(x) =$ $\quad \mathrm{f}'''(x) =$ $\quad \mathrm{f}^{(4)}(x) =$

$\mathrm{f}(0) =$ $\quad \mathrm{f}'(0) =$ $\quad \mathrm{f}''(0) =$ $\quad \mathrm{f}'''(0) =$ $\quad \mathrm{f}^{(4)}(0) =$

Therefore, $\mathrm{f}(x) =$ $\quad \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$

Remark: The Maclaurin's series may be used for approximations. In general, an approximation becomes better when more terms of the Maclaurin's series of $\mathrm{f}(x)$ are included as illustrated below.

Let $p_n(x)$ denote a polynomial of up to degree *n*.

By successively differentiating e^x, we find the Maclaurin's series expansion of e^x as a polynomial of

(i) degree 0, $p_0(x)$,

(ii) degree 1, $p_1(x)$,

(iii) degree 2, $p_2(x)$,

(iv) degree 3, $p_3(x)$.

$$\begin{aligned}
\text{Let } \mathrm{f}(x) &= \mathrm{e}^x && \Rightarrow \mathrm{f}(0) = 1 \\
\mathrm{f}'(x) &= \mathrm{e}^x && \Rightarrow \mathrm{f}'(0) = 1 \\
\mathrm{f}''(x) &= \mathrm{e}^x && \Rightarrow \mathrm{f}''(0) = 1 \\
\mathrm{f}'''(x) &= \mathrm{e}^x && \Rightarrow \mathrm{f}'''(0) = 1
\end{aligned}$$

From MF26,

$$\mathrm{f}(x) = \mathrm{f}(0) + x\,\mathrm{f}'(0) + \frac{x^2}{2!}\mathrm{f}''(0) + \frac{x^3}{3!}\mathrm{f}'''(0) + \ldots$$

$$\begin{aligned}
\therefore \quad \mathrm{e}^x &= 1 + x(1) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(1) + \ldots \\
&= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots
\end{aligned}$$

(i) $\mathrm{e}^x \approx p_0(x) =$

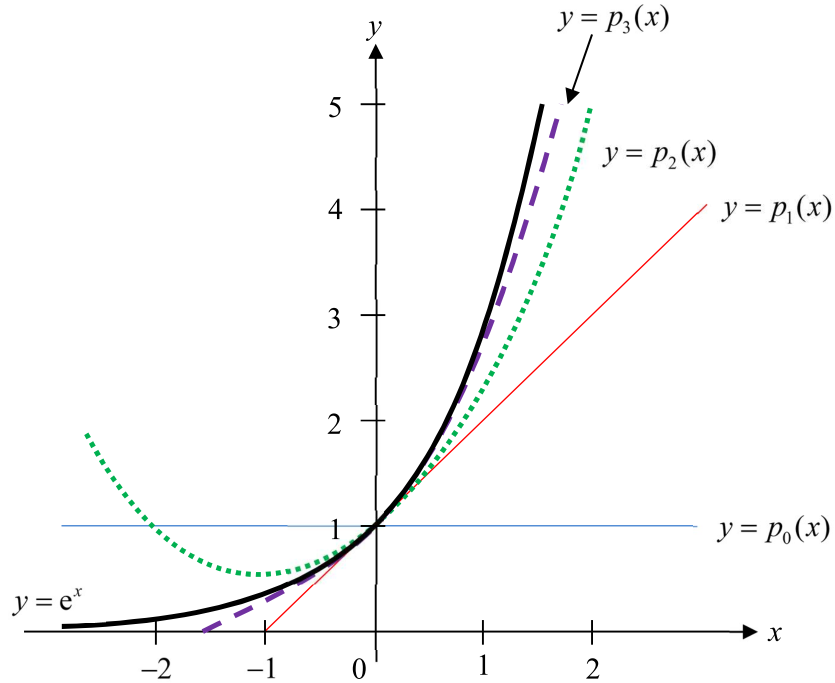
(ii) $\mathrm{e}^x \approx p_1(x) =$

(iii) $\mathrm{e}^x \approx p_2(x) =$

(iv) $\mathrm{e}^x \approx p_3(x) =$

Question: Which of the above polynomials would give us the best approximation for e^x?

We now use graphs to compare the accuracy of the approximations of e^x made by these four polynomials.



Compare the graph of $y = e^x$ and the graphs of the four polynomials. Note that the graphs of $y = p_1(x)$, $y = p_2(x)$ and $y = p_3(x)$ are virtually indistinguishable from the graph of $y = e^x$ near $x = 0$. So these polynomials are good approximations of e^x for values of x which are close to 0. However, the farther x is from 0, the poorer these approximations become.

For a larger interval of x from 0, the approximation made by $p_n(x)$ gets better as n gets larger. In other words, the approximation becomes better when the polynomial includes more terms of the Maclaurin's series of e^x.

Thus, ________ give us the best approximation for e^x.

Example 2

Find the first 3 non-zero terms in the Maclaurin's expansion for

(i) $\mathrm{f}(x) = \sin x$ (ii) $y = \ln(1 + x)$

Solution:

(i) $\mathrm{f}(x) = \sin x$ $\mathrm{f}(0) =$

$\mathrm{f}'(x) =$ $\mathrm{f}'(0) =$

$\mathrm{f}''(x) =$ $\mathrm{f}''(0) =$

$\mathrm{f}'''(x) =$ $\mathrm{f}'''(0) =$

$\mathrm{f}^{(4)}(x) =$ $\mathrm{f}^{(4)}(0) =$

$\mathrm{f}^{(5)}(x) =$ $\mathrm{f}^{(5)}(0) =$

Therefore, $\mathrm{f}(x) = 0 + x(1) + \dfrac{x^2}{2!}(0) + \dfrac{x^3}{3!}(-1) + \dfrac{x^4}{4!}(0) + \dfrac{x^5}{5!}(1) + \ldots$

$$\approx x - \frac{x^3}{6} + \frac{x^5}{120}$$

(ii) $y = \mathrm{f}(x) = \ln(1 + x)$ $\mathrm{f}(0) =$

$\mathrm{f}'(x) =$ $\mathrm{f}'(0) =$

$\mathrm{f}''(x) =$ $\mathrm{f}''(0) =$

$\mathrm{f}'''(x) =$ $\mathrm{f}'''(0) =$

Therefore, $y = 0 + x(1) + \dfrac{x^2}{2!}(-1) + \dfrac{x^3}{3!}(2) + \ldots$

$$\approx x - \frac{x^2}{2} + \frac{x^3}{3}$$

Example 3

Given that $y=\cos\left[\ln(1+x)\right]$, prove that

(i) $(1+x)\dfrac{\mathrm{d}y}{\mathrm{d}x}=-\sin\left[\ln(1+x)\right]$,

(ii) $(1+x)^2\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+(1+x)\dfrac{\mathrm{d}y}{\mathrm{d}x}+y=0$.

Obtain an equation relating $\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}$, $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x}$. Hence find Maclaurin's series for y, up to and including the term in x^3.

Solution

(i) Let $y=\cos\left[\ln(1+x)\right]$

Then $\dfrac{\mathrm{d}y}{\mathrm{d}x}=$

$\left(1+x\right)\dfrac{\mathrm{d}y}{\mathrm{d}x}=-\sin\left[\ln(1+x)\right]$ (shown)

(ii) Differentiate (i) w.r.t. x:

$\left(1+x\right)^2\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(1+x\right)\dfrac{\mathrm{d}y}{\mathrm{d}x}+y=0$ (shown) **<u>Note</u>:** $y=\cos\left[\ln(1+x)\right]$.

Differentiate (ii) wrt x :

$$\left(1+x\right)^2\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+3\left(1+x\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2\frac{\mathrm{d}y}{\mathrm{d}x}=0$$

When $x=0$, $y=1$, $\dfrac{\mathrm{d}y}{\mathrm{d}x}=0$, $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=-1$ and $\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}=3$

By Maclaurin's Series, $y=1-\dfrac{x^2}{2!}+\dfrac{3x^3}{3!}+\ldots$

$$\approx 1-\frac{x^2}{2}+\frac{x^3}{2}$$

3. Standard Series Expansions

The following are standard series expansions derived from the Maclaurin's series (discussed in Section 2) and the ranges of values of x for which the expansions are valid. They are found in MF26 and may be quoted without proof unless their derivation is asked for.

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots \qquad (|x| < 1)$$

$$\mathrm{e}^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^r}{r!} + \ldots \qquad (\text{all } x)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + \frac{(-1)^r x^{2r+1}}{(2r+1)!} + \ldots \qquad (\text{all } x)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + \frac{(-1)^r x^{2r}}{(2r)!} + \ldots \qquad (\text{all } x)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + \frac{(-1)^{r+1} x^r}{r} + \ldots \qquad (-1 < x \le 1)$$

Note that when we deal with the sum or product of standard Maclaurin's series, the range of values of x for which the expansion is valid is the **intersection of the ranges of values of *x* for which each standard Maclaurin's series used is valid**.

Example 4

By using the standard series, expand $y = \mathrm{e}^x \sin x$ as a series of ascending powers of x up to and including the term in x^3.

Solution

$y = \mathrm{e}^x \sin x$

$$= x - \frac{x^3}{3!} + x^2 + \frac{x^3}{2!} + \ldots$$

$$\approx x + x^2 + \frac{x^3}{2} - \frac{x^3}{6}$$

$$= x + x^2 + \frac{x^3}{3}$$

<u>Example 5</u>

Use the standard series expansions to expand $e^{-2x}\ln(2+x)$ in ascending powers of x, up to and including the term in x^3. State the range of values of x for which the expansion is valid.

Solution:

From MF26, we have $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^r}{r!} + \ldots$ and

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + \frac{(-1)^{r+1}x^r}{r} + \ldots$$

Now,

$$= 1 - 2x + 2x^2 - \frac{4}{3}x^3 + \ldots$$

Also, $\ln(2+x) =$

$\therefore \ln(2+x) =$

$$= \ln 2 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{24} + \ldots$$

$$\therefore \quad e^{-2x}\ln(2+x) = \left(1 - 2x + 2x^2 - \frac{4}{3}x^3 + \ldots\right)\left(\ln 2 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{24} + \ldots\right)$$

$$= \ln 2 + \left(\frac{1}{2} - 2\ln 2\right)x + \left(-\frac{9}{8} + 2\ln 2\right)x^2 + \left(\frac{31}{24} - \frac{4}{3}\ln 2\right)x^3 + \ldots$$

The expansion of e^{-2x} is valid for ____________.

Range of values of x for which the expansion of $\ln(2+x)$ is valid is

______________, i.e. ___________.

Hence, range of values of x for which the expansion of $e^{-2x}\ln(2+x)$ is valid is ____________.

Example 6

Using the Maclaurin's expansion, we obtained in Example 3, $\cos\left[\ln(1+x)\right]\approx 1-\frac{x^2}{2}+\frac{x^3}{2}$, verify that the same result is obtained if the standard series expansions for $\ln(1+x)$ and $\cos x$ are used.

Solution

$y=\cos\left[\ln(1+x)\right]$

$\approx$

$\approx$

$\approx 1-\frac{\left(x-\frac{x^2}{2}\right)^2}{2!}$ ($\frac{x^3}{3}$ can be ignored as question only requires up to x^3 term)

$=1-\frac{\left(x^2-x^3+...\right)}{2!}$

$\approx 1-\frac{x^2}{2}+\frac{x^3}{2}$ (verified)

<u>Example 7</u>

(i) Given that $y = \tan x$, show that $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 2y\dfrac{\mathrm{d}y}{\mathrm{d}x}$. Hence find Maclaurin's series for y, up to and including the term in x^3.

(ii) Using the standard series expansion for $\ln(1+x)$ and Maclaurin's series for y, find the series expansion of $\ln(1+\tan x)$, in ascending powers of x up to and including the term in x^3.

(iii) Hence show that the first three non-zero terms in the expansion of $\dfrac{\sec^2 x}{1+\tan x}$ are $1 - x + 2x^2$.

Solution:

(i) $y = \tan x$

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = \sec^2 x$

$\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 2y\dfrac{\mathrm{d}y}{\mathrm{d}x}$ (shown)

Differentiate $\boxed{\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 2y\dfrac{\mathrm{d}y}{\mathrm{d}x}}$ w.r.t. x:

$$= 2\left[y\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]$$

When $x = 0$, $y = 0$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 1$, $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 0$, $\dfrac{\mathrm{d}^3 y}{\mathrm{d}x^3} = 2$.

Maclaurin's series for y is $y = 0 + x(1) + \dfrac{x^2}{2!}(0) + \dfrac{x^3}{3!}(2) + \ldots \approx x + \dfrac{1}{3}x^3$

(ii) From MF26, $\ln(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3}$.

$\ln(1+\tan x) =$

$=$

$= x - \dfrac{1}{2}x^2 + \dfrac{2}{3}x^3 + \ldots$

(iii) From part (ii), $\ln(1+\tan x) = x - \dfrac{1}{2}x^2 + \dfrac{2}{3}x^3 + \ldots$

________________ throughout with respect to x, we have

(shown)

4. Series Expansion of $(1 + x)^n$, where n is a positive integer

Recall, if n is a **positive integer** ($n \in \mathbb{Z}^+$), we can use the following formula, known as **Binomial Theorem** to expand the terms:

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \ldots + \binom{n}{n}b^n,$$

where n is a positive integer and $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$

Important Points to Note:

- **This formula is given in MF 26**. It is used to expand $(a+b)^n$ where ***n* is a positive integer**, e.g. $(2+x)^5$, $(3x+2)^6$, $(1+y+y^2)^3$ etc.
- There are $(n + 1)$ terms in the expansion.
- The sum of the powers of a and b in each term is n. The power of a decreases from n to 0; the power of b increases from 0 to n.
- The $(r + 1)^{\text{th}}$ term is $\binom{n}{r}a^{n-r}b^r$.
- By symmetry, $\binom{n}{r} = \binom{n}{n-r}$. E.g. $\binom{5}{0} = \binom{5}{5} = 1$, $\binom{10}{2} = \binom{10}{8} = 45$.
- The expansion is **valid for all values of *a* and *b*** since it has finite number of terms.
- The expansion of $(a - b)^n$ can be obtained by **replacing *b* by –*b*.**

Example 8

Obtain the first 4 terms of $\left(3 - \frac{1}{2}x\right)^7$ in ascending powers of x.

Solution:

$$\left(3 - \frac{1}{2}x\right)^7 =$$

$$\approx 2187 - \frac{5103}{2}x + \frac{5103}{4}x^2 - \frac{2835}{8}x^3 \quad \text{(up to first 4 terms)}$$

Example 9

Expand $\left(1+x+x^2\right)^8$ in ascending powers of x, up to and including the term in x^3.

Solution:

$\left(1+x+x^2\right)^8 =$

$=$

$= 1 + 8\left(x+x^2\right) + 28\left(x^2+2x^3+...\right) + 56\left(x^3+...\right) + \ldots$ (up to x^3 terms)

$\approx 1 + 8x + 36x^2 + 112x^3$

4.1 Binomial Expansion of $(1 + x)^n$, where $n \in \mathbb{Q}$ and not a positive integer

Let $f(x) = (1+x)^n$. Then $f(0) = 1$, and

$$f'(x) = n(1+x)^{n-1} \qquad f'(0) = n$$
$$f''(x) = (n)(n-1)(1+x)^{n-2} \qquad f''(0) = n(n-1)$$
$$f'''(x) = (n)(n-1)(n-2)(1+x)^{n-3} \qquad f'''(0) = n(n-1)(n-2)$$

Maclaurin's series for $(1+x)^n$ is given by

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots$$
$$(1+x)^n = 1 + x(n) + \frac{x^2}{2!}\left[n(n-1)\right] + \frac{x^3}{3!}\left[n(n-1)(n-2)\right] + \ldots$$
$$= 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

When the **power *n* is a rational number which is not a positive integer**, the series expansion of $(1+x)^n$ called the **Binomial Expansion** of $(1+x)^n$ is given by

$$\boxed{(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots, \quad (|x|<1)}$$

Important Points to Note:

This formula is given in the MF 26 under Maclaurin's expansions. It is used to expand $(1+x)^n$ where n is a **rational number and not a positive integer**. e.g. $(1+x)^{-1}$, $\sqrt{1-3x}$, $\dfrac{2}{1+x^2}$ etc. It is an extension from the earlier formula where n is a **positive integer.**

- The **powers of *x* are ascending**.
- The term in x^r is given by $\dfrac{n(n-1)\ldots(n-r+1)}{r!}x^r$.
- There are an **infinite number of** terms (when the power $n \in \mathbb{Q}$ and not a positive integer).
- The expansion is **valid only for** $|x|<1$**,** i.e. $-1<x<1$
- To expand $(1+kx)^n$ in ascending powers of x, simply replace x by kx as follows:

$$(1+kx)^n = 1+n(kx)+\frac{n(n-1)}{2!}(kx)^2+\ldots+\frac{n(n-1)\ldots(n-r+1)}{r!}(kx)^r+\ldots$$

In this case, the expansion is **valid for** $|kx|<1 \Rightarrow |x|<\dfrac{1}{|k|}$**,** i.e. $-\dfrac{1}{|k|}<x<\dfrac{1}{|k|}$**.**

Example 10

Expand the following expressions as a series in ascending powers of x, up to and including the term in x^3. State the set of values of x for which the expansion is valid.

(a) $\dfrac{1}{1+x}$, (b) $(1+2x)^{-\frac{1}{2}}$ (c) $\dfrac{1}{(2+x)^2}$

Solution:

(a) $\dfrac{1}{1+x}=(1+x)^{-1}=$

$$\approx 1-x+x^2-x^3$$

The expansion is valid for _________, ∴ ____________

(b) $(1+2x)^{-\frac{1}{2}}=$

$$\approx 1-x+\frac{3}{2}x^2-\frac{5}{2}x^3$$

The expansion is valid for _______________, ∴ _____________.

(c) $\dfrac{1}{(2+x)^2} = (2+x)^{-2} =$

$=$

$\approx \dfrac{1}{4}\left(1 - x + \dfrac{3}{4}x^2 - \dfrac{1}{2}x^3\right)$

The expansion is valid for _______________, $\therefore$ __________

Example 11

Expand $\dfrac{4-x}{\sqrt{1-2x^2}}$ as a series of ascending powers of x up to and including the term in x^3.
State the range of values of x for which the expansion is valid.

Solution:

$$\frac{4-x}{\sqrt{1-2x^2}} = (4-x)(1-2x^2)^{-\frac{1}{2}}$$

$$= (4-x)\left(1 + \left(-\frac{1}{2}\right)(-2x^2) + \ldots\right)$$

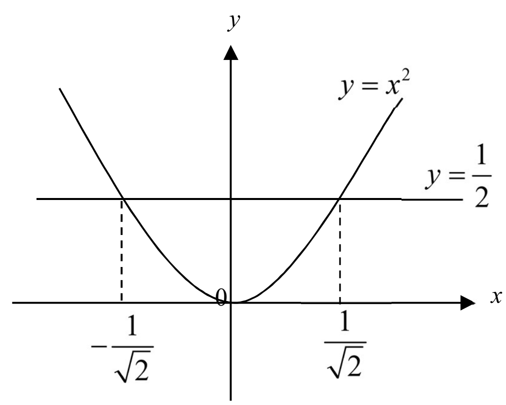
$$\approx (4-x)\left(1+x^2\right)$$

$$= 4 - x + 4x^2 - x^3$$

The expansion is valid for ________

$$\Rightarrow \left|x^2\right| < \frac{1}{2}$$

$$\Rightarrow x^2 < \frac{1}{2}$$



Hence, _______________.

Note: $(4-x)$ is valid for all x.

Example 12

It is given that $y=\left(2-\mathrm{e}^{2x}\right)^{\frac{1}{4}}$, where $y \neq 0$.

(i) Show that $y\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}+3\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2=2y\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

(ii) Hence find the Maclaurin's series for *y*, up to and including the term in x^2.

(iii) Given that the first two terms in the Maclaurin's series for *y* are equal to the first two terms in the series expansion of $\dfrac{1}{a+bx}$, where *a* and *b* are constants, find *a* and *b*.

Solution:

(i) $y=\left(2-\mathrm{e}^{2x}\right)^{\frac{1}{4}}$

----- (1)

Differentiating (1) w.r.t. x

$4y^3\dfrac{\mathrm{d}y}{\mathrm{d}x}=-2\mathrm{e}^{2x}=$

----- (2)

Differentiating (2) w.r.t. x

$$4y^3\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}+12y^2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=8y^3\frac{\mathrm{d}y}{\mathrm{d}x}$$

Since $y \neq 0$, $y\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}+3\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2=2y\dfrac{\mathrm{d}y}{\mathrm{d}x}$ (shown)

(ii) When $x=0$, $y=1$, $\dfrac{\mathrm{d}y}{\mathrm{d}x}=-\dfrac{1}{2}$, $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}=-\dfrac{7}{4}$.

The Maclaurin's series for y is

$$y=1+x\left(-\frac{1}{2}\right)+\frac{x^2}{2}\left(-\frac{7}{4}\right)+\ldots \quad \approx 1-\frac{1}{2}x-\frac{7}{8}x^2$$

(iii) $\dfrac{1}{a+bx}=$

$$=\frac{1}{a}\left(1+(-1)\frac{b}{a}x+\frac{(-1)(-2)}{2!}\left(\frac{b}{a}x\right)^2+\dots\right)$$

$$=\frac{1}{a}-\frac{b}{a^2}x+\frac{b^2}{a^3}x^2+\dots \qquad \text{----- (3)}$$

From (ii), $y=1-\frac{1}{2}x-\frac{7}{8}x^2+\dots.$ ----- (4)

Comparing the coefficients of the first two terms of series (3) and (4), we have

Therefore, $a=1$ and $b=\frac{1}{2}$.

4.2 <u>Expansions Involving Two or More Binomial Expansions</u>

When the sum or product of binomial expansions are involved, the range of values of x for which the expansion is valid is the <u>**intersection**</u> **of the ranges of values of x for which each binomial expansion used is valid**.

<u>Example 13</u>

(i) Express $\mathrm{f}(x)=\dfrac{1+x}{(1-x)(2+x)}$ in partial fractions.

(ii) Given that x is sufficiently small for x^3 and higher powers of x to be neglected, show that $\mathrm{f}(x)\approx\frac{1}{2}+\frac{3}{4}x+\frac{5}{8}x^2$.

(iii) State the set of values of x for which the expansion is valid. **(N76/P2/Q1)**

Solution:

(i) $\mathrm{f}(x)=\dfrac{1+x}{(1-x)(2+x)}=$

So,

Subs. $x=-2$, $-1=3B$ $\Rightarrow B=-\frac{1}{3}$

Subs. $x=1$, $2=3A$ $\Rightarrow A=\frac{2}{3}$

Hence, $\mathrm{f}(x)=\dfrac{1+x}{(1-x)(2+x)}=\dfrac{2}{3(1-x)}-\dfrac{1}{3(2+x)}$

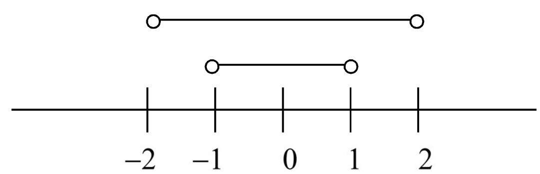
(ii) $$(1-x)^{-1} = 1 + (-1)(-x) + \frac{(-1)(-2)}{2!}(-x)^2 + \ldots$$
$$\approx 1 + x + x^2$$
$$(2+x)^{-1} = (2)^{-1}\left(1+\frac{x}{2}\right)^{-1}$$
$$= \frac{1}{2}\left[1 + (-1)\left(\frac{x}{2}\right) + \frac{(-1)(-2)}{2!}\left(\frac{x}{2}\right)^2 + \ldots\right]$$
$$\approx \frac{1}{2} - \frac{1}{4}x + \frac{1}{8}x^2$$
$$\mathrm{f}(x) = \frac{2}{3(1-x)} - \frac{1}{3(2+x)}$$
$$= \frac{2}{3}(1-x)^{-1} - \frac{1}{3}(2+x)^{-1}$$
$$\approx \frac{2}{3}(1+x+x^2) - \frac{1}{3}\left(\frac{1}{2} - \frac{1}{4}x + \frac{1}{8}x^2\right)$$
$$= \frac{1}{2} + \frac{3}{4}x + \frac{5}{8}x^2 \quad \text{(Shown)}$$

(iii) For the expansion to be valid,

and

$\Rightarrow$ **and**

Hence, __________.



Partial Fractions Decomposition (Given in MF26)

$$\frac{px+q}{(ax+b)(cx+d)} = \frac{A}{(ax+b)} + \frac{B}{(cx+d)}$$
$$\frac{px^2+qx+r}{(ax+b)(cx+d)^2} = \frac{A}{(ax+b)} + \frac{B}{(cx+d)} + \frac{C}{(cx+d)^2}$$
$$\frac{px^2+qx+r}{(ax+b)(x^2+c^2)} = \frac{A}{(ax+b)} + \frac{Bx+C}{(x^2+c^2)}$$

Example 14

Express $\mathrm{f}(x) = \dfrac{2x^2+x+2}{(2+x)(1+x^2)}$ in partial fractions.

Expand f(x) in ascending powers of x, up to and including the term in x^3.

State the set of values of x for which the expansion is valid.

Hence estimate the value of $\int_{-0.1}^{0.1} \mathrm{f}(x)\,\mathrm{d}x$.

Solution:

$$\mathrm{f}(x) = \frac{2x^2+x+2}{(2+x)(1+x^2)} =$$

So,

Subs. $x = -2,\ \ A = \dfrac{8}{5}$

Subs. $x = 0,\ C = \dfrac{1}{5}$

Subs. $x = 1,\ B = \dfrac{2}{5}$

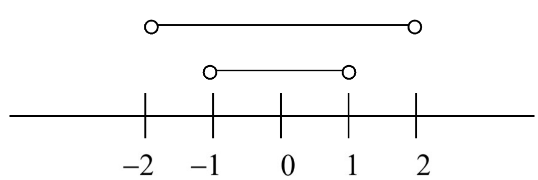
Hence, $\mathrm{f}(x) = \dfrac{2x^2+x+2}{(1+2x)(1+x^2)} = \dfrac{8}{5(2+x)} + \dfrac{2x+1}{5(1+x^2)}$

$$\begin{aligned}
\mathrm{f}(x) &= \frac{8}{5(2+x)} + \frac{2x+1}{5(1+x^2)} \\
&= \frac{8}{5}(2+x)^{-1} + \frac{1}{5}(2x+1)(1+x^2)^{-1} \\
&= \frac{8}{5}(2)^{-1}\left(1+\frac{x}{2}\right)^{-1} + \frac{1}{5}(2x+1)(1+x^2)^{-1} \\
&= \frac{4}{5}\left(1+(-1)\frac{x}{2} + \frac{(-1)(-2)}{2!}\left(\frac{x}{2}\right)^2 + \frac{(-1)(-2)(-3)}{3!}\left(\frac{x}{2}\right)^3 + \dots\right) + \frac{1}{5}(2x+1)\left(1+(-1)x^2+\dots\right) \\
&\approx \frac{4}{5} - \frac{2x}{5} + \frac{x^2}{5} - \frac{x^3}{10} + \frac{2x}{5} - \frac{2x^3}{5} + \frac{1}{5} - \frac{x^2}{5} \\
&= 1 - \frac{1}{2}x^3
\end{aligned}$$

For expansion to be valid,

and

and



Hence, _________.

Now, $\int_{-0.1}^{0.1} \mathrm{f}(x)\,\mathrm{d}x \approx$ $\qquad = \left[x - \dfrac{1}{8}x^4\right]_{-0.1}^{0.1} = 0.2$

5. Approximations

5.1 Approximations using Binomial Expansion

The binomial expansion of $(1+x)^n$, in ascending powers of x,

$$\boxed{(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots} \quad (|x|<1)$$

can be used for approximations when appropriate values of x are chosen. The value of x chosen must fall within the range of values of x for which the expansion is valid.

The idea of approximation is that when the value of x is close to zero, higher powers of x will have values that are very small/negligible and can be neglected. Under this situation, we will perform approximation using the first few terms of the binomial expansion.

Example 15

Show that, if x is sufficiently small for x^4 and higher powers to be neglected, then the expansion of $(1-x)^{\frac{1}{2}}$ in ascending powers of x is $1-\frac{1}{2}x-\frac{1}{8}x^2-\frac{1}{16}x^3$. By taking $x=\frac{1}{64}$, use this expansion to estimate $\sqrt{7}$ correct to five decimal places.

The substitution $x=-\frac{3}{4}$ can also be used to estimate $\sqrt{7}$. Explain whether this will give a better approximation than using $x=\frac{1}{64}$.

Solution:

$$(1-x)^{\frac{1}{2}} = 1+\frac{1}{2}(-x)+\frac{(\frac{1}{2})(-\frac{1}{2})}{2!}(-x)^2+\frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3!}(-x)^3+...$$

$$\approx 1-\frac{1}{2}x-\frac{1}{8}x^2-\frac{1}{16}x^3 \quad \text{(shown)}$$

Substituting $x=\frac{1}{64}$ into the above expansion,

$$\left(1-\frac{1}{64}\right)^{\frac{1}{2}} \approx 1-\frac{1}{2}\left(\frac{1}{64}\right)-\frac{1}{8}\left(\frac{1}{64}\right)^2-\frac{1}{16}\left(\frac{1}{64}\right)^3$$

$$\left(\frac{63}{64}\right)^{\frac{1}{2}} =$$

$$\therefore\ \sqrt{7} \approx \frac{8}{3}(0.9921567) = 2.64575 \text{ (to 5 d.p.)}$$

Since $x=\frac{1}{64}$ is closer to zero than $x=-\frac{3}{4}$, the approximated value when substituting $x=\frac{1}{64}$ will be closer to the actual value of $\sqrt{7}$ compared to substituting with $x=-\frac{3}{4}$.

Remark: When multiple values of x can be used for approximation, as a general rule of thumb, the one with a value closest to zero will give the best approximate.

5.2 Approximations using Maclaurin's Series

The Maclaurin's series may also be used for approximations. In general, an approximation becomes better when more terms of the Maclaurin's series of $\mathrm{f}(x)$ are included as illustrated in Example 1.

5.3 Small Angle Approximations

We have derived the Maclaurin's series of sin x and cos x:

Maclaurin's series of $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - .... + \frac{(-1)^r x^{2r+1}}{(2r+1)!} + ...$ all x (in rad)

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - .... + \frac{(-1)^r x^{2r}}{(2r)!} + ...$ all x (in rad)

If x is small (close to 0) and measured in radians, then

	up to the term in x^2
$\sin x$	$\approx x$
$\cos x$	$\approx 1 - \frac{x^2}{2}$
$\tan x$	$\approx x$

Taken from the series expansion of $\sin x$ and $\cos x$ in MF26.

Not in MF26 !
Must Remember !

Remarks:

1. When x is sufficiently small, such that nx is also small, we have:

$$\sin 2x \approx 2x, \quad \sin 3x \approx 3x, \quad \cos 2x \approx 1 - \frac{(2x)^2}{2}, \quad \tan \frac{1}{2}x \approx \frac{1}{2}x .$$

2. If α is not small, we cannot assume that $(x \pm \alpha)$ is small (even when x is small).

   Hence the small angle approximation cannot be applied here.

   e.g. $\sin\left(x + \frac{\pi}{6}\right)$ is **not** approximately equal to $\left(x + \frac{\pi}{6}\right)$.

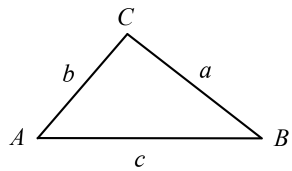
   In these cases, we need to use the compound angle formulae (given in MF26):

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

   to expand $\sin(x \pm \alpha)$, $\cos(x \pm \alpha)$ and $\tan(x \pm \alpha)$ first before applying the small angle formulae.

3. We may use the Sine Rule and Cosine Rule to solve problems:

For any triangle ABC,



(i) **Sine Rule**:

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$

(ii) **Cosine Rule**:

$$a^2=b^2+c^2-2bc\cos A$$

Example 16

Given that θ (measured in radians) is small enough for θ^3 and higher powers of θ to be ignored, express

(i) $\tan\left(\theta+\frac{\pi}{3}\right)$ as a quadratic expression in θ,

(ii) $\dfrac{\sin 3\theta}{1+\cos 2\theta}$ as a linear expression in θ.

Solution:

(i) $\tan\left(\theta+\frac{\pi}{3}\right)=$

$$\begin{aligned}
&=\left(\theta+\sqrt{3}\right)\left[1+(-1)(-\sqrt{3}\theta)+\frac{(-1)(-2)}{2!}\left(-\sqrt{3}\theta\right)^2+\ldots\right]\\
&=\left(\theta+\sqrt{3}\right)\left(1+\sqrt{3}\theta+3\theta^2+\ldots\right)\\
&=\theta+\sqrt{3}\theta^2+\sqrt{3}+3\theta+3\sqrt{3}\theta^2+\ldots\\
&\approx\sqrt{3}+4\theta+4\sqrt{3}\theta^2
\end{aligned}$$

(ii) $\dfrac{\sin 3\theta}{1+\cos 2\theta}$

$$=\frac{3\theta}{2\left(1-\theta^2\right)}$$
$$=\frac{3\theta}{2}\left(1-\theta^2\right)^{-1}$$

Example 17

Given that $\cos x-4\sin x=6x$. If x is sufficiently small for x^3 and higher powers of x to be ignored, find an approximate value for x.

Solution:

When x is small, $\cos x-4\sin x=6x$ can be approximated by

$$\therefore \quad x^2+20x-2\approx 0$$

$$\approx -20.099504,\ 0.099504$$

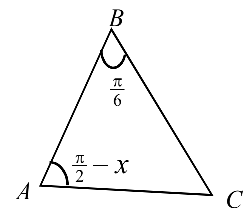
Example 18

In a triangle ABC, angle B is $\frac{\pi}{6}$ radians and angle A is almost a right angle, so that $A = \left(\frac{\pi}{2} - x\right)$ radians, where x is small. Show that $\frac{AC}{BC} \approx \frac{1}{2} + \frac{x^2}{4}$.

Solution

Using Sine Rule, $\frac{AC}{\sin B} = \frac{BC}{\sin A}$

$$\Rightarrow \frac{AC}{BC} = \frac{\sin\frac{\pi}{6}}{\sin\left(\frac{\pi}{2} - x\right)}$$



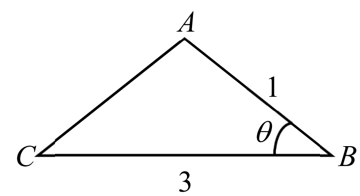
$$= \frac{1}{2}\left(1 + (-1)\left(-\frac{x^2}{2}\right) + ...\right) = \frac{1}{2}\left(1 + \frac{x^2}{2} + ...\right) \approx \frac{1}{2} + \frac{x^2}{4}$$

Example 19

In the triangle *ABC*, *AB* = 1, *BC* = 3 and angle *ABC* = θ radians. Given that θ is a sufficiently small angle, show that $AC \approx (4 + 3\theta^2)^{\frac{1}{2}} \approx a + b\theta^2$, for constants *a* and *b* to be determined.

Solution:

By cosine rule,



$AC^2 = 10 - 6\cos\theta$

$\therefore AC \approx (4 + 3\theta^2)^{\frac{1}{2}}$ (Shown)

$AC =$

$= 2\left[1 + \frac{1}{2}\left(\frac{3\theta^2}{4}\right) + ...\right]$

$= 2 + \frac{3\theta^2}{4} + \ldots$

$\therefore a = 2, \ b = \frac{3}{4}$

Appendix (Proof of Maclaurin's series key result)

A **Maclaurin's series** in x is a series of ascending powers of x of the form

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n + \ldots$$

where $a_0, a_1, a_2, a_3, \ldots, a_n, \ldots$ are constants.

If $f(x)$ is differentiable and can be expressed as a series in ascending powers of x, then

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots + a_nx^n + \ldots, \text{ ——— (I)}$$

where $a_0,\ a_1,\ a_2,\ a_3,\ a_4,\ \ldots,\ a_n,\ \ldots$ are constants to be found.

When $x = 0$, $a_0 = f(0)$

Differentiate (I) with respect to x,

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 + \ldots + na_nx^{n-1} + \ldots \qquad \text{——— (II)}$$

When $x = 0$, $a_1 = f'(0)$

Differentiate (II) with respect to x,

$$f''(x) = 2a_2 + 3(2)a_3x + \ldots + n(n-1)a_nx^{n-2} + \ldots \qquad \text{——— (III)}$$

When $x = 0$, $2a_2 = f''(0)$

$$a_2 = \frac{f''(0)}{2} = \frac{f''(0)}{2!}$$

Differentiate (III) with respect to x,

$$f'''(x) = 3(2)a_3 + \ldots + n(n-1)(n-2)a_nx^{n-3} + \ldots \qquad \text{——— (IV)}$$

When $x = 0$, $3(2)a_3 = f'''(0)$

$$a_3 = = \frac{f'''(0)}{(3)(2)} = \frac{f'''(0)}{3!}$$

By differentiating again and again, we will get in general,

$$\boxed{a_n = \frac{f^{(n)}(0)}{n!}}$$

Substituting these values of $a_0, a_1, a_2, a_3, \ldots, a_n, \ldots$ into (I), we have

$$\boxed{f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \ldots}$$