H2 Topic 06 – Motion In a Circle

We often experience the effects of centripetal ("centre-seeking") motion on amusement rides. Thrill-seekers can opt for vertical loops in fast roller coasters, while those looking to chill can opt for carousel rides.

Content:

- Kinematics of uniform circular motion
- Centripetal acceleration
- Centripetal force

Learning Objectives:

Candidates should be able to:

(a) express angular displacement in radians

(b) show an understanding of and use the concept of angular velocity to solve problems.

(c) recall and use $v = r\omega$ to solve problems.

(d) describe qualitatively motion in a curved path due to a perpendicular force, and understand the centripetal acceleration in the case of uniform motion in a circle.

(e) recall and use centripetal acceleration $a = r\omega^2$ and $a = \frac{v^2}{r}$ to solve problems.

(f) recall and use centripetal force $F = mr\omega^2$ and $F = \frac{mv^2}{r}$ to solve problems.

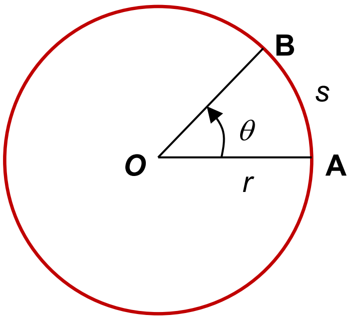
6.0 Introduction

So far, our analysis has been limited to that for *linear* motion. Here we will work with a simplified view of *rotational* motion. Nonetheless, we will be able to explain the workings of a centrifuge, the need for inclination of the road on high speed race tracks, the reason behind the thrill-factor of a roller coaster ride, and even the possibility of simulating gravity in space.

6.1 Kinematics of Uniform Circular Motion

6.1.1 Angular Displacement θ

Consider an object moving in a typical anti-clockwise circular path of radius r from **A** to **B**.

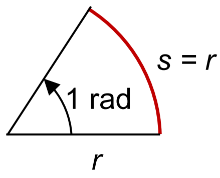


Legend
s = arc length
r = radius

Angular displacement θ is
the angle swept out by a radius.

$$\theta = \frac{s}{r}$$
(or $s = r\theta$)

θ : angular displacement (rad)
s : arc length (m)
r : radius (m)



π rad = 180°

Since $\theta = \frac{s}{r}$ is a ratio, angular displacement θ is technically dimensionless, though we refer to values of angular displacement as having a "unit" of radian:

One **radian** is the

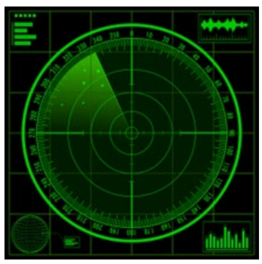
angle subtended at the centre of a circle by an arc length that is equal to the radius

6.1.2 Angular Velocity ω

Angular velocity ω is

the rate of change of angular displacement swept out by a radius

$$\omega = \frac{\mathrm{d}\theta}{\mathrm{d}t}$$



A screen for the read-out of a radar. *Quite often seen in popular culture such as movies, the radar readout screen, where a radius sweeps out and reveals "enemies", is nonetheless a useful image to base your understanding of angular velocity on.*

ω : angular velocity (rad s^{-1})

$\frac{\mathrm{d}\theta}{\mathrm{d}t}$: rate of angular displacement swept out by radius (rad s^{-1})

In an analogous sense, the angular *speed* of an object is the rate of change of *angle* (no mention of "displacement") swept out by a radius.s

6.1.3 Uniform Circular Motion

Uniform circular motion occurs when an object moves round a circle with the **same speed**. The time it takes to complete one revolution is called the *period* T within which the angular velocity is $\omega = \dfrac{d\theta}{dt} = \dfrac{2\pi}{T} = 2\pi f$. f is the frequency of the rotation: the number of rotations per unit time, $T = \dfrac{1}{f}$.

$$\omega = \frac{2\pi}{T} = 2\pi f$$

6.1.4 Relationship between Angular Velocity ω and Linear Velocity v

rate of change with respect to time↓

quantity	linear	angular	relation
displacement	s	θ	$s = r\theta$
velocity	v	ω	$v = r\omega$

An object in uniform circular motion travels a full circumference in a period:

$$v = \frac{\text{circumference}}{\text{period}} = \frac{2\pi r}{T} = r\frac{2\pi}{T} = r\omega$$

$$v = r\omega$$

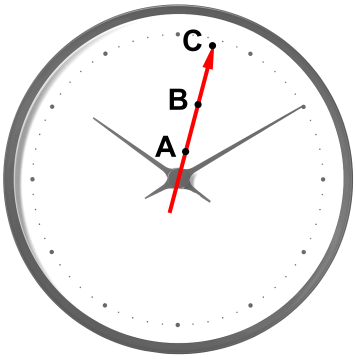
v : linear velocity *or* tangential velocity (m s^{-1})
r : radius of the circular path (m)
ω : angular velocity (rad s^{-1})

Example 1

Points A, B and C lie on the second hand of an analogue watch.

(a) State and explain the point which has the largest
  (i) angular velocity ω,
  (ii) linear speed v.

(b) Point **C** is 2.0 cm from the centre of the face, calculate the
  (i) angular velocity of point **C**,
  (ii) linear velocity of point **C**.



Solution

(a)(i) all 3 points have same angular velocity as they lie on the same radius and is swept through the same rate of change of angular displacement

(a)(ii) $v = r\omega$ so linear velocity is directly proportional to distance from centre of circular motion. C is furthest away so has largest linear velocity

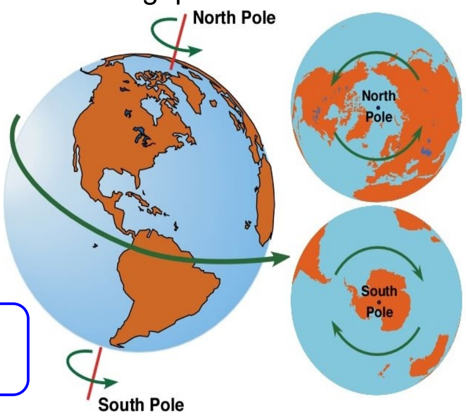
(b)(i) $\omega = \dfrac{2\pi}{T} = \dfrac{2\pi}{60}$
$= 0.105$ rad s^{-1}

The second hand takes 60 s to complete one cycle.

(b)(ii) $v = r\omega = (2 \times 10^{-2})\left(\dfrac{2\pi}{60}\right)$
$= 0.00209$ m s^{-1}

Example 2

Singapore lies approximately on the Earth's equator. The radius of the Earth is about 6370 km. Calculate the **(i)** angular velocity and **(ii)** linear speed of Singapore.



Solution

(i) $\omega = \dfrac{2\pi}{T}$

$= \dfrac{2\pi}{(24)(60)^2} = 7.27 \times 10^{-5}\ \text{rad s}^{-1}\ (\text{west to east})$

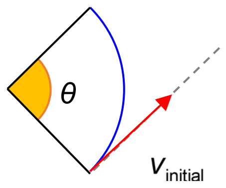
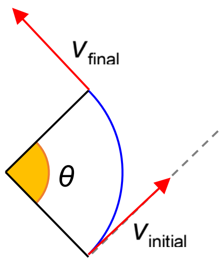
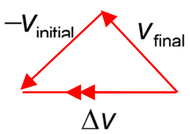
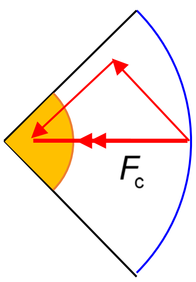
The Earth takes 24 h to rotate one cycle about its axis.

(ii) $v = r\omega$

$= (6370 \times 10^3)\left(\dfrac{2\pi}{(24)(60)^2}\right) = 463\ \text{m s}^{-1}\ (\text{faster than speed of sound in air!})$

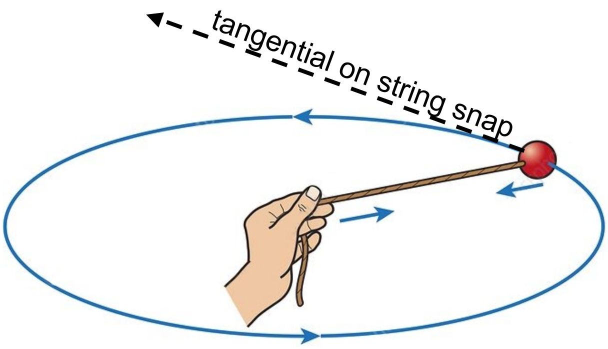
6.2 Centripetal Acceleration and Centripetal Force

Centripetal means "centre-seeking". In circular motion, the force (also acceleration) is directed towards the centre of the circular path.

	By Newton's 1st Law, if there is no net force on an object, it will continue in its state of rest or constant velocity. Inertia will cause the object to continue moving with $v_{initial}$ along a straight line (- - - -).
	For an object in uniform circular motion, speed remains constant but direction changes so velocity changes. Therefore, acceleration must exist.
	$\Delta v = v_{final} - v_{initial} = v_{final} + (-v_{initial})$ By Newton's 2nd Law, the object has to experience a net force that is given by its rate of change of momentum.
	The net force (also the acceleration) is directed towards the centre of the circular path.

centripetal acceleration a_c	centripetal force F_c
$a_c = \frac{v^2}{r}$	$F_c = \frac{mv^2}{r}$
$a_c = r\omega^2$	$F_c = mr\omega^2$

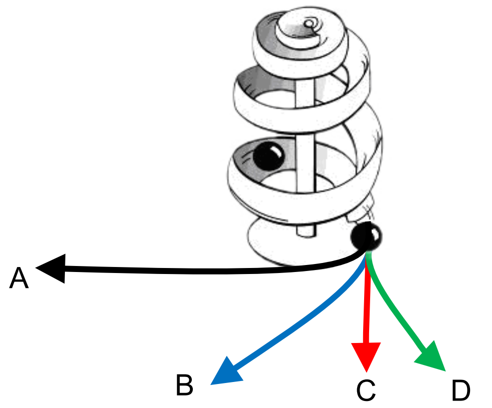
- Centripetal force F_c is a *resultant* force
  F_c is the "leftover" net force resulting from *real forces* such as contact, tension, weight etc.
  Therefore **do not draw centripetal force** in a free body diagram.
- For uniform circular motion, constant linear speed implies zero tangential force.
  The centripetal force only changes the direction of motion but not the speed.



Tension in the string provides the centripetal force to keep the ball in circulation motion.

If the string were to snap suddenly, by Newton's 1st Law, the ball will tend to continue in its state of constant velocity just before the string snaps:

Ball will continue moving off tangential to the original circular motion.



A toy tower has a curved track for a marble to roll down along a circular path onto the floor smoothly.

Ball will continue on the floor along path C, which is tangential to the curved track just before the track ends.

Example 3
Explain why is there no work done during circular motion by a centripetal force.

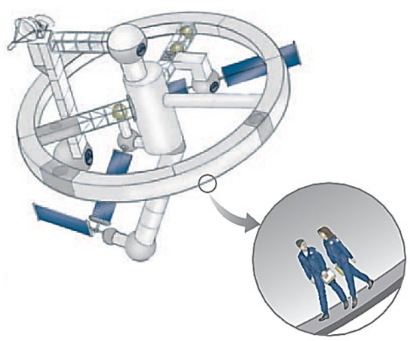
Solution
[direction] Force* that provides centripetal force for the object to undergo uniform circular motion is always perpendicular to the direction of motion of the object

[w.d. definition] Hence no displacement in the direction of the force and therefore no work done.

Note: need to specify the type of *real* force when answering such questions to demonstrate the knowledge that centripetal force is a resultant force and not a new type of force. e.g. "tension provides the centripetal force" / "gravitational force provides the centripetal force " etc.

Example 4

The Stanford Torus is a proposed design for a space habitat. It is suggested that gravity be simulated in the habitat by means of circular motion by rotating a "donut"-shaped ring of 1.8 km diameter, 1 revolution per minute (RPM). Determine the magnitude of the gravity being simulated.



Solution

$$a_c = r\omega^2 = \left(\frac{d}{2}\right)\left(\frac{2\pi}{T}\right)^2 = \left(\frac{1.8\times10^3}{2}\right)\left(\frac{2\pi}{60}\right)^2$$

$$= 9.87 \text{ m s}^{-2}$$

6.3 General Approach to Solving Problems relating to Circular Motion

1. Draw a free body diagram showing all the forces acting on the body.
2. Determine the centre of the circular path.
3. Resolve the forces in the direction towards the centre of the circle and the perpendicular direction.
4. Identify the force(s) that provide(s) the centripetal force and write down the statement. e.g. **Friction provides the centripetal force.**
5. Apply Newton's Second Law:

$$F_{\text{net}} = mr\omega^2 = \frac{mv^2}{r}$$

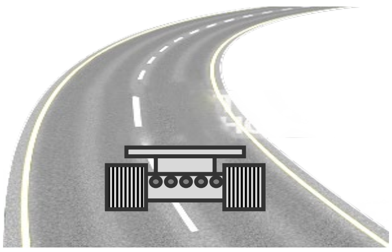
6.3.1 Uniform circular motion along a horizontal plane

Example 5

A race car of mass 800 kg speeds along a track shaped like the arc of a circle. The maximum friction between all tyres and road is 12 000 N.

(a) Explain which force provides the centripetal force.

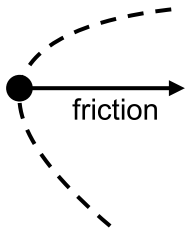
(b) Find the maximum speed if the curve has a radius of 30 m



Solution

(a) friction acts laterally on a car's tyres to provide centripetal force. the force is constant in magnitude and acts normal to the car's forward velocity.

(b) friction on tyres provides centripetal force



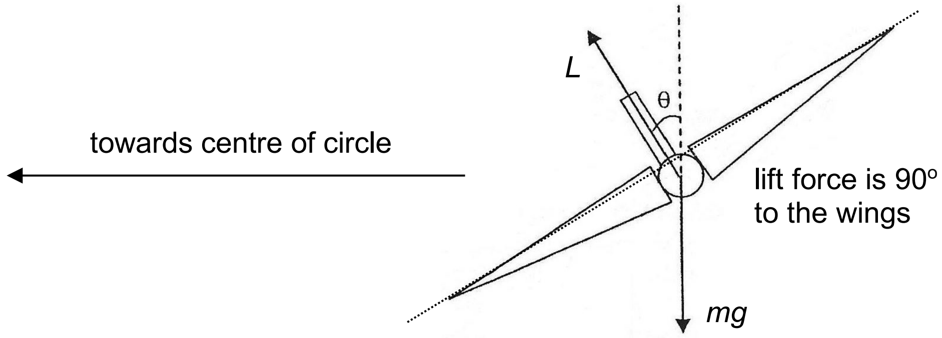
$$F_{\text{friction}} = \frac{mv^2}{r}$$

$$v = \sqrt{\frac{F_{\text{friction}}r}{m}} = \sqrt{\frac{(12000)(30)}{800}} = 21.2 \text{ m s}^{-1}$$

Note: the magnitude of friction can vary up to a maximum value (see H204 Forces). For a same circular path, the radius r of the path is fixed–a higher speed will need a higher force of friction to provide the centripetal force, failing which the car skids.

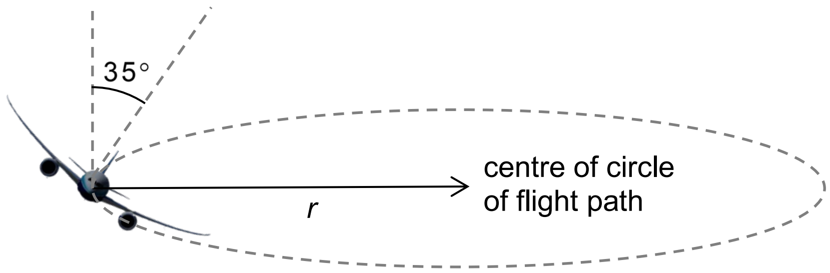
Airplane making a circular turn

When an airplane turns, it tilts the body and wings at angle to the vertical. This allows a *component* of the lift force L to provide for the centripetal force.



Example 6

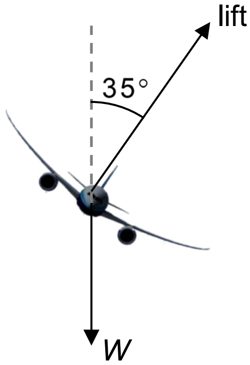
An aircraft of mass 4.00×10^4 kg flies at a constant speed of 250 m s^{-1}. To make a particular turn, it tilts its body $35°$ away from the vertical as shown below. Find the radius of the flight path.



Solution

horizontal component of lift force provides centripetal force

$$L \sin(35°) = \frac{mv^2}{r} \quad \_\_\_(1)$$



vertical component of lift balances the weight

$$L \cos(35°) = mg \quad \_\_\_(2)$$

solving $\frac{(1)}{(2)}$: $\tan(35°) = \frac{v^2}{gr}$

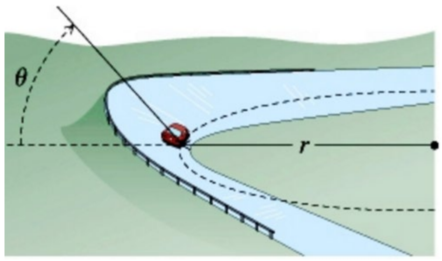
$$r = \frac{v^2}{g(\tan(35°))} = \frac{250^2}{9.81(\tan(35°))}$$

$= 9099$ m

≈ 9100 m

Example 7

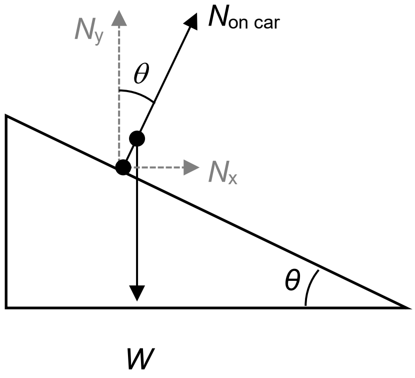
A car drives along an icy road banked at angle $\theta = 25°$ and a radius *r* of 50 m as shown. Calculate the maximum speed that the car can travel at while maintaining the circular radius.



Solution

Assuming an icy road provides no lateral friction,
horizontal component of normal force on car provides centripetal force.

$$N \sin\theta = \frac{mv^2}{r} \quad \_\_\_(1)$$



vertical component of normal force balances the weight

$$N \cos\theta = mg \quad \_\_\_(2)$$

solving $\frac{(1)}{(2)}$: $\tan(\theta) = \frac{v^2}{gr}$

$$v = \sqrt{gr \tan\theta} = \sqrt{(9.81)(50)\tan(25°)}$$
$$= 15.1 \text{ m s}^{-1}$$

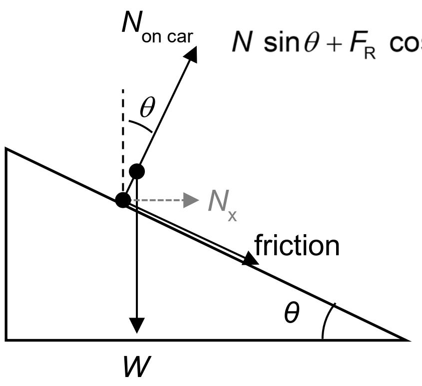
Note:

(a) the magnitude of the normal contact force responds to the motion of the car. If the car does not have to navigate a curve on a banked road, the normal contact force is given by $N = W$ (see H204 Forces). However, here the car is accelerating towards the centre of the circular path from the horizontal net force provided by the normal contact force. A similar thinking is the tension in the leather strap supporting a bus grab-handle. If there is no acceleration, the tension has the same magnitude as weight and is vertical. When the bus accelerates, there is more tension in the inclined strap to provide a net horizontal force.

(b) if there is lateral friction, horizontal component of friction adds to resultant horizontal force:

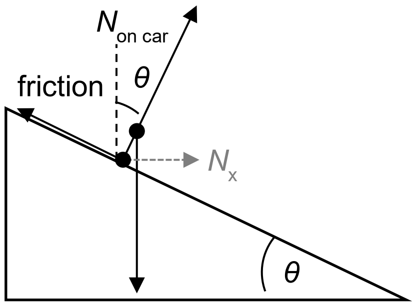
"high" speed so car tends to skid out

$$N \sin\theta + F_R \cos\theta = \frac{mv^2}{r}$$



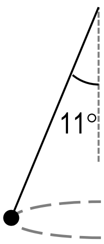
"low" speed so car tends to slide down-ramp

$$N \sin\theta - F_R \cos\theta = \frac{mv^2}{r}$$



Example 8

A pendulum bob is suspended by a light inextensible string and made to perform uniform circular motion of radius 7.0 cm in a horizontal plane. The string is inclined 11° from the vertical. Find the period of circular motion.



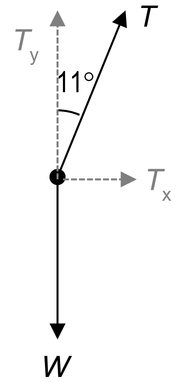
Solution

horizontal component of tension provides centripetal force

$$T \sin\theta = mr\omega^2 \quad \_\_\_(1)$$

vertical component of tension balances the weight

$$T \cos\theta = mg \quad \_\_\_(2)$$



solving $\frac{(1)}{(2)}: \tan(\theta) = \frac{r\omega^2}{g} = \frac{r}{g}\left(\frac{2\pi}{T}\right)^2$

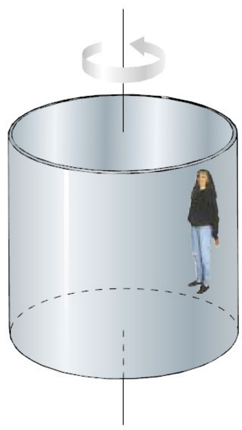
$$T = (2\pi)\sqrt{\frac{r}{g\ (\tan\theta)}} = (2\pi)\sqrt{\frac{7\times10^{-2}}{(9.81)\ (\tan(11°))}}$$

$$= 1.20 \text{ s}$$

Note: the tension responds to the motion of the bob. The magnitude of the tension is that of the bob's weight when the bob is stationary, and increases to provide centripetal force.

Example 9

An amusement ride involves a large hollow cylinder with radius 5.0 m spinning fast enough that the passengers are held up against the inner walls by friction. The magnitude of friction F_r is related to the normal contact force at the inner walls $F_r = 0.2(N)$. Find the period of revolution at which the floor can be safely dropped.



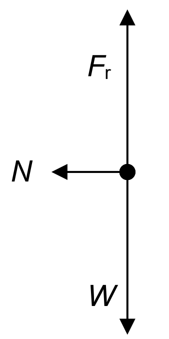
Solution

normal force with inner walls provides centripetal force

$$N = mr\omega^2 \quad \_\_\_(1)$$

friction balances the weight

$$F_r = mg \quad \_\_\_(2)$$



solving $\frac{F_r}{0.2} = mr\omega^2$

$$\frac{mg}{0.2} = mr\left(\frac{2\pi}{T}\right)^2$$

$$T = (2\pi)\sqrt{\frac{0.2r}{g}} = (2\pi)\sqrt{\frac{0.2(5.0)}{9.81}}$$

$$= 2.01 \text{ s}$$

6.3.2 Circular motion along a vertical plane

When an object moves along a vertical circular path, there is a natural tendency for the speed of the object to vary. The object will:

- accelerate when moving down
- slow down when moving up

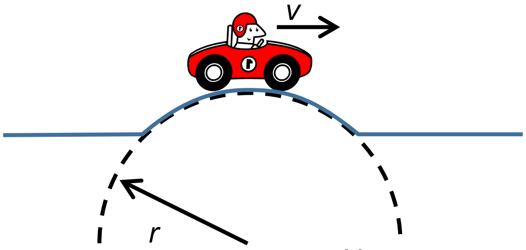
because of the effects the role of weight in the conservation of energy (converting gravitational potential energy into kinetic energy).

Therefore, if an object shows *uniform* circular motion when moving vertically, there is a cyclic input of energy (to do work against gravity) and dissipation of energy (to "brake" when moving down).

It is hence common to apply conservation of energy when solving circular motion questions with vertical circular motion.

Example 10

A car moves with constant speed *v* over a road hump shaped as a circular arc of radius *r*. Find the expression for *v* when the car just loses contact from the top of the hump.

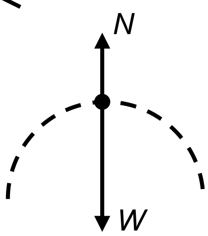


Solution

vector sum of normal force and weight provides centripetal force

$$W - N = \frac{mv^2}{r}$$

$$N = W - \frac{mv^2}{r}$$



For the car to stay in contact with the hump,

$$N \geq 0$$

$$W - \frac{mv^2}{r} \geq 0 \quad \text{(since } W = mg\text{)}$$

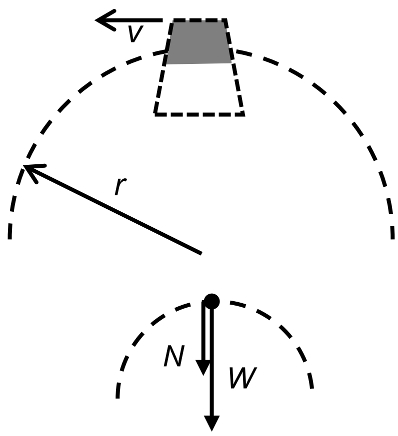
$$v \leq \sqrt{rg}$$

$$v_{max} = \sqrt{rg}$$

Note: it is "easier" for a car to lose contact for a small radius, making for an uncomfortable "bump". This is why larger, flatter speed bumps are more forgiving than very small and rounded ones.

Example 11

Water is spun vertically in a bucket attached to a string. Find the minimum speed such that the water stays in the bucket.



Solution

vector sum of normal force & weight provides centripetal force

$$W + N_{\text{on water by bucket}} = \frac{mv^2}{r}$$

$N \geq 0$ for water to stay (in contact with bucket)

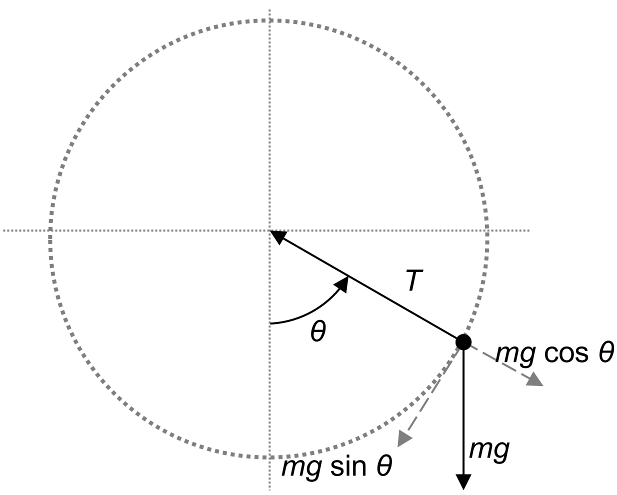
$$mg \leq \frac{mv^2}{r}$$

$$\sqrt{rg} \geq v$$

$$v_{\min} = \sqrt{rg}$$

Note: the faster the bucket is swung, the more likely the person doing the experiment stays dry.

The general relationships for an object in vertical motion swung along a string:



$$T - mg\cos\theta = \frac{mv^2}{r}$$

in general, $T = \frac{mv^2}{r} + mg\cos\theta$

at bottom, $T = \frac{mv^2}{r} + mg\cos(0)$

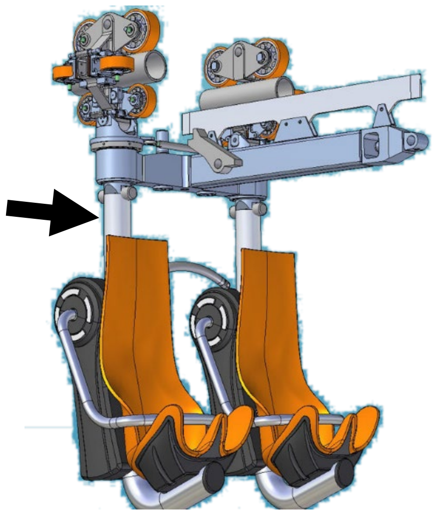
$$= \frac{mv^2}{r} + mg$$

at top, $T = \frac{mv^2}{r} + mg\cos(180°)$

$$= \frac{mv^2}{r} - mg$$

Example 12

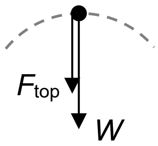
The Battlestar Galactica roller coaster ride in Universal Studios Singapore was shut down for 11 months in 2010. It had been reported that one of the seats was detached from the frame during a routine test. The seats are attached to the frame by means of a rod under stress as indicated.



Determine if the rod is subject to greater stress at the top of a vertical loop or at the bottom.

Solution

Consider the combined weight of passenger and chair:

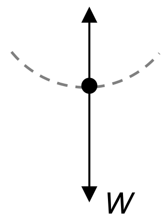


Rod is compressed by F_{top} at top of loop.
Resultant force of F_{top} and weight provides centripetal force:

$$F_{top} + W = \frac{mv^2}{r}$$

$$F_{top} = \frac{mv^2}{r} - mg$$

$$= m\left(\frac{v^2}{r} - g\right)$$



Rod is under tension F_{bottom} at bottom of loop.
Resultant force of F_{bottom} and weight provides centripetal force:

$$F_{bottom} - W = \frac{mv^2}{r}$$

$$F_{bottom} = \frac{mv^2}{r} + mg$$

$$= m\left(\frac{v^2}{r} + g\right)$$

Rod is subject to more force at bottom of loop for the same speed.

Example 13

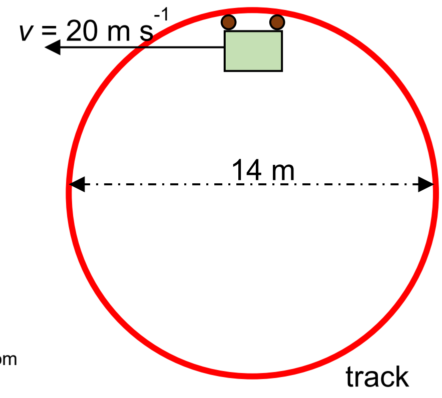
In a particular ride in an amusement park, a cart of mass m = 55 kg reaches the top of a circular frictionless track of diameter 14 m at a speed of v = 20 m s^{-1}, as shown in the diagram below.

(a) Draw a labelled diagram of the forces acting on the cart at the top of the track.

(b) Find the magnitude of the force exerted by the cart on the track at the top of the track.

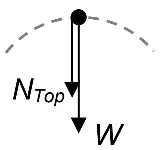
(c) Find the velocity of the cart at the bottom of the track.

(d) Find the magnitude of the normal force on the cart at the bottom of the track.



Solution

(a)



(b) Resultant force of normal force *by track on cart* and weight provides centripetal force:

$$N_{Top} + W = \frac{mv^2}{r}$$

$$N_{Top} = \frac{mv^2}{\frac{d}{2}} - mg = m\left(\frac{2v^2}{d} - g\right)$$

$$= (55)\left(\frac{2(20)^2}{14} - 9.81\right)$$

$$= 2600 \text{ N}$$

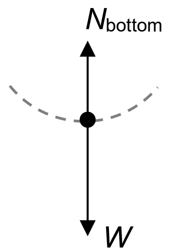
By Newton's 3rd Law, the normal force *on track by cart* is 2600 N *upwards*

(c) By principle of conservation of energy, loss in GPE = gain in KE:

$$mg\Delta h = \frac{1}{2}mv_{\text{bottom}}^2 - \frac{1}{2}mv_{\text{top}}^2$$

$$v_{\text{bottom}} = \sqrt{2gd + v_{\text{top}}^2} = \sqrt{2(9.81)(14) + (20)^2}$$

$$= 26.0 \text{ m s}^{-1}$$



(d) Resultant force of N_{bottom} and weight provides centripetal force:

$$N_{\text{bottom}} - W = \frac{mv^2}{r}$$

$$N_{\text{bottom}} = \frac{mv^2}{r} + mg = m\left(\frac{2v^2}{d} + g\right)$$

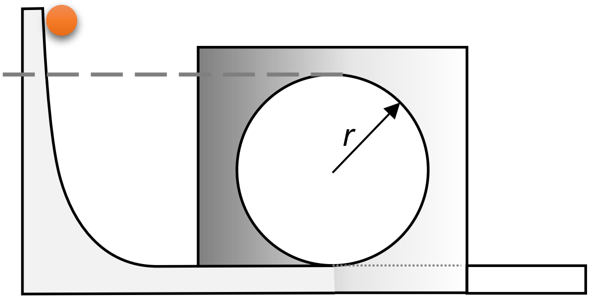
$$= (55)\left(\frac{2(25.975)^2}{14} + 9.81\right)$$

$$= 5840 \text{ N}$$

Note: recall that you do not feel your weight, but perceive the sensation through the normal contact force. The large magnitude of normal force at the bottom of the loop is the "squashed" sensation at the bottom of a roller coaster loop. The reduced magnitude of normal force gives the "dropping" sensation.

Example 14

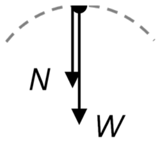
A marble rolls down a smooth track from the height as indicated. It did not loop-the-loop successfully. Find the height from which the marble can be dropped so that it can loop-the-loop successfully.



Solution

To loop-the-loop, the marble must be **in contact with the track at the highest point**.

Consider the force diagram at the top of the loop:



Resultant force of normal contact force *by track on marble* and weight provides centripetal force:

$$N + W = \frac{mv^2}{r}$$

For the marble to be in contact with the track:

$$N \geq 0$$

$$\frac{mv^2}{r} - mg \geq 0$$

$$v^2 \geq rg$$

Hence, min velocity at the top of the loop:

$$v_{min} \geq \sqrt{rg}$$

This is like example 10 and example 11.

By Conservation of Energy:

Initial total energy = final total energy at top of loop

$$E_{p,initial} + E_{k,initial} = E_{p,final} + E_{k,final}$$

$$mgh + 0 = mg(2r) + \frac{1}{2}mv_{\min}^2$$

$$mgh + 0 = mg(2r) + \frac{1}{2}m(rg)$$

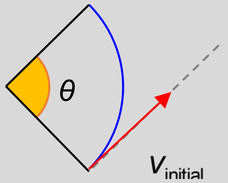
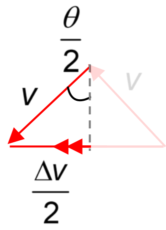
$$h = \frac{5}{2}r$$

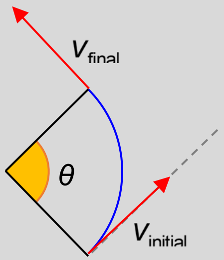
6.4 Ending Notes

Consider this topic as an extension of Dynamics specific to resultant forces (Newton's 2nd Law) directing a moving object to accelerate towards the centre of a circular path.

Use the space on the next page for your own summary and/or mindmap.

Annex: Derivation for Centripetal Acceleration (not in syllabus)

<table>
<tr>
<td>
</td>
<td>By Newton's 1st Law, if there is no net force on an object,
it will continue in its state of rest or constant velocity.
Inertia will cause the object to continue moving with v_{initial} along a straight line (- - - -).</td>
<td rowspan="4">from third row, consider triangular geometry:
uniform circular motion so no change in speed: $|v_{\text{final}}| = |v_{\text{initial}}|$, let be v


$$v \sin\left(\frac{\theta}{2}\right) = \frac{\Delta v}{2}$$
[differentiation concept] we consider the limit where the variable θ tends to being infinitesimally small, $\theta \to \Delta\theta$ and $\sin(x) \approx x$ when x is small
$$\frac{\Delta v}{2} = v \sin\left(\frac{\theta}{2}\right) \approx v\left(\frac{\Delta\theta}{2}\right)$$
$$\Delta v = v\Delta\theta$$
[kinematics]
$$a = \frac{\mathrm{d}v}{\mathrm{d}t} \approx \frac{\Delta v}{\Delta t} = \frac{v\Delta\theta}{\Delta t} \to v\frac{\mathrm{d}\theta}{\mathrm{d}t} = v\omega$$
for uniform circular motion,
$$v\omega = (r\omega)\omega \quad \text{or} \quad v\left(\frac{v}{r}\right)$$</td>
</tr>
<tr>
<td>
</td>
<td>For an object in uniform circular motion,
speed remains constant
but direction changes so velocity changes.
Therefore, acceleration must exist.</td>
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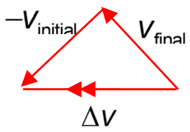
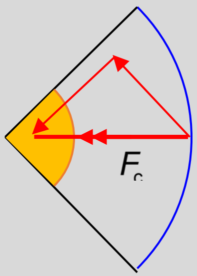
<tr>
<td>
</td>
<td>$\Delta v = v_{\text{final}} - v_{\text{initial}} = v_{\text{final}} + (-v_{\text{initial}})$
By Newton's 2nd Law, the object has to experience a net force that is given by its rate of change of momentum.</td>
</tr>
<tr>
<td>
</td>
<td>The net force (also the acceleration) is directed towards the centre of the circular path.</td>
</tr>
</table>