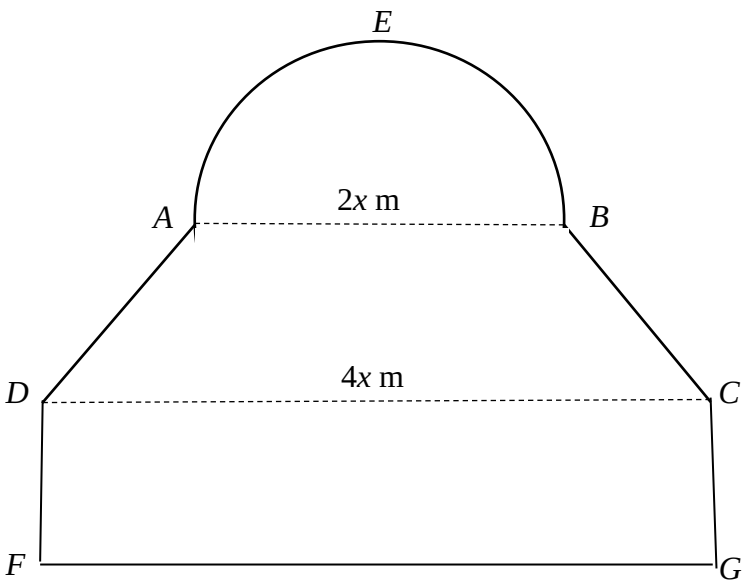
**1** Determine the set of values of $k$ such that the equation $x^2 - kx + 3 + k = 0$ has real roots. [4]

**2** **(a)** Find $\int \frac{\sqrt[3]{x} - \left(\sqrt{x}\right)^3}{\sqrt{x}} \, \mathrm{d}x$. [2]

**(b) (i)** Differentiate $6\ln\sqrt[3]{3x^2+4}$ with respect to $x$, simplifying your answer. [3]

**(ii)** Hence find $\int \frac{x}{3x^2+4} \, \mathrm{d}x$. [2]

**3** A window at a grand church has the shape of a semicircle $AEB$ joined to a regular trapezium $ABCD$ of height $x$ m, and a rectangle $DCGF$ as shown in the diagram. It is given that $AB = 2x$ m, $DC = 4x$ m and the angle ADC and the angle BCD are both equal to $45^\circ$. The total perimeter $AEBCGFD$ of the window is 10 m.

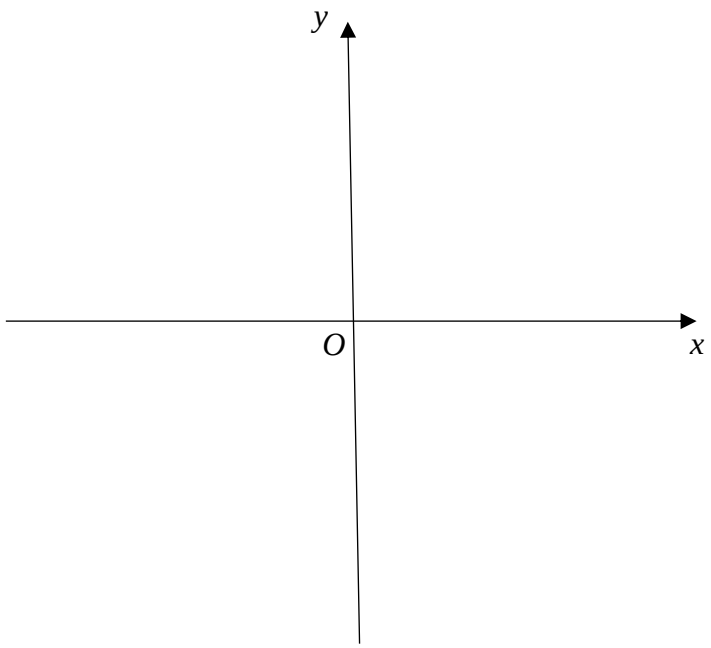



**(i)** Show that $DF = 5 - \left(\frac{\pi}{2} + \sqrt{2} + 2\right)x$ m. [2]

**(ii)** Hence use a non-calculator method to find the exact value of $x$ such that the area of the window is maximum. [5]

**4** The curve $C$ has equation $y = 3 - e^{-3x}$.

**(i)** Sketch the graph of $C$, stating the exact coordinates of the points of intersections with the axes and the equation of the asymptote. [3]




**(ii)** Find the equation of the tangent to $C$ at the point where the curve $C$ meets the $x$-axis, giving your answer in the form $y = ax + b\ln 3$, where $a$ and $b$ are integers to be determined. [3]

**(iii)** Hence find the exact value of the area enclosed by the curve $C$, the $y$-axis and the tangent at the point where the curve meets the $x$-axis [4]

**5** A company sells three types of toys: boats, cars and planes.

- The selling price of a toy boat is twice as much as that of a toy car.
- The total selling price of 9 toy boats, 5 toy cars and 3 toy planes is \$335.
- The total selling price of 3 toy boats and 4 toy cars is \$38 more than the total selling price of a toy boat and 2 toy planes.

**(i)** Find the selling price of a toy plane. [3]

The company decides to trial the production of a new type of toy: trains. Based on market research, the manager estimates that when the toy trains are priced at $p$ dollars apiece, the number sold each month, $D$ thousands, can be modelled by

$$D = 3 + a\mathrm{e}^{-0.08p}$$

where $a$ is a constant. When the toy trains were given out for free to spark public interest, the demand for the toy was 20 thousand units.

**(ii)** Find the exact value of $a$. [1]

**(iii)** Sketch the graph of $D$ against $p$, stating the coordinates of any point(s) of intersection with the axes and the equation(s) of any asymptote(s). [2]

**(iv)** Find $\dfrac{\mathrm{d}D}{\mathrm{d}p}$. [1]

The manager estimates that $t$ months from its production, the selling price of a toy train, \$$p$, will be

$$p = \sqrt{0.5(t-9)^2 + 312}, \quad 0 \le t \le 30.$$

**(v)** Use differentiation to find the rate of change of monthly demand for toy trains at 20 months from its production. [4]

**(vi)** Explain in context what the answer in **(v)** suggests about the demand for toy trains at 20 months from its production. [1]

## Section B: Probability and Statistics [60 marks]

**6** A company has 8 vacancies to fill from 17 applicants. The applicants consist of 8 women and 9 men. One of the 8 women is the wife of one of the 9 men.

**(i)** How many different ways can these vacancies be filled if there are no restrictions? [1]

On the day of the interview, all 17 applicants are seated in a row in the waiting room.

**(ii)** How many different ways can the 17 applicants be arranged such that the men and women alternate and the husband and wife are seated together? [3]

The company then decides that four of the eight vacancies shall be filled by women and four by men.

**(iii)** How many different ways can the eight vacancies be filled under this new decision? [1]

**(iv)** Of all the possible selections of four women and four men, one selection is picked at random. Find the probability that this selection includes either the wife or her husband, but not both. [2]

**7** A group of customers in a restaurant are asked which fruits they like from a choice of mangoes, strawberries and apples. The results are as follows.

65 like mangoes
40 like strawberries
45 like apples
21 like mangoes and apples
14 like strawberries and apples
26 like mangoes and strawberries
$x$ like all three fruits
3 like none of these three fruits.

One of the customers is chosen at random.

$M$ is the event that the customer likes mangoes.
$S$ is the event that the customer likes strawberries.
$A$ is the event that the customer likes apples.

**(i)** Represent this information on a Venn diagram. [3]

**(ii)** Given that events $M$ and $S$ are independent, show that $x = 8$. [2]

**(iii)** Explain, in the context of this question, what is meant by $\mathrm{P}(M \mid A)$, and find its value. [2]

Four customers are chosen at random, without replacement.

**(iv)** Find the probability that three customers like exactly two of these three fruits and one customer likes only one of these fruits. [3]

**8** Alice sold homemade Nutella brownies when she was jobless during the Covid-19 outbreak to earn a living. The numbers of brownies, $x$, sold in the first eight months of the year 2022, together with the profits, $y$ dollars, on the sale of the Nutella brownies are given in the following table.

| | Jan | Feb | Mar | Apr | May | Jun | Jul | Aug |
|---|---|---|---|---|---|---|---|---|
| $x$ | 400 | 560 | 290 | 360 | 620 | 190 | 520 | 600 |
| $y$ | 800 | 1100 | 560 | $k$ | 1200 | 390 | 970 | 1100 |

**(i)** It is known that the regression line of $y$ on $x$ is given by $y = 1.834352x + 40.8$. Show that the value of $k$ is 700, correct to the nearest integer. [3]

Take $k = 700$ for the rest of the question.

**(ii)** Find the product moment correlation coefficient and comment on its value in the context of the data. [2]

**(iii)** On the same diagram, give a sketch of the regression line of $y$ on $x$ given in part **(i)** and the scatter diagram of the data as shown in your calculator. [2]

**(iv)** Use the equation of regression line in part **(i)** to calculate an estimate of the profit when 480 brownies were sold. Explain whether you would expect this estimate to be reliable. [2]

**(v)** Alice realised that she had recorded her profits wrongly. There is a shortfall of 80 dollars for each of the monthly profit in the table above. Comment on whether this recording error will affect the value of the product moment correlation coefficient found in **(ii)**. [1]

**9** In epidemiology, particularly in the discussion of infectious disease dynamics, the duration of infection period is the time interval during which a host (individual or patient) is infectious. Researchers claim that the mean duration of a particular viral infection among children is 4.3 days. In a random sample of 120 children infected with the particular virus, the average duration of infection is 4.07 days and the standard deviation is 1.215 days.

**(i)** Test at the 5% significance level whether the researchers' claim is supported by the data. [4]

**(ii)** State, giving a reason, whether it is necessary to assume that the durations of the infection for the particular virus among children are normally distributed for this test to be valid. [1]

**(iii)** Given that there is sufficient evidence from the sample given above to conclude that the mean duration of the particular infection has decreased at $k$% level of significance, find the range of values of $k$. [2]

A doctor claims that medicine $M$ is able to reduce the average duration of the particular infection among children. A second random sample of 60 children infected with the particular virus is treated with medicine $M$. The durations of infection, $y$, are summarised by

$$\bar{y} = 4 \qquad \sum (y-4)^2 = 77.88$$

**(iv)** Find unbiased estimate of the population variance using this second sample. [1]

The population mean duration of infection after being treated with medicine $M$ is $\mu$ days.

**(v)** In a 1-tail test of the null hypothesis $\mu = 4.3$, there is not enough evidence to reject the null hypothesis at the $\beta$ % level of significance. State the alternative hypothesis and explain in the context of the question, the meaning of "$\beta$ % level of significance." [2]

**In this question, you should state clearly all the distributions that you use, together with the values of the appropriate parameters.**

**10** The masses of Honeycrisp apples are normally distributed with mean 78 grams and standard deviation 13 grams. The apples are sold in boxes containing 24 apples. The masses of the empty boxes are normally distributed with mean 10 grams and standard deviation 0.8 grams.

**(i)** Find the probability that the mass of a randomly chosen Honeycrisp apple is between 70 and 90 grams. [1]

**(ii)** Find the probability that the total mass of a box containing 24 apples is less than 2000 grams. [2]

**(iii)** State an assumption you have made in your calculation in **(ii)**. [1]

**(iv)** Find, to the nearest whole number, the largest mass $k$, such that more than 55% of the apple is at least $k$ grams. [3]

A Honeycrisp apple cost \$0.07 per gram and an empty box cost \$0.20.

**(v)** Find the probability that the total cost of 3 boxes containing 24 apples each is between \$370 and \$420. [4]

**In this question, you should state clearly all the distributions that you use, together with the values of the appropriate parameters.**

**11** In Hwa Chong Institution (College Section), the height, in cm, of male students and female students have independent normal distributions with means and standard deviations as shown in the table.

| | Mean height | Standard deviation |
|---|---|---|
| Male students | 171 | 3.58 |
| Female students | 160 | 5.11 |

**(i)** Two female students are randomly selected.

**(a)** Find the probability that each of the two female students is taller than 158 cm. [2]

**(b)** Find the probability that one female student is taller than 158 cm and the other is shorter than 158 cm. [2]

**(ii)** Find the probability that the total height of 3 randomly selected male students differs from the total height of 4 randomly selected female students by at most 130 cm. [3]

**(iii)** Find the probability that there are at least 50 female students from a random sample of 80 female students whose height is more than 158 cm. [2]

**(iv)** A sample of $n$ male students is randomly selected. Find the largest value of $n$ such that the percentage of at most 10 male students who are each shorter than 175 cm is more than 40%. [3]