RAFFLES INSTITUTION
H2 Mathematics 9758
2023 Year 6 Term 3 Revision 8 (Summary and Tutorial)

Topic: Applications of Differentiation

Summary for Applications of Differentiation

1 Strictly Increasing/Strictly Decreasing & Concavity

IF	Then f is	
$\mathrm{f}'(x) > 0$	Strictly increasing	over the interval (a,b)
$\mathrm{f}'(x) < 0$	Strictly decreasing	
$\mathrm{f}''(x) > 0$	Concave up	
$\mathrm{f}''(x) < 0$	Concave down	

To show "< 0" or "> 0", we can complete the square of the expression if it is quadratic otherwise we can try to "build up the expression" from the given interval.

For example to show $\mathrm{f}'(x) = 2 - \frac{1}{x} > 0$ for $x > \frac{1}{2}$.

We can rewrite the expression: $\mathrm{f}'(x) = \frac{2x-1}{x}$

If $x > \frac{1}{2}$, then $2x > 1$ which implies that $2x - 1 > 0$

If $x > \frac{1}{2}$, then the denominator must be "> 0"

Therefore if the numerator is "> 0" and the denominator is also "> 0", we can conclude that for $x > \frac{1}{2}$, $\mathrm{f}'(x) > 0$

So the **strategy** here is to try to examine the signs the numerator and denominator from the given condition of $x > \frac{1}{2}$.

2 Methods for determining the nature of a stationary point

Method 1: Second Derivative Test

IF		Then $(k, \mathrm{f}(k))$ is
$\mathrm{f}''(x) > 0$	at $x = k$	a minimum turning point
$\mathrm{f}''(x) < 0$		a maximum turning point

Note that if $\mathrm{f}''(x) = 0$ then there is no conclusion about the nature of the stationary point. The first derivative test must then be used.

Method 2: First Derivative Test

To determine the nature of the stationary point at $(k,\ \mathrm{f}(k))$, we need to check the signs of $\mathrm{f}'(x)$ for $x = k^-$ and $x = k^+$.

Important note: you need to fully factorise $\mathrm{f}'(x)$ where possible BEFORE discussing the signs of $\mathrm{f}'(x)$.

Example: $\mathrm{f}(x) = 2x^5 - 5x^3$. It is not sufficient to just differentiate and get $\mathrm{f}'(x) = 10x^4 - 15x^2$. You need to factorise further to get $\mathrm{f}'(x) = 10x^2\left(x - \sqrt{\frac{3}{2}}\right)\left(x + \sqrt{\frac{3}{2}}\right)$, BEFORE discussing stationary points at $(0,0)$, $\left(\sqrt{\frac{3}{2}}, -3\sqrt{\frac{3}{2}}\right)$ and $\left(-\sqrt{\frac{3}{2}}, 3\sqrt{\frac{3}{2}}\right)$ making reference to the table below.

x	k^- $\quad$ k $\quad$ k^+	k^- $\quad$ k $\quad$ k^+	k^- $\quad$ k $\quad$ k^+
$\mathrm{f}'(x)$	$+$ve $\quad$ 0 $\quad$ $-$ ve	$-$ve $\quad$ 0 $\quad$ $+$ ve	$+$ve $\quad$ 0 $\quad$ $+$ ve or $-$ve $\quad$ 0 $\quad$ $-$ ve
Nature of Stationary Point	Maximum	Minimum	Stationary point of inflexion

3 TANGENTS AND NORMALS

To find the equations of the tangent or normal at a point $(k,\ \mathrm{f}(k))$ on a curve, we need:

Equation of a (straight) line $y - \mathrm{f}(k) = m\ (x - k)$ with gradient *m* passing through $(k,\ \mathrm{f}(k))$.

- Obtain *m* by finding $\mathrm{f}'(k)$ which will the **gradient of the tangent** at $(k,\ \mathrm{f}(k))$.
- If the question needs equation of the normal at $(k,\ \mathrm{f}(k))$, **gradient of the normal** at $(k,\ \mathrm{f}(k))$ is $-\frac{1}{\mathrm{f}'(k)}$.

When gradient at a point on a curve $\mathrm{f}(x)$ is **parallel** to the		
x-axis	$\mathrm{f}'(x) = 0$	(Usually mean that the **numerator** of $\mathrm{f}'(x) = 0$)
y-axis	$\mathrm{f}'(x)$ is undefined.	(Usually mean that the **denominator** of $\mathrm{f}'(x) = 0$)

4 MAXIMIZATION AND MINIMIZATION

Some guidelines to solve problems involving maximization and minimization :

- Denote each changing quantity by a variable.
- Write down a formula for the quantity to be maximized or minimized.
- Express the quantity to be maximized or minimized in terms of 1 variable only.
- Differentiate and equate derivative to zero for stationary values.
- Justify if quantity is a maximum or minimum (using 2^{nd} or 1^{st} derivative test).
- Answer the question.

Some strategies

1. Identify any right angle triangle by drawing a diagram. Use Pythagoras theorem to relate the two variables.
2. Identify similar triangles to relate the two variables.
3. Read questions carefully to identify the **constant(s)** used in the questions. That will reduce confusion on what variable to differentiate with respect to.

5 CONNECTED RATES OF CHANGE

Some guidelines to solve questions involving rate of change:

- Denote each changing quantity by a variable.
- Find the equations relating the variables.
- Use the **chain rule** to link up the derivatives.
- Write down the values of the variables and the given rates of change.
- Solve for the unknown rate.

Some strategies

1. Write down the rate of change that is given in the question. Use the units to guide you. For example m^3s^{-1} is the rate of change of volume per unit second.
2. Write down the rate of change required by the question and use the chain rule to link up the derivative.
   For example to find $\frac{\mathrm{d}V}{\mathrm{d}t}$ and you are given $\frac{\mathrm{d}V}{\mathrm{d}r}$ and $\frac{\mathrm{d}r}{\mathrm{d}t}$. Then using chain rule, we can get $\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}r} \times \frac{\mathrm{d}r}{\mathrm{d}t}$.

6 MACLAURIN SERIES

Reminder: the formulas are found in MF26.

If $\mathrm{f}(x)$ can be expanded as a power series for a given range of x including zero, then

$$\mathrm{f}(x) = \mathrm{f}(0) + \mathrm{f}'(0)x + \frac{\mathrm{f}''(0)}{2!}x^2 + \frac{\mathrm{f}^{(3)}(0)}{3!}x^3 + \ldots + \frac{\mathrm{f}^{(n)}(0)}{n!}x^n + \ldots \text{ [in MF26]}$$

Binomial Series

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots$$

The binomial expansion is valid for $\boldsymbol{|x| < 1}$, i.e. $\boldsymbol{-1 < x < 1}$. [in MF26]

Note that we need to have "1" before applying the formula, so if we need to expand $(3+x)^{\frac{1}{2}}$, we can do the following depending on what is required.

- For **ascending** powers of x, we rewrite

  $$(3+x)^{\frac{1}{2}} = 3^{\frac{1}{2}}\left(1+\frac{x}{3}\right)^{\frac{1}{2}}$$

  before applying the binomial expansion.

  In this case, the expansion is valid for $\left|\frac{x}{3}\right| < 1$ (i.e. small x)

- For **descending** powers of x, we rewrite

  $$(3+x)^{\frac{1}{2}} = (x+3)^{\frac{1}{2}} = x^{\frac{1}{2}}\left(1+\frac{3}{x}\right)^{\frac{1}{2}}$$

  before applying the binomial expansion.

  In this case, the expansion is valid for $\left|\frac{3}{x}\right| < 1$ (i.e. large x)

Small angle approximations

For all small angles, positive or negative, we have (1) $\sin x \approx x$ (2) $\cos x \approx 1-\frac{x^2}{2}$ (3) $\tan x \approx x$, where x is measured in **radians**.

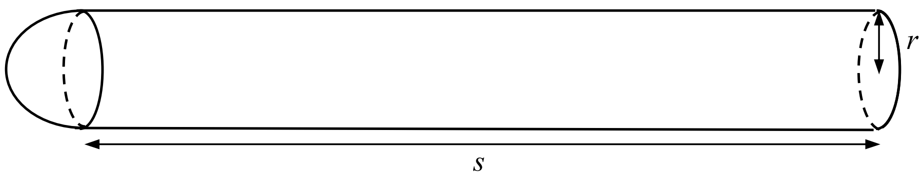
Note that

- If angle x is small, addition or subtraction to angle x may not remain small. i.e $\sin(x \pm a) \not\approx x \pm a$. We need to use addition formula to simplify the expression before applying the small angle approximations.
  - For example: $\sin\left(x+\frac{\pi}{4}\right)=\sin x\cos\frac{\pi}{4}+\cos x\sin\frac{\pi}{4} \approx x\left(\frac{1}{\sqrt{2}}\right)+\left(1-\frac{x^2}{2}\right)\left(\frac{1}{\sqrt{2}}\right)$
- If x is small, multiple of x is still small.
  - For example: $\cos(2x) \approx 1-\frac{(2x)^2}{2}$

Revision Tutorial Questions

Source of Question: VJC/Promo/2018/01/Q13

1 [It is given that a sphere of radius r has surface area $4\pi r^2$ and volume $\frac{4}{3}\pi r^3$.]



A touchscreen pen is made up of three parts.

- The head is modelled by the curved surface of a hemisphere of radius r mm.
- The body is modelled by the curved surface of a cylinder of radius r mm and length s mm.
- The base is modelled by a circular disc of radius r mm.

The three parts are joined together as shown in the diagram. The model is made of material of negligible thickness.

(i) It is given that the volume of the model is a fixed value $k\text{ mm}^3$, and the external surface area is a minimum. Use differentiation to find the values of r and s in terms of k, simplifying your answers. [7]

(ii) It is given instead that the volume of the model is 1270 mm^3 and its external surface area is 1290 mm^2. Show that there is only one possible value of r and find this value. [5]

Solution:

1(i) [7]	Volume, $k = \pi r^2 s + \frac{2}{3}\pi r^3 \quad \Rightarrow s = \frac{k}{\pi r^2} - \frac{2r}{3}$ External surface area is given by $A = 2\pi rs + \pi r^2 + 2\pi r^2$ $= 2\pi r\left(\frac{k}{\pi r^2} - \frac{2r}{3}\right) + 3\pi r^2$ $= \frac{2k}{r} + \frac{5}{3}\pi r^2$ Let $\frac{\mathrm{d}A}{\mathrm{d}r} = 0, \quad \Rightarrow \frac{10}{3}\pi r - \frac{2k}{r^2} = 0$ $\Rightarrow \frac{10}{3}\pi r^3 = 2k$ $\Rightarrow r = \sqrt[3]{\frac{3k}{5\pi}}$

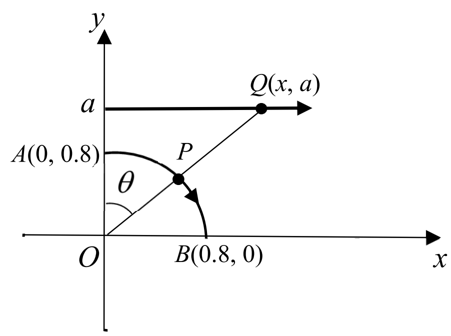
	$k=\pi r^2 s+\frac{2}{3}\pi r^3$ $=\pi\left(\sqrt[3]{\frac{3k}{5\pi}}\right)s+\frac{2}{3}\pi\left(\frac{3k}{5\pi}\right)$ $\Rightarrow \pi\left(\sqrt[3]{\frac{3k}{5\pi}}\right)^2 s=\frac{3k}{5}$ $\Rightarrow s=\frac{3k}{5\pi}\left(\frac{5\pi}{3k}\right)^{\frac{2}{3}}=\sqrt[3]{\frac{3k}{5\pi}}$
(ii) **[5]**	**<u>Method 1</u>** $k=1270,\quad A=1290$ $1290=2\pi rs+3\pi r^2$ ——— (1) $1270=\pi r^2 s+\frac{2}{3}\pi r^3$ ——— (2) From (1), $s=\frac{1290-3\pi r^2}{2\pi r}$ Substitute into (2), $1270=\pi r^2\left(\frac{1290-3\pi r^2}{2\pi r}\right)+\frac{2}{3}\pi r^3$ $\Rightarrow \frac{5}{6}\pi r^3-645r+1270=0$ Using GC, r = 2.0015, 14.599 or -16.601 Since $r>0$, r = -16.6 is rejected If $r=2.0015$, $s=\frac{k}{\pi(2.0015)^2}-\frac{2(2.0015)}{3}=99.573$ If $r=14.599$, $s=-7.8364$ Hence $r=14.599$ is rejected since s cannot be negative. So the only value of r is 2.00. **<u>Method 2 (similar to Method 1 actually)</u>** Given: $k=1270,\quad A=1290$ From (i), $A=\frac{2k}{r}+\frac{5}{3}\pi r^2$ $\therefore 1290=\frac{2(1270)}{r}+\frac{5}{3}\pi r^2$ $\Rightarrow \frac{5}{3}\pi r^3-1290r+2540=0$ Using GC, r = 2.0015, 14.599 or -16.601 Since $r>0$, $r=-16.6$ is rejected If $r=2.0015$, $s=\frac{k}{\pi(2.0015)^2}-\frac{2(2.0015)}{3}=99.573$

	If $r = 14.599$, $s = -7.8364$ Hence $r = 14.599$ is rejected since s cannot be negative. So the only value of r is 2.00.

Source of Question: HCI/Promo/2018/01/Q2

2 [It is given that the arc length of a circle is $r\theta$, where r is the radius of the circle and θ is the angle in radians subtended by the arc and the centre of the circle.]

The points A (0, 0.8) and B (0.8, 0) lie on the circumference of a circle with centre at the origin O and radius 0.8 units. Point P moves from A, along the arc AB in the clockwise direction. Point $Q(x, a)$ moves in the positive x-direction along the line $y = a$, where a is a constant such that $a > 0.8$. At time t seconds, points P and Q move at a rate such that O, P and Q are always on a line making an angle of θ radians with the positive y-axis, where $0 \le \theta < \frac{\pi}{2}$.

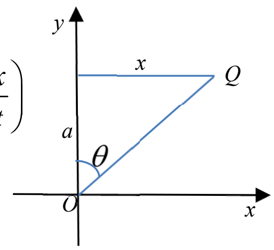


(i) Show that point P is moving along the arc at a rate of $0.8\frac{\mathrm{d}\theta}{\mathrm{d}t}$ units per second. [1]

(ii) Express $\tan\theta$ in terms of x and a. Given that $\frac{\mathrm{d}x}{\mathrm{d}t} = 0.6$ units per second, find in terms of a and x, the rate at which the point P is moving along the arc. [4]

Solution:

2(i) **[1]**	Arc Length = $S = r\theta$ Speed at which Q is moving = $\frac{\mathrm{d}S}{\mathrm{d}t} = r\frac{\mathrm{d}\theta}{\mathrm{d}t} = 0.8\frac{\mathrm{d}\theta}{\mathrm{d}t}$ (since r is a constant)

(ii) **[4]**	$\tan\theta = \dfrac{x}{a}$  $\sec^2\theta\left(\dfrac{d\theta}{dt}\right) = \dfrac{1}{a}\left(\dfrac{dx}{dt}\right)$ $\dfrac{d\theta}{dt} = \dfrac{\cos^2\theta}{a}\left(\dfrac{dx}{dt}\right)$ $\dfrac{d\theta}{dt} = \dfrac{0.6\cos^2\theta}{a}$ From (i), $\dfrac{dS}{dt} = 0.8\left[\dfrac{0.6\cos^2\theta}{a}\right]$ Since $\tan\theta = \dfrac{x}{a}$ $\sec^2\theta = 1+\tan^2\theta = 1+\dfrac{x^2}{a^2}$ $\dfrac{dS}{dt} = 0.8\left[\dfrac{0.6}{a\left(1+\dfrac{x^2}{a^2}\right)}\right]$ $\dfrac{dS}{dt} = \dfrac{0.48a}{a^2+x^2} = \dfrac{12a}{25\left(a^2+x^2\right)}$

Source of Question: CJC/Promo/2018/01/Q7

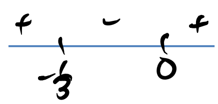
3 A curve C has equation $kx^2+2xy-3y^2=5$ where k is a non-zero constant.

(i) Show that $\dfrac{dy}{dx} = \dfrac{kx+y}{3y-x}$. [2]

(ii) Find the range of values of k such that tangents to the curve C are parallel to the x –axis. [4]

(iii) For the case where $k=13$, a point $P(x, y)$ moves along the curve C in such a way that its x–coordinate is increasing at a constant rate of 5 units per second. Find the rate of change of its y–coordinate at the instant when $x=1$ and $y=2$. [2]

Solution:

3(i) **[2]**	$kx^2+2xy-3y^2=5$ $2kx+\left(2x\frac{\mathrm{d}y}{\mathrm{d}x}+2y\right)-6y\frac{\mathrm{d}y}{\mathrm{d}x}=0$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2kx+2y}{6y-2x}$ $=\frac{kx+y}{3y-x}\text{(shown)}$
(ii) **[4]**	For tangents parallel to x-axis, $\frac{\mathrm{d}y}{\mathrm{d}x}=0$, $kx+y=0$ $y=-kx$ Substitute $y=-kx$ into C, $kx^2+2x(-kx)-3(-kx)^2=5$ $x^2\left(-k-3k^2\right)=5$ $x^2=\frac{-5}{k+3k^2}$ Since k is a non-zero constant, $k+3k^2<0$ $k(1+3k)<0$  $-\frac{1}{3}<k<0$
(iii) **[2]**	$k=13$, $x=1$ and $y=2$, $\frac{\mathrm{d}x}{\mathrm{d}t}=5$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{13(1)+(2)}{3(2)-(1)}=3$ $\frac{\mathrm{d}y}{\mathrm{d}t}=\frac{\mathrm{d}y}{\mathrm{d}x}\cdot\frac{\mathrm{d}x}{\mathrm{d}t}=15$ units per second.

Source of Question: EJC/Promo/2018/01/Q6

4 A curve D has parametric equations $x = a\theta - a\sin\theta$, $y = a - a\cos\theta$, where $a > 0$ and $0 < \theta < \pi$.

(i) Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \cot\frac{\theta}{2}$. [3]

(ii) Find the exact equation of the normal to the curve at the point for which $\theta = \frac{\pi}{3}$. [4]

Solution:

4(i) **[3]**	$\frac{\mathrm{d}x}{\mathrm{d}\theta} = a - a\cos\theta$ and $\frac{\mathrm{d}y}{\mathrm{d}\theta} = a\sin\theta$ $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}\theta} \div \frac{\mathrm{d}x}{\mathrm{d}\theta}$ $= \frac{a\sin\theta}{a - a\cos\theta}$ $= \frac{\sin\theta}{1-\cos\theta}$ $= \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}$ (double angle formula) $= \cot\frac{\theta}{2}$ (shown)
(ii) **[4]**	When $\theta = \frac{\pi}{3}$, $x = a\left(\frac{\pi}{3}\right) - a\sin\left(\frac{\pi}{3}\right) = \left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)a$ and $y = a - a\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}a$ Gradient of normal $= -\tan\frac{\theta}{2} = -\tan\frac{\pi}{6} = -\frac{1}{\sqrt{3}}$ Equation of normal: $y - \frac{1}{2}a = -\frac{1}{\sqrt{3}}\left[x - \left(\frac{\pi}{3}a - \frac{\sqrt{3}}{2}a\right)\right]$ $y = -\frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{3}}\left(\frac{\pi}{3}a - \frac{\sqrt{3}}{2}a\right) + \frac{1}{2}a$ $y = -\frac{1}{\sqrt{3}}x + \frac{a\pi}{3\sqrt{3}}$

Source of Question: MJC/Promo/2018/01/Q4

5 [It is given that velocity is the rate of change of displacement.]

A particle travels in a straight line so that its displacement, s cm, from a fixed point A is given by $s=\frac{17}{2}t^2-\frac{1}{4}t^3$, where t is the time in seconds after the start of travel.

(i) Find the exact values of t at which the velocity of the particle is zero. [2]

(ii) Use differentiation to find the exact maximum velocity of the particle, proving that it is a maximum. [4]

(iii) Sketch the velocity-time graph of the particle over the time interval of t seconds, where $t \geq 0$, labelling the axial intercepts and the turning point. [2]

(iv) Point B is at a displacement of 252 cm from the fixed point A. Find the time taken for the particle to first reach point B. [2]

Solution:

5(i) **[2]**	$s=\frac{17}{2}t^2-\frac{1}{4}t^3$ $v=\frac{ds}{dt}$ $=17t-\frac{3}{4}t^2$ When $v=0$, $17t-\frac{3}{4}t^2=0$ $t\left(17-\frac{3}{4}t\right)=0$ $\therefore t=0 \quad \text{or} \quad t=\frac{68}{3}$ Note: Accept solution using GC
(ii) **[4]**	$\frac{dv}{dt}=-\frac{3}{2}t+17$ When $\frac{dv}{dt}=0$, $-\frac{3}{2}t+17=0$ $t=\frac{34}{3}$ $\frac{d^2v}{dt^2}=-\frac{3}{2}<0$

$\therefore t = \dfrac{34}{3}$ gives a maximum velocity.

t	$\left(\dfrac{34}{3}\right)^-$	$\dfrac{34}{3}$	$\left(\dfrac{34}{3}\right)^+$
$\dfrac{\mathrm{d}V}{\mathrm{d}t}$	+ve /	0 —	-ve \

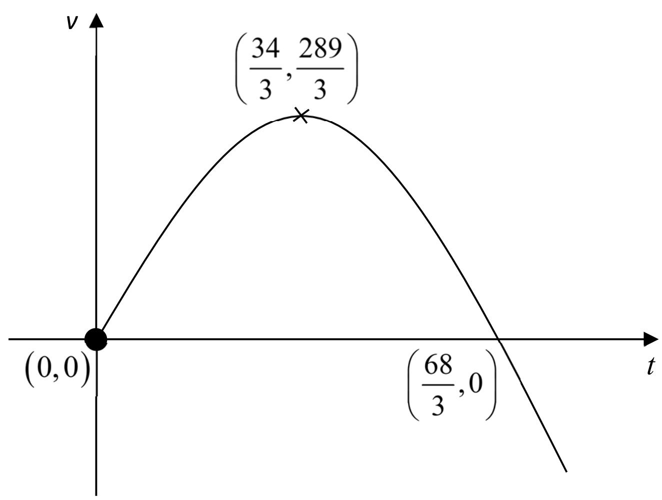
When $t = \dfrac{34}{3}$,

$$v = -\frac{3}{4}\left(\frac{34}{3}\right)^2 + 17\left(\frac{34}{3}\right)$$

$$= \frac{289}{3}$$

$\therefore$ Maximum velocity $= \dfrac{289}{3}$ cm/s

(iii)
[2]



(iv)
[2]

When $s = 252$,

$$-\frac{1}{4}t^3 + \frac{17}{2}t^2 = 252$$

$$\frac{1}{4}t^3 - \frac{17}{2}t^2 + 252 = 0$$

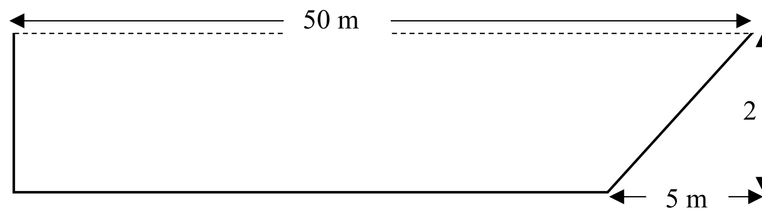
Using GC,

$t = -5.08$ (rej. $\because t \geq 0$) or $t = 6$ or $t = 33.1$

Time taken to first reach point B is 6 seconds.

Source of Question: NJC/Promo/2018/01/Q12

6 **(a)** The diagram below (not drawn to scale) shows a cross-section of an empty swimming pool measuring 50 m long, 20 m wide and 2 m deep.



Suppose water is being pumped into the pool at a rate of 100 m^3 per min. How fast is the water level in the pool rising when the depth of water is 1.6 m? [4]

(b) Figure A shows a dining table that consists of 3 parts:

- a cylindrical table top of radius r m,
- a cylindrical base of height $\dfrac{1}{20\sqrt{r}}$ m, that is similar to the table top,
- and a cylindrical body of a fixed radius 0.3 m and height 1 m, attached to the table top and the base.

Figure B shows a side view of the table which is symmetrical about a vertical axis.

Figure A

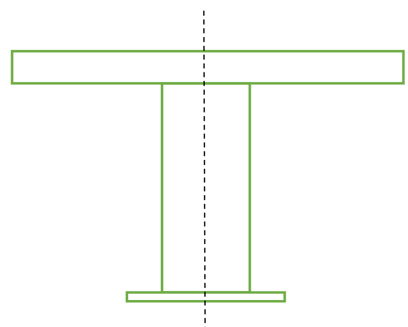


Figure B

The ratio of the volume of the table top to that of the base is to be kept at $8:r^3$.

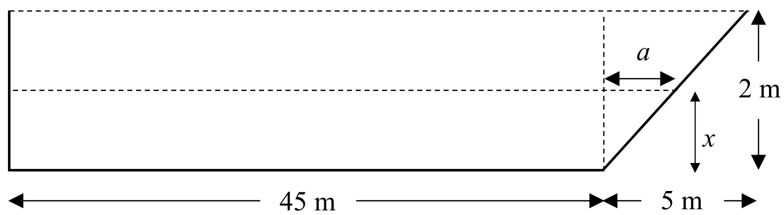
The curved surface of a cylinder is known as the lateral surface. Show that the total area S, in m^2, of the lateral surfaces of the three cylinders that form the table is given by

$$\pi\left(\frac{1}{5\sqrt{r}}+\frac{r\sqrt{r}}{20}+K\right),$$

where the constant K is to be determined. [4]

Use differentiation to find the exact value of r that minimises S. [3]

Solution:

6(a) **[4]**	 Let V be the volume of water in the pool when the depth of water is x m. By similar triangles, $\frac{a}{5}=\frac{x}{2} \Rightarrow a=\frac{5}{2}x$. $\therefore V=\frac{1}{2}\left(45+45+\frac{5}{2}x\right)(x)(20)=900x+25x^2$ $\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{\mathrm{d}x}{\mathrm{d}V}\times\frac{\mathrm{d}V}{\mathrm{d}t}$ $=\frac{1}{900x+50x^2}\times 100$ $=\frac{10}{90+5x}$ When x = 1.6, $\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{10}{90+5(1.6)}=\frac{5}{49}$ m/min
(b) **[7]**	Let x m be the radius of the base and h m be the height of the table top. Since the table top and base are similar, both ratio of the radii and ratio of the heights are equal to $\sqrt[3]{\frac{8}{r^3}}=\frac{2}{r}$, i.e. $\frac{r}{x}=\frac{h}{\frac{1}{20r^{0.5}}}=\frac{2}{r}$ Thus, $x=\frac{r^2}{2}$ and $h=\frac{1}{20r^{0.5}}\times\frac{2}{r}=\frac{1}{10r^{1.5}}$ Lateral surface area of the table top $=2\pi r\left(\frac{1}{10r^{1.5}}\right)$ $=\frac{\pi}{5r^{0.5}}$

	Lateral surface area of the base $= 2\pi\left(\frac{1}{2}r^2\right)\left(\frac{1}{20r^{0.5}}\right)$ $= \frac{\pi r^{1.5}}{20}$ Lateral surface area of the body $= 2\pi(0.3)(1)$ $= 0.6\pi$ $S = \frac{\pi}{5r^{0.5}} + \frac{\pi r^{1.5}}{20} + 0.6\pi$ $\quad = \pi\left(\frac{1}{5\sqrt{r}} + \frac{r\sqrt{r}}{20} + 0.6\right)$ (Shown) $\frac{\mathrm{d}S}{\mathrm{d}r} = \pi\left(\frac{-0.5r^{-1.5}}{5} + \frac{1.5r^{0.5}}{20}\right) = \pi\left(-\frac{1}{10r^{1.5}} + \frac{3r^{0.5}}{40}\right)$ $\frac{\mathrm{d}S}{\mathrm{d}r} = 0$ $\Rightarrow \frac{1}{10r^{1.5}} = \frac{3r^{0.5}}{40}$ $\Rightarrow 30r^2 = 40$ $\Rightarrow r = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}}$ $\frac{\mathrm{d}^2S}{\mathrm{d}r^2} = \pi\left(\frac{1.5r^{-2.5}}{10} + \frac{1.5r^{-0.5}}{40}\right) > 0$ for all $r > 0$. Thus, S is minimum when $r = \frac{2}{\sqrt{3}}$ (or $\frac{2\sqrt{3}}{3}$).

Source of Question: SAJC/Promo/2018/01/Q3

7 For the curve with equation $40x^2 - 36xy + 9y^2 - 25 = 0$,

(i) show that $\dfrac{dy}{dx} = \dfrac{2(9y-20x)}{9(y-2x)}$; [3]

(ii) find the coordinates of the point(s) at which the tangent is parallel to the y-axis. [3]

Solution:

7(i) **[3]**	$40x^2 - 36xy + 9y^2 - 25 = 0$ Differentiating with respect to x, $80x - 36x\dfrac{dy}{dx} - 36y + 18y\dfrac{dy}{dx} = 0$ $40x - 18x\dfrac{dy}{dx} - 18y + 9y\dfrac{dy}{dx} = 0$ $(9y - 18x)\dfrac{dy}{dx} = 18y - 40x$ $\dfrac{dy}{dx} = \dfrac{2(9y-20x)}{9(y-2x)}$
(ii) **[3]**	At the point where tangent is parallel to y-axis, $\dfrac{dy}{dx}$ is undefined. i.e. $y - 2x = 0$ Substitute $y = 2x$ into the equation of the curve: $40x^2 - 36x(2x) + 9(2x)^2 - 25 = 0$ $40x^2 - 72x^2 + 36x^2 - 25 = 0$ $x^2 = \dfrac{25}{4} \quad \Rightarrow \quad x = \pm\dfrac{5}{2}, y = \pm 5$ Coordinates of the points are $\left(\dfrac{5}{2}, 5\right)$ and $\left(-\dfrac{5}{2}, -5\right)$.

Source of Question: MI/Promo/2018/01/Q5(a)

8 Given a function $\mathrm{f}(x)=x^2\mathrm{e}^{x^2}$, for $x\in\mathbb{R}$.

(i) By differentiation, find the range of values of x for which the function is increasing. [4]

(ii) Hence find the equation of the tangent to the curve, $\mathrm{f}(x)=x^2\mathrm{e}^{x^2}$ for $x\in\mathbb{R}$, at the point where $x=1$, giving your answer in terms of e. [2]

Solution:

8(i) **[4]**	$\mathrm{f}(x)=x^2\mathrm{e}^{x^2}$, for $x\in\mathbb{R}$, $\mathrm{f}'(x)=x^2\left(2x\mathrm{e}^{x^2}\right)+2x\mathrm{e}^{x^2}$ $=2x\mathrm{e}^{x^2}\left(x^2+1\right)$ Since $x^2+1>0$ and $\mathrm{e}^{x^2}>0$, for all $x\in\mathbb{R}$, Therefore when $x>0$, we have $\mathrm{f}'(x)>0$ $\mathrm{f}'(x)>0\Rightarrow$ f is increasing when $x>0$.
(ii) **[2]**	When $x=1$, $\mathrm{f}'(1)=2(1)\mathrm{e}(2)=4\mathrm{e}$, $\mathrm{f}(1)=\mathrm{e}$ Equation of tangent at $x=1$: $y-\mathrm{e}=4\mathrm{e}(x-1)$ $\therefore y=4\mathrm{e}x-3\mathrm{e}$

Source of Question: ACJC/Promo/2018/01/Q8

9 **(i)** Given that $y=\mathrm{e}^{\sqrt{1+2x}}$, show that

$$\left(\ln y\right)\frac{\mathrm{d}y}{\mathrm{d}x}=y\,. \qquad [1]$$

By further differentiation of this expression, find the series expansion for *y*, up to and including the term in x^3, giving all coefficients in exact form. [3]

(ii) Find the series expansion for $\sqrt{1+2x}$, up to and including the term in x^3. [1]

By using the Maclaurin expansion for e^x found in the List of Formulae (MF26) and the expansion for $\sqrt{1+2x}$, verify your answer in **(i)**. [2]

Solution:

9(i) **[4]**	$y=\mathrm{e}^{\sqrt{1+2x}} \quad\Rightarrow\quad \ln y=\sqrt{1+2x}$ differentiating w.r.t. *x*, $\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{\sqrt{1+2x}}$ $=\frac{1}{\ln y}$ Hence $\left(\ln y\right)\frac{\mathrm{d}y}{\mathrm{d}x}=y$ (shown). Differentiating w.r.t. *x*, $\left(\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}y}{\mathrm{d}x}+\left(\ln y\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\mathrm{d}y}{\mathrm{d}x} \quad\Rightarrow\quad \frac{1}{y}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+\left(\ln y\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\mathrm{d}y}{\mathrm{d}x}.$ Differentiating w.r.t. *x*, $\left(-\frac{1}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+\frac{1}{y}\left(2\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)+\left(\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(\ln y\right)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=\frac{\mathrm{d}^2y}{\mathrm{d}x^2}.$ When $x=0$, $y=\mathrm{e}$, $\frac{\mathrm{d}y}{\mathrm{d}x}=\mathrm{e}$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0$, and $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=\mathrm{e}$. Hence by Maclaurin expansion, $y\approx\mathrm{e}+\mathrm{e}x+\frac{\mathrm{e}}{3!}x^3=\mathrm{e}+\mathrm{e}x+\frac{\mathrm{e}}{6}x^3.$
(ii) **[3]**	$\sqrt{1+2x}=\left(1+2x\right)^{\frac{1}{2}}$ $=1+\frac{1}{2}\left(2x\right)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2}\left(2x\right)^2+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}\left(2x\right)^3+\ldots$ $\approx 1+x-\frac{1}{2}x^2+\frac{1}{2}x^3.$ Hence $\mathrm{e}^{\sqrt{1+2x}}$

	$\approx e^{\left(1+x-\frac{1}{2}x^2+\frac{1}{2}x^3\right)}$ $= e \cdot e^{\left(x-\frac{1}{2}x^2+\frac{1}{2}x^3\right)}$ $\approx e\left[1+\left(x-\frac{1}{2}x^2+\frac{1}{2}x^3\right)+\frac{\left(x-\frac{1}{2}x^2+\frac{1}{2}x^3\right)^2}{2!}+\frac{\left(x-\frac{1}{2}x^2+\frac{1}{2}x^3\right)^3}{3!}\right]$ $\approx e\left[1+x-\frac{1}{2}x^2+\frac{1}{2}x^3+\frac{1}{2}x^2-\frac{1}{4}x^3-\frac{1}{4}x^3+\frac{1}{6}x^3\right]$ $= e\left(1+x+\frac{1}{6}x^3\right) = e+ex+\frac{e}{6}x^3$ (verified).

Source of Question: VJC/Promo/2018/01/Q5

10 It is given that $y = -\sec 2x$.

(i) Show that $\frac{d^2y}{dx^2} = -4$ when $x = 0$. [3]

(ii) Find the first two non-zero terms in the Maclaurin series for y. [2]

It is given further that the first two non-zero terms in the series expansion of y are equal to the first two non-zero terms in the series expansion of $(a+bx^2)^{\frac{1}{3}}$ where a and b are constants. Find the values of a and b. [3]

Solution:

10(i) **[3]**	$\frac{dy}{dx} = -2\sec 2x \tan 2x$ $\frac{d^2y}{dx^2} = -2\left\{\sec 2x\left(2\sec^2 2x\right)+\tan 2x\left(2\sec 2x \tan 2x\right)\right\}$ $= -4\sec 2x\left(\sec^2 2x+\tan^2 2x\right)$ When $x=0$, $\tan 2x = 0$, $\sec 2x = 1$ $\therefore \frac{d^2y}{dx^2} = -4(1+0) = -4$
(ii) **[2]**	At $x = 0$, $y = -1$, $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = -4$ First two non-zero terms in the Maclarin's series for y is $y = -1+0(x)+\frac{-4}{2!}x^2+...$ $y = -1-2x^2+...$

(iii) **[3]**	$(a+bx^2)^{\frac{1}{3}}$ $=a^{\frac{1}{3}}(1+\frac{b}{a}x^2)^{\frac{1}{3}}$ $=a^{\frac{1}{3}}(1+\frac{bx^2}{3a})+...$ Since the first two non-zero terms are equal $a=-1,$ $\frac{bx^2}{3}=-2x^2, b=-6$ $\therefore a=-1, b=-6$

Source of Question: MI/Promo/2018/02/Q6

11 It is given that $\mathrm{f}(x)=\cos^4 x+\sin x$, where $x \geq 0$.

(i) Given that x is a sufficiently small angle, show that

$$\mathrm{f}(x) \approx 1+ax+bx^2,$$

for constants a and b to be determined. [4]

(ii) Joel uses the answer to part **(i)** to give an approximation for $\int_0^{\frac{\pi}{3}} \mathrm{f}(x)\,\mathrm{d}x$. Explain why the approximation is not very good. [1]

(iii) Suggest a method to improve the approximation in part **(ii)**. [1]

Solution:

11(i) **[4]**	$\mathrm{f}(x)=\cos^4 x+\sin x$ $\approx \left(1-\frac{x^2}{2}\right)^4+x$ $=1+4\left(-\frac{x^2}{2}\right)+x+...$ $\approx 1+x-2x^2$ Therefore, $a=1$, $b=-2$.
(ii) **[1]**	The approximation is not very good because this is only **valid for small values of** x. $x=\frac{\pi}{3}$ is not small. **OR** The approximation is not very good because the approximation for the integrand $\mathrm{f}(x)$ **consists of only a few terms**.

(iii) **[1]**	One can increase the number of terms for the approximation for $\mathrm{f}(x)$ to make the integral approximation better for larger values of x.

12 Let f be a continuous and differentiable function.

(a) Explain with the aid of a sketch, why $\mathrm{f}'(0)=\lim_{h\to 0}\mathrm{S}(h)$, where $\mathrm{S}(h)=\dfrac{\mathrm{f}(h)-\mathrm{f}(-h)}{2h}$. [2]

(b) A student proposes to find $\mathrm{f}'(0)$ using the function $\mathrm{T}(h)=\dfrac{\mathrm{f}(-2h)-8\mathrm{f}(-h)+8\mathrm{f}(h)-\mathrm{f}(2h)}{kh}$, where k is a constant.

(i) Assuming that the Maclaurin series of f exists, write down the Maclaurin series of $\mathrm{f}(h)$ and $\mathrm{f}(2h)$ up to and including the term in h^5. [2]

(ii) Hence or otherwise, determine k such that $\mathrm{f}'(0)=\lim_{h\to 0}\mathrm{T}(h)$. [3]

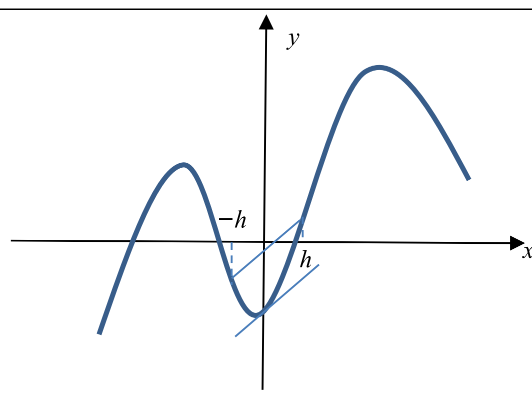
(c) Let $\mathrm{f}(x)=\sin x$. Determine the largest m such that

(i) $\mathrm{S}(h)$ differs from $\mathrm{f}'(0)$ by at most 10^{-3} for $|h|\le m$. [2]

(ii) $\mathrm{T}(h)$ differs from $\mathrm{f}'(0)$ by at most 10^{-3} for $|h|\le m$. [2]

(iii) Comment on the values found above in parts **(c)(i)** and **(c)(ii)**. [1]

Solution

3 **(a)**	 From the sketch, $\dfrac{\mathrm{f}(h)-\mathrm{f}(-h)}{2h}$ represents the gradient of the line joining the two points $(h,\mathrm{f}(h))$ and $(-h,\mathrm{f}(-h))$. This gradient approaches the value of the tangent to the curve at $x=0$ when h tends to 0.

(b) **(i)**	$\mathrm{f}(h)=\mathrm{f}(0)+h\mathrm{f}'(0)+\frac{h^2}{2}\mathrm{f}''(0)+\frac{h^3}{6}\mathrm{f}'''(0)+\frac{h^4}{24}\mathrm{f}^{(4)}(0)+\frac{h^5}{120}\mathrm{f}^{(5)}(0)+\dots$ $\mathrm{f}(2h)=\mathrm{f}(0)+2h\mathrm{f}'(0)+2h^2\mathrm{f}''(0)+\frac{4h^3}{3}\mathrm{f}'''(0)+\frac{2h^4}{3}\mathrm{f}^{(4)}(0)+\frac{4h^5}{15}\mathrm{f}^{(5)}(0)+\dots$
(ii)	Therefore, $\mathrm{T}(h)=\frac{\mathrm{f}(-2h)-8\mathrm{f}(-h)+8\mathrm{f}(h)-\mathrm{f}(2h)}{kh}$ $=\frac{1}{kh}\begin{pmatrix} \mathrm{f}(0)-2h\mathrm{f}'(0)+2h^2\mathrm{f}''(0)-\frac{4h^3}{3}\mathrm{f}'''(0)+\frac{2h^4}{3}\mathrm{f}^{(4)}(0)-\frac{4h^5}{15}\mathrm{f}^{(5)}(0)+\dots \\ -8\mathrm{f}(0)+8h\mathrm{f}'(0)-4h^2\mathrm{f}''(0)+\frac{4h^3}{3}\mathrm{f}'''(0)-\frac{h^4}{3}\mathrm{f}^{(4)}(0)+\frac{h^5}{15}\mathrm{f}^{(5)}(0)+\dots \\ +8\mathrm{f}(0)+8h\mathrm{f}'(0)+4h^2\mathrm{f}''(0)+\frac{4h^3}{3}\mathrm{f}'''(0)+\frac{h^4}{3}\mathrm{f}^{(4)}(0)+\frac{h^5}{15}\mathrm{f}^{(5)}(0)+\dots \\ -\mathrm{f}(0)-2h\mathrm{f}'(0)-2h^2\mathrm{f}''(0)-\frac{4h^3}{3}\mathrm{f}'''(0)-\frac{2h^4}{3}\mathrm{f}^{(4)}(0)-\frac{4h^5}{15}\mathrm{f}^{(5)}(0)+\dots \end{pmatrix}$ $=\frac{1}{kh}\left(12h\mathrm{f}'(0)-\frac{2h^5}{5}\mathrm{f}^{(5)}(0)+\dots\right)$ $=\frac{12}{k}\mathrm{f}'(0)-\frac{2}{5k}h^4\mathrm{f}^{(5)}(0)+\dots$ Hence for $\mathrm{f}'(0)=\lim_{h\to 0}\mathrm{T}(h)$, we need $k = 12$.
(c) **(i)**	$\mathrm{f}'(0)=1$. Solving the inequality $\left\lvert\frac{\mathrm{f}(h)-\mathrm{f}(-h)}{2h}-1\right\rvert\le 10^{-3}$ $\Rightarrow\left\lvert\frac{\sin h}{h}-1\right\rvert\le 10^{-3}$ From GC, $-0.0774\le h\le 0.0774$ (accept 0.0775) Thus largest m is 0.0774
(ii)	$\left\lvert\frac{16\sin h-2\sin 2h}{12h}-1\right\rvert\le 10^{-3}$ From GC, $-0.418\le h\le 0.418$ Thus largest m is 0.418
(iii)	This shows that to approximate the derivative of sin x at 0 to an accuracy of 0.001, the second method does not require such a small h compared to the first method.

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