Polar Coordinates [FM]

1. [04/AJC/FM/I/6]

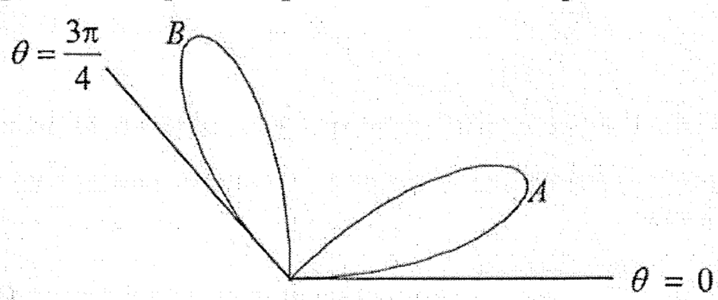
A curve C has equation, in polar coordinates, $r = \dfrac{a}{1+\cos\theta}$, $0 \le \theta \le 2\pi$, where a is a positive constant.

Let the pole be denoted by O, the points $P(r_1, \theta_1)$ and $Q(r_2, \theta_2)$ on the curve C are such that POQ is a straight line and $0 < \theta_1 < \pi$.

If the length of the chord PQ is $\dfrac{8a}{3}$, find the coordinates of point P. **[5]**

2. [04/NJC/FM/I/7]

The diagram shows part of a polar curve C with equation $r = a\sin b\theta$, where $b \in \mathbb{Z}^+$.



State, with a reason, the least possible value of b. **[2]**

(i) Prove that $r^2 + \left(\dfrac{dr}{d\theta}\right)^2 = a^2\left(1+\left(b^2-1\right)\cos^2 b\theta\right)$. Using the value of b found above, show that the arc length of one loop of C is at most πa. **[3]**

(ii) A and B are two points on C furthest from the pole. Show that the cartesian coordinates of A is $\left(a\cos\dfrac{\pi}{8}, a\sin\dfrac{\pi}{8}\right)$. Hence, find the cartesian equation of the line AB in the form $y = mx + c$. **[3]**

[least $b = 4$; $y = -\left(\cot\dfrac{3\pi}{8}\right)x + a\cot\dfrac{3\pi}{8}\cos\dfrac{\pi}{8} + a\sin\dfrac{\pi}{8}$]

3. [04/TJC/FM/I/5]

The curve C has polar equation $r = \dfrac{1}{\sqrt{\sin\theta}}$, for $0 < \theta < \pi$.

(i) Show that C has the following characteristics:

(a) $r \geq 1$ for the given range of θ;

(b) the curve C is symmetrical about the line $\theta = \dfrac{\pi}{2}$;

(c) the curve C has an asymptote $\theta = 0$.
[Hint: consider the x- and y- coordinates of C as $\theta \to 0$.] **[3]**

(ii) Give a sketch of the curve. **[1]**

(iii) Find the area of the closed region bounded by C and the line with cartesian equation $y = \dfrac{1}{\sqrt{2}}$. **[5]**

4 The curve C_1 has polar equation

$$r = a + a\cos 2\theta \text{ for } 0 \leq \theta < 2\pi \text{, where } a \text{ is a positive constant.}$$

The curve C_2 is obtained by rotating C_1 through an angle of $\dfrac{\pi}{2}$ anticlockwise about the pole.

(i) Find a polar equation for C_2 in terms of $\cos 2\theta$. [1]
(ii) On the same diagram, sketch the curves C_1 and C_2, indicating the polar coordinates of the points of intersections of the curves. [2]
(iii) Find the exact area of the regions that lie within both C_1 and C_2. [4]
(iv) A design for a necklace is made by taking the combined graphs of C_1 and C_2, but with the perimeter of the regions described in part **(iii)** removed. Find the perimeter of the necklace. [2]

[CJC/FM/2017/Promo/Q4]

5 Show that the gradient of tangent of the polar curve with equation $r = \dfrac{1}{1+\cos\theta}$ is $-\cot\dfrac{\theta}{2}$. [5]

[EJC/FM/2017/Promo/Q8b]

6 The curve C has equation $2x^2 - xy + 2y^2 = 150$.

(a) Show that a polar equation for C can be expressed in the form

$$r^2 = \frac{P}{Q + R\sin 2\theta},$$

where P, Q and R are integers to be found and $-\pi < \theta \leq \pi$. [3]

(b) Hence, find the polar coordinates of the points on C which are the furthest from the pole O. [3]

[JJC/FM/2017/Promo/Q2]

7 The straight line with polar equation

$$r_1 = \frac{1}{a\sin\theta + b\cos\theta},$$

is a tangent to the circle with the polar equation,

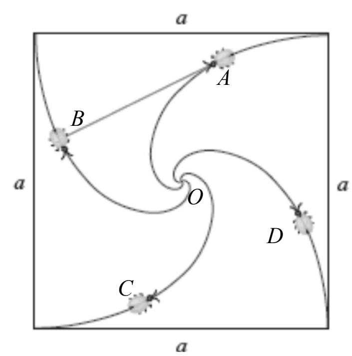
$$r_2 = 2c\cos\theta,$$

where a, b and c are real numbers, $a^2 + b^2 \neq 0$ and $c \neq 0$.

By first finding the Cartesian equations of the respective polar equations, find the possible value(s) of c in terms of a and/or b. [8]

[MJC/FM/2017/Promo/Q2]

8



In the above diagram, four bugs A, B, C and D are placed at the four corners of a square with side of length a. The bugs crawl counter clockwise towards the centre of the square, O, along the spiral paths. Bug A starts from the corner $\left(\frac{a}{2}, \frac{a}{2}\right)$. The line joining the bug A to the bug B is tangent to the path of the bug A.

(i) Taking O as the origin and the coordinates of the bug A to be (x, y), explain why the Cartesian coordinates of the bug B are $(-y, x)$. **[1]**

It is given that $OA = r$ and OA makes an angle θ with the x-axis.

(ii) Show that the gradient of the line AB is $\dfrac{\tan\theta - 1}{\tan\theta + 1}$. **[2]**

(iii) By expressing $\dfrac{dx}{d\theta}$ and $\dfrac{dy}{d\theta}$ in terms of r, θ and $\dfrac{dr}{d\theta}$, show that $\dfrac{dr}{d\theta} = -r$. **[4]**

(iv) It is known that the polar equation of the path of the bug A is in the form $r = ke^{-\theta}$, show that $k = \dfrac{\sqrt{2}a}{2}e^{\frac{\pi}{4}}$, assuming the pole is at the centre of the square. **[2]**

(v) Find the exact distance travelled by the bug A for $\dfrac{\pi}{4} \leq \theta \leq 2\pi$. **[4]**

[SAJC/FM/2017/Promo/Q10]

9 A curve C has polar equation given by

$$r^2 = a\cos^2\theta + b\sin^2\theta\text{, where } a \text{ and } b \text{ are non-zero constants.}$$

(a) State the condition(s) on a and b so that C has tangent(s) through the pole. [1]

(b) State the condition(s) on a and b so that C will never come back to the pole. [1]

(c) Sketch C, where $0 \le \theta \le 2\pi$, for the following cases:

(i) $\sqrt{a} > \sqrt{b}$, [2]

(ii) $a > 0$ and $b < 0$. [2]

(d) Given that $a = 4$ and $b = 2$, find the exact area of the region enclosed by C. [3]

(e) Write down the polar equation of the curve if C is

(i) rotated 90° anti-clockwise about the pole, [1]

(ii) reflected about the line $y = x$. [1]

[TJC/FM/2017/Promo/Q8]

10 Sketch the curve $r = \sin 2\theta$, for $r \ge 0$, $0 \le \theta < 2\pi$. [2]

Describe the curves $r = \sin 2n\theta$, where $r \ge 0$, n are positive integers and show that the area enclosed by such a curve is independent of n. [6]

[VJC/FM/2018/P1/4]

11 The curve D has polar equation

$$r = 6\sin\frac{1}{2}\theta\text{, where } 0 \le \theta < 2\pi.$$

(i) Sketch D, indicating clearly all key features and symmetries of the curve. [2]

(ii) Find the arc length of D. [2]

The locus of points (r, θ) satisfying $6\sin\frac{1}{2}\theta \le r \le 3$ forms a region R.

(iii) Find the exact area of R. [5]

[NJC/FM/2018/P2/4]

12 The curve Γ has polar equation

$$r = 1 + \cos\theta, \quad 0 \le \theta \le 2\pi.$$

The circle with equation $(x-2)^2 + y^2 = 4$ intersects Γ at O, A and B, where O is the pole.

Determine the perimeter of the sector OAB, where OA and OB are straight line segments and AB is an arc on Γ that lies within the circle.

Leave your answer in an exact surd form. [8]

[VJC/FM/2019/MCT/6]

13 The curve T_1 has polar equation $r = 3 + 2\cos 3\theta$, $0 \le \theta < 2\pi$.

(i) Sketch T_1, indicating all the key features and equations of lines of symmetry of the curve. [3]

(ii) A piece of wire of length 32 units long is used to bend into the shape of T_1. State, with a reason, whether the wire is long enough to do so. [2]

Another curve T_2 has polar equation $r = 2 + \cos 3\theta$, $0 \le \theta < 2\pi$.

(iii) Use calculus to evaluate the exact area enclosed in between T_1 and T_2. [3]

[RI/FM/2019/MCT/3]

14 [It is given that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.]

A polar curve C has equation $r = \sin 2\theta$ where $0 \le \theta \le \frac{\pi}{2}$. O is the pole and the tangent at a point $P(r, \theta)$ on the curve is parallel to the line $\theta = \frac{\pi}{2}$.

(i) Sketch *C*. [2]

(ii) Show that $\cos\theta = \sqrt{\frac{2}{3}}$ at *P*. [3]

(iii) Find the area bounded by *C*, the line $\theta = 0$ and the tangent at *P*. [5]

[TJC/FM/2018/P1/6]

15 **(a)** The Archimedean spiral, *S*, which was first studied by the Greek mathematician Archimedes in the 3rd century BC, has polar equation given by

$$r = a + b\theta$$

where *a* and *b* are non-negative real constants and $\theta \ge 0$.

The Archimedean spiral has the property that any ray from the pole intersects successive turnings of the spiral at points with a constant separation distance *d*, hence also the name “arithmetic spiral”.

(i) Prove the above property and state the value of *d*. [2]

(ii) For the case when $a = b$, prove that the angle which the tangent to *S* at the point (r, θ) makes with the initial line is given by

$$\tan^{-1}(1 + \theta) + \theta$$

and hence write down the cartesian equation of the tangent to *S* at the point where $\theta = 0$. [5]

(b) Let A and B be two points on a polar curve corresponding to $\theta = \alpha$ and $\theta = \beta$ respectively. The area of the curved surface generated when the arc AB is rotated completely about the initial line is given by the integral

$$2\pi \int_{\alpha}^{\beta} r \sin\theta \left\{ r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta} \right)^2 \right\}^{\frac{1}{2}} \mathrm{d}\theta .$$

Use the above integral to derive the formula for the surface area of a sphere of radius a, explaining your working clearly. [3]

[NYJC/FM/2017/P2/4]

16 The curve T has polar equation $r = \sqrt{2} - \sin\theta,\ 0 \le \theta < 2\pi$.

(i) Sketch T. Show on your diagram, the exact polar coordinates of the point of intersection of T with the initial line. [2]

(ii) The tangent to T at the point A where $\theta = \alpha,\ 0 < \alpha < \frac{\pi}{2}$ is parallel to the initial line.

Show that $\alpha = \frac{\pi}{4}$ (do not merely verify) and find the exact distance from A to the pole. [4]

(iii) Find the exact area of the region bounded by T, the tangent in part **(ii)** and the half-line $\theta = \frac{\pi}{6}$. [5]

[HCI et al/FM/2018/P1/6]

17 The rotating blades of a fan can be modelled by the curve C with polar equation

$$r = a\left|\cos 2\theta\right|$$

where $-\pi < \theta \le \pi$ and a is a positive constant.

(i) The region R not containing the pole is bounded between the part of C for which $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$ and the line with polar equation $r = \frac{\sqrt{3}a}{4\cos\theta}$ $\left(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\right)$. Find, in terms of a, the exact area of R. [4]

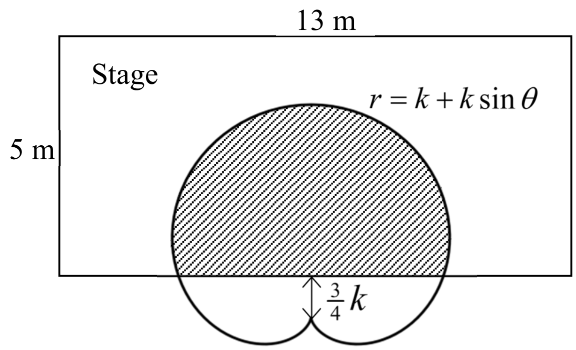
(ii) The fan is rotated through an acute angle α radians about its centre. The rotated fan can be modelled by the curve with polar equation $r = a\left|\sin 2\theta\right|$.

Without the aid of the graphing calculator, determine the exact value of α in terms of π, justifying your answer with relevant working. [2]

(iii) A thin straight wire, which can be modelled by a line ℓ, rests horizontally on top of two blades of the rotated fan in **(ii)**. Find the cartesian equation of ℓ. [4]

[NYJC/FM/2019/MYE/P1/6]

18 Jenny is holding her vocal live performance on a rectangular stage of length 13 m and breadth 5 m. The technical crew uses a microphone with a cardioid pickup pattern so that it minimizes the pickup of noise from the audience. The crew places the microphone at a distance of $\frac{3}{4}k,\ k \geq 0$, from the front of the stage as shown in the figure.



The boundary of the optimal pickup region is given by the cardioid $r = k + k\sin\theta$, where r is measured in meters and the microphone is at the pole. The optimal pickup region on the stage is indicated by the shaded area in the figure. Find

(a) the furthest distance, in terms of k, that Jenny can be on the stage from the microphone so that she is within the optimal pickup region, and [2]

(b) the minimum value of k if the optimal pickup region that Jenny has on stage is at least 75% of the stage area. [6]

[ACJC/FM/2017/P2/3]

Answers:

1. $P = \left(\tfrac{2}{3}a,\ \tfrac{1}{3}\pi\right)$ or $\left(2a,\ \tfrac{2}{3}\pi\right)$

3. **(iii)** $\ln\left(2+\sqrt{3}\right)-\frac{\sqrt{3}}{2}$

4. **(i)** $r = a - a\cos\left(2\theta\right)$ **(iii)** $\left(\frac{3}{2}\pi-4\right)a^2$ units^2 **(iv)** 13.7 a units

6. **b.** $\left(10,\frac{\pi}{4}\right)$ and $\left(10,-\frac{3\pi}{4}\right)$

7. $\left(x-c\right)^2+y^2=c^2$; $c=\frac{-b\pm\sqrt{b^2+a^2}}{a^2}$, $a\neq 0$ or $c=\frac{1}{2b}, a=0$

8. **(v)** $a\left(1-\mathrm{e}^{-\frac{7\pi}{4}}\right)$

9. **(d)** 3π [answers for the rest omitted]

11. **(ii)** 29.1 **(iii)** $9\sqrt{3}-3\pi$

12. $\frac{8}{3}\left(1+\sqrt{3}\right)$

13. **(iii)** $\frac{13\pi}{2}$ units^2

14. **(iii)** 0.0949

15 **(a)(i)** $d = 2b\pi$ **(ii)** $y = x - a$

16. **(ii)** $\frac{1}{\sqrt{2}}$ units **(iii)** $\frac{\sqrt{6}}{2}+\frac{\sqrt{3}}{16}-1-\frac{5\pi}{48}$ units^2

17. **(i)** $\frac{\pi}{12}a^2$ **(ii)** $\alpha=\frac{\pi}{4}$ **(iii)** $y=\frac{4}{3\sqrt{3}}a$

18. **a.** Furthest distance, $r = 2k$ **b.** $k \geq 4.38$