RAFFLES JUNIOR COLLEGE

JC2 Preliminary Examination 2007

MATHEMATICS
Higher 2

9740/01

Paper 1

12 September 2007

3 hours

Additional materials : Answer Paper
List of Formulae (MF15)

READ THESE INSTRUCTIONS FIRST

Write your name and CT group on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer all the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
At the end of the examination, fasten all your work securely together.

1 Find the set of values of x for which

$$\frac{e^{x^2}}{x-1} - 8x > 0.$$

Hence solve $\dfrac{e^{x^2}}{|x|-1} - |8x| > 0$. **[4]**

2 Prove by induction that, for $n \in \mathbf{Z}^+$,

$$n^2 - (n-1)^2 + (n-2)^2 - (n-3)^2 + \ldots + (-1)^{n-1}(1)^2 = \frac{n}{2}(n+1).$$ **[5]**

3 A sequence of negative numbers is defined by $x_{n+1} = \dfrac{2-3x_n}{x_n - 4}$, where $x_1 = -\dfrac{1}{7}$.

(i) Write down the values of x_2 and x_3, giving your answers correct to 3 significant figures. **[2]**

(ii) Given that as $n \to \infty$, $x_n \to \ell$, find, without the use of a graphic calculator, the value of ℓ. **[3]**

4 The graph of $y = \mathrm{f}(x)$ undergoes in succession, the following three transformations:

A: A translation of 1 unit in the negative x-direction.

B: A reflection about the y-axis.

C: A stretch parallel to the x-axis (with y-axis invariant) with a scale factor of 2.

The equation of the resulting curve is $y = \mathrm{g}(x)$ where $\mathrm{g}(x) = \dfrac{x^2-2x-3}{x+2}$.

(i) Express f(x) in the form g(ax+b), where a and b are constants to be found. **[2]**

(ii) Sketch the graph of y = g(x), showing clearly all the intersections with the axes, the asymptotes and the coordinates of turning points (if any). **[4]**

5 Given that $\mathrm{f}(r) = \dfrac{1}{r}$, where r is a positive integer, find a single expression for $\mathrm{f}(r) - \mathrm{f}(r+1)$.

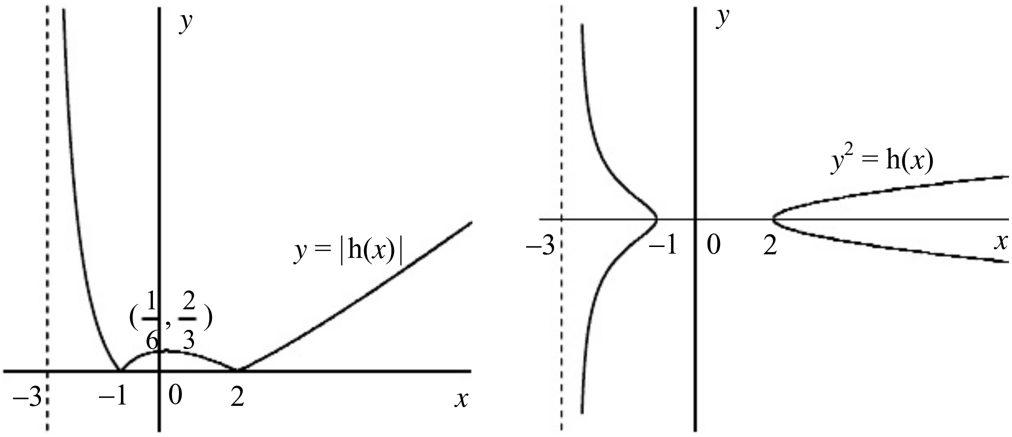
Hence, find the sum to $2n$ terms of the series $\dfrac{1}{2} + \dfrac{1}{6} + \dfrac{1}{12} + \dfrac{1}{20} + \ldots$. **[4]**

Deduce that the sum to $2n$ terms of the series $\dfrac{1}{2^2} + \dfrac{1}{3^2} + \dfrac{1}{4^2} + \dfrac{1}{5^2} + \ldots$ is less than 1. **[2]**

6 Given that $|z+3-2i| \le 2$ and $|z+5+2i| \ge |z-3i|$, illustrate the locus of the point representing the complex number z in an Argand diagram. **[4]**

Hence find the least possible value of arg z. **[2]**

7 The graphs of $y = |\,\mathrm{h}(x)\,|$ and $y^2 = \mathrm{h}(x)$ for $x > -3$ are as shown below:



(i) Explain why $\mathrm{h}(x) < 0$ for $-1 < x < 2$. **[1]**

Hence sketch the graph of $y = \mathrm{h}(x)$ for $x > -3$, showing clearly the asymptote and the coordinates of the stationary point. **[1]**

(ii) Sketch the graphs of

(a) $y = \mathrm{h}'(x)$ where h' is the derivative function of h, **[2]**

(b) $y = \dfrac{1}{\mathrm{h}(x)}$ for $x > -3$, **[3]**

showing clearly all the asymptote(s) and the coordinates of the stationary point(s).

8 Find

a) $\int \tan^{-1}\dfrac{1}{x}\,\mathrm{d}x$, **[3]**

b) $\int \dfrac{1}{x\sqrt{2x+1}}\,\mathrm{d}x$, by using the substitution $u = \sqrt{2x+1}$. **[4]**

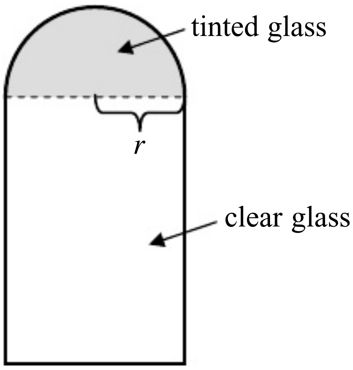
9 **(i)** The region R is bounded by the curves $y = \dfrac{4}{x^4+1}$, $y = \ln x$, the x-axis and the line $x = 2$. Calculate the area of R. **[3]**

(ii) The region Q is bounded by the curves $y = \dfrac{4}{x^4+1}$, $y = \ln x$ and the line $x = 1$. Find the volume of the solid of revolution formed when Q is rotated completely about the y-axis. **[4]**

10 A glass window, with fixed perimeter P, is in the shape of a semicircle with radius r, and a rectangle as shown in the diagram below. The semicircle is made of tinted glass and the rectangle is made of clear glass. The clear glass lets through a constant amount of light per unit area, L, while the tinted glass lets through $\frac{1}{3}$ as much light per unit area as the clear glass. Let X be the total amount of light that the entire window lets through.

(i) Show that $X = Lr\left[P - r\left(\frac{5}{6}\pi + 2\right)\right]$. **[3]**

(ii) Prove that, to let maximum amount of light through, the radius r of the window is $\frac{3P}{(5\pi + 12)}$. **[4]**



11 **(i)** Find $\int_0^x \frac{1}{\sqrt{1-4t^2}}\,\mathrm{d}t$. **[2]**

(ii) Expand $\left(1-4x^2\right)^{-\frac{1}{2}}$ as a series in ascending powers of x up to and including the term in x^4, simplifying the coefficients. Write down the set of values of x for which the expansion is valid. **[3]**

(iii) By using the results from parts **(i)** and **(ii)** above, show that the Maclaurin's series for $\sin^{-1}(2x)$ is given by

$$2x + \frac{4}{3}x^3 + \frac{12}{5}x^5 + \ldots .$$ **[2]**

(iv) Hence, show that $\frac{\pi}{6} \approx \frac{2009}{3840}$. **[2]**

12 The line l_1 has equation $x-1=2-y=-z$.
The line l_2 passes through the point $A(-2,1,7)$ and is parallel to the vector $-3\mathbf{i}+\mathbf{j}+5\mathbf{k}$.

(i) Write down the vector equations of lines l_1 and l_2. **[2]**

(ii) Show that the lines l_1 and l_2 intersect and find the coordinates of E, the point of intersection of l_1 and l_2. **[3]**

(iii) The acute angle between the lines l_1 and l_2 is denoted by θ. By finding $\cos\theta$, show that $\sin\theta=\dfrac{2\sqrt{2}}{\sqrt{35}}$. Hence find the shortest distance from A to the line l_1, leaving your answer in exact form. **[4]**

13 On 1st Jan 2000, Selina deposits \$1000 into an account which pays a *fixed* interest of \$200 per year, credited into the account at the end of every year.

On 1st Jan 2005, Hebe deposits \$1000 into an investment account and receives an interest of \$100 on 31st Dec 2005. Thereafter, the interest returned at the end of the year is 1.5 times the interest returned in the previous year.

Taking year 2000 as the first year,

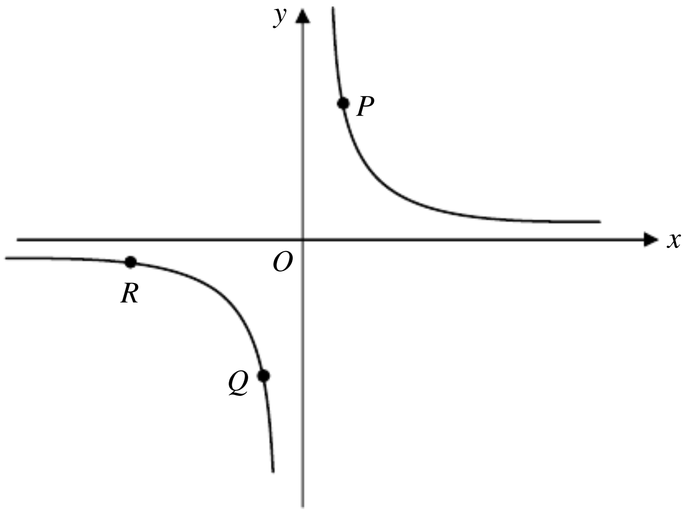
(i) Write down the amount of savings that Selina has in her account at the end of the n^{th} year. **[1]**

(ii) Show that the amount of savings that Hebe has in her account at the end of n^{th} year is \$$(1000 + 200(1.5^{n-5} - 1))$. **[2]**

(iii) At the end of the k^{th} year, Hebe saw that her savings finally exceeds Selina's savings for the first time. Find the value of k and the interest that Hebe receives in the k^{th} year. **[4]**

After k complete years, the interest that Hebe receives for subsequent years is 1/3 of the interest received in the previous year. Determine, showing your reasons clearly, the maximum value of Hebe's investment. **[3]**

14



A curve has parametric equations $x = ct$, $y = \frac{c}{t}$, where c is a positive constant. Three points, $P\left(cp, \frac{c}{p}\right)$, $Q\left(cq, \frac{c}{q}\right)$ and $R\left(cr, \frac{c}{r}\right)$ on the curve are shown in the diagram.

(a) Show that the equation of the tangent at the point $P\left(cp, \frac{c}{p}\right)$ to the curve is given by $x + p^2 y - 2cp = 0$. **[2]**

(i) This tangent meets the x-axis and the y-axis at A and B respectively. Write down the coordinates of A and B. Prove that the area of triangle AOB is $2c^2$. **[3]**

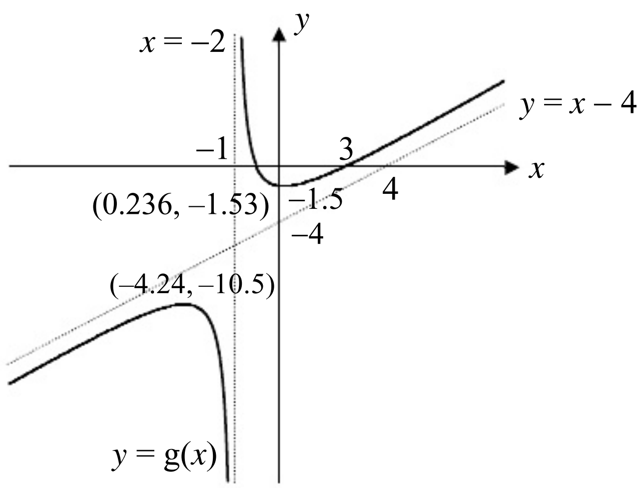
(ii) Given that the line PO meets the curve again at Q and the straight line BQ meets the x-axis at L. Express q in terms of p and find the area of the triangle QOL in terms of c. **[4]**

(b) If the normal at $R\left(cr, \frac{c}{r}\right)$ meets the curve again at $S\left(cs, \frac{c}{s}\right)$, show that $r^3 s + 1 = 0$. **[3]**

END

2007 JC2 Preliminary Examination
H2 Mathematics (9740/01)
Paper 1
Solutions

1	From GC, Solution Set $= (-1.96, -0.114) \cup (1, \infty)$. $\dfrac{e^{x^2}}{\lvert x\rvert - 1} - \lvert 8x \rvert > 0 \Rightarrow \dfrac{e^{\lvert x\rvert^2}}{\lvert x\rvert - 1} - 8\lvert x \rvert > 0$ Hence, $-1.96 < \lvert x\rvert < -0.114$ (reject) or $\lvert x\rvert > 1$ $\therefore$ Solution Set for $x = (-\infty, -1) \cup (1, \infty)$
2	Let P_n be the statement "$n^2-(n-1)^2+(n-2)^2-(n-3)^2+...+(-1)^{n-1}(1)^2=\frac{n}{2}(n+1)$, for $n\in\mathbf{Z}^+$." When $n=1$, $\text{L.H.S}=(-1)^0(1)^2=1,\quad \text{R.H.S}=\frac{1}{2}(1+1)=1 \Rightarrow \text{L.H.S}=\text{R.H.S}$ $\therefore P_1$ is true. Assume that P_k is true for some $k\in\mathbf{Z}^+$, i.e. $k^2-(k-1)^2+(k-2)^2-(k-3)^2+...+(-1)^{k-1}(1)^2=\frac{k}{2}(k+1)$. To prove that P_{k+1} is true. i.e. $(k+1)^2-k^2+(k-1)^2-(k-2)^2+...+(-1)^{(k+1)-1}(1)^2=\frac{(k+1)}{2}[(k+1)+1]$, i.e. $(k+1)^2-k^2+(k-1)^2-(k-2)^2+...+(-1)^{k}(1)^2=\frac{1}{2}(k+1)(k+2)$. $\text{L.H.S}=(k+1)^2-k^2+(k-1)^2-(k-2)^2+...+(-1)^k(1)^2$ $=(k+1)^2-[k^2-(k-1)^2+(k-2)^2-...-(-1)^{k-1}(1)^2]$ $=(k+1)^2-\frac{k}{2}(k+1)$ $=\frac{(k+1)}{2}[2(k+1)-k]$ $=\frac{1}{2}(k+1)(k+2)=\text{R.H.S.}$ $\therefore P_k$ is true $\Rightarrow P_{k+1}$ is true. Since P_1 is true, by mathematical induction, P_n is true for all $n\in\mathbf{Z}^+$.

3(i)	$x_2 = -0.586$ and $x_3 = -0.820$ (to 3 s.f.)
(ii)	As $n \to \infty$, $x_n \to \ell$, $x_{n+1} \to \ell$ $\ell = \dfrac{2-3\ell}{\ell-4}$ $\ell(\ell-4) = 2-3\ell$ $\ell^2 - \ell - 2 = 0$ $(\ell+1)(\ell-2) = 0$ $\ell = -1$ or $\ell = 2$ (reject $\because$ ℓ is negative)
4(i)	Before C, we have $y = \mathrm{g}(2x)$. Before B, we have $y = \mathrm{g}(-2x)$. Before A, we have $y = \mathrm{g}[-2(x-1)] = \mathrm{g}(2-2x) = \mathrm{f}(x)$, i.e. $a = -2$, $b = 2$. □ **Alternatively,** After A, we have $y = \mathrm{f}(x+1)$. After B, we have $y = \mathrm{f}(-x+1)$. After C, we have $y = \mathrm{f}(-\frac{x}{2}+1) = \mathrm{g}(x)$ $\Rightarrow \mathrm{g}(-2x) = \mathrm{f}(x+1) \Rightarrow \mathrm{g}[-2(x-1)] = \mathrm{f}(x)$, i.e. $a = -2$, $b = 2$. □
(ii)	$y = \mathrm{g}(x) = \dfrac{x^2-2x-3}{x+2} = \dfrac{(x-3)(x+1)}{x+2} = x - 4 + \dfrac{5}{x+2}$. 

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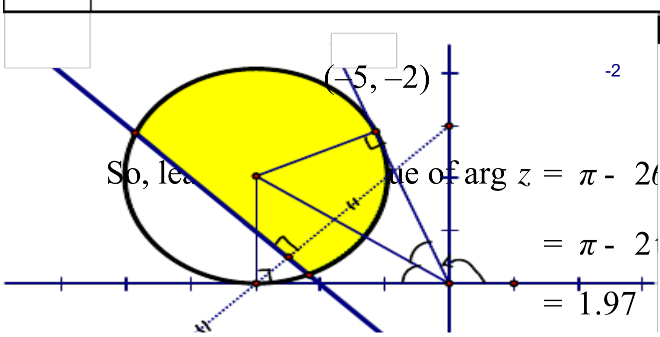
5(i)	$\mathrm{f}(r)-\mathrm{f}(r+1)=\dfrac{1}{r}-\dfrac{1}{r+1}=\dfrac{1}{r(r+1)}$. $\dfrac{1}{2}+\dfrac{1}{6}+\dfrac{1}{12}+\dfrac{1}{20}+\ldots 2n^{\text{th}}$ term $=\displaystyle\sum_{r=1}^{2n}\frac{1}{r(r+1)}$ $=\displaystyle\sum_{r=1}^{2n}[\mathrm{f}(r)-\mathrm{f}(r+1)]$ $=[\mathrm{f}(1)-\mathrm{f}(2)]+[\mathrm{f}(2)-\mathrm{f}(3)]+[\mathrm{f}(3)-\mathrm{f}(4)]+\ldots+[\mathrm{f}(2n)-\mathrm{f}(2n+1)]$ $=\mathrm{f}(1)-\mathrm{f}(2n+1)$ $=1-\dfrac{1}{2n+1}$ Consider $\dfrac{1}{(r+1)^2}=\dfrac{1}{(r+1)(r+1)}<\dfrac{1}{r(r+1)}$
	Hence $\dfrac{1}{2^2}+\dfrac{1}{3^2}+\dfrac{1}{4^2}+\dfrac{1}{5^2}+\ldots 2n^{\text{th}}$ term $=\displaystyle\sum_{r=1}^{2n}\frac{1}{(r+1)^2}<\sum_{r=1}^{2n}\frac{1}{r(r+1)}=1-\frac{1}{2n+1}<1$
6	$\lvert z+3-2\mathrm{i}\rvert=2$ $A(-3, 2)$ 3 2 θ θ Least $\arg z$ -3 O



$\lvert z+5+2\mathrm{i}\rvert=\lvert z-3\mathrm{i}\rvert$

So, least value of $\arg z=\pi-2\theta$

$=\pi-2\ldots$

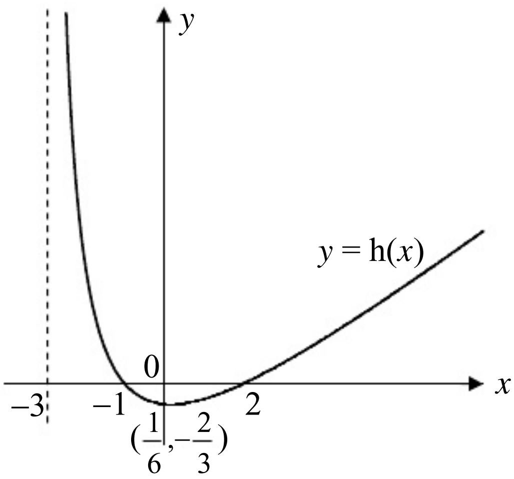
$=1.97$

7(i)

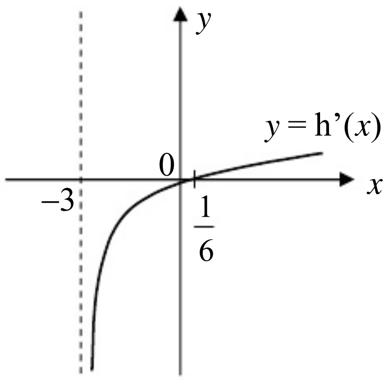
In the graph of $y = |\mathrm{h}(x)|$, there is a graph for $-1 < x < 2$, indicating that $\mathrm{h}(x) > 0$ or $\mathrm{h}(x) < 0$ for $-1 < x < 2$.

In the graph of $y^2 = \mathrm{h}(x)$, there is no graph for $-1 < x < 2$, indicating that $\mathrm{h}(x) < 0$ for $-1 < x < 2$.

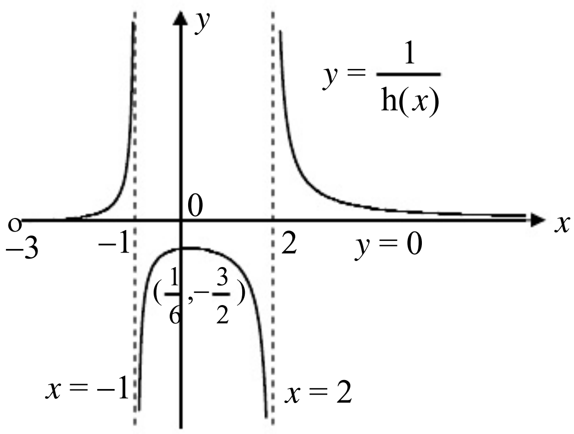
Hence, $\mathrm{h}(x) < 0$ for $-1 < x < 2$. □

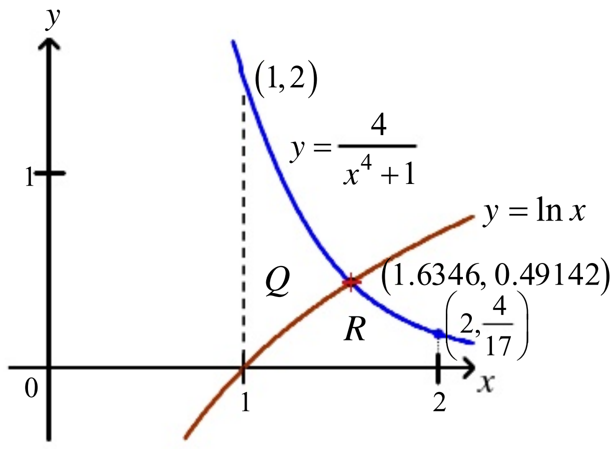


(ii)(a)



(ii)(b)



8(i)	$\displaystyle\int \tan^{-1}\left(\frac{1}{x}\right)\,\mathrm{d}x = x\tan^{-1}\left(\frac{1}{x}\right) - \int x\,.\,\frac{1}{1+\left(\frac{1}{x}\right)^2}\left(-\frac{1}{x^2}\right)\mathrm{d}x$ $\displaystyle = x\tan^{-1}\left(\frac{1}{x}\right) - \int \frac{-x}{x^2+1}\,\mathrm{d}x$ $\displaystyle = x\tan^{-1}\left(\frac{1}{x}\right) + \frac{1}{2}\ln\left\|x^2+1\right\| + c$
(ii)	$\displaystyle\int \frac{1}{x\sqrt{2x+1}}\,\mathrm{d}x$ $\displaystyle = \int \frac{1}{\frac{u^2-1}{2}\times u}\times u\,\mathrm{d}u$ $\displaystyle = 2\int \frac{1}{u^2-1}\,\mathrm{d}u$ $\displaystyle = (2)\left(\frac{1}{2}\ln\left\|\frac{u-1}{u+1}\right\|\right)+c$ $\displaystyle = \ln\left\|\frac{\sqrt{2x+1}-1}{\sqrt{2x+1}+1}\right\|+c$ Let $u=\sqrt{2x+1}$ $u^2=2x+1$ $\displaystyle 2u\frac{\mathrm{d}u}{\mathrm{d}x}=2$ $\displaystyle \frac{\mathrm{d}x}{\mathrm{d}u}=u$
9(i)	 Area of R $\displaystyle = \int_1^{1.6346}\ln x\,\mathrm{d}x + \int_{1.6346}^{2}\frac{4}{x^4+1}\,\mathrm{d}x$ $= 0.16864 + 0.12607$ $= 0.295\text{units}^2$

9(ii)	Volume of solid formed when Q is rotated completely about y-axis $=\pi\int_0^{0.49142}\left(e^y\right)^2 dy+\pi\int_{0.49142}^{2}\sqrt{\frac{4}{y}-1}\,dy-\pi(1)^2(2)$ $=\pi(0.83602+2.3961-2)$ $=1.2321\pi$ $=3.87$ units^3
10(i)	Area of semicircle $=\frac{1}{2}\pi r^2$ Length of rectangle $=\frac{P-\pi r-2r}{2}$ Area of rectangle $=(2r)\left(\frac{P-\pi r-2r}{2}\right)=r(P-\pi r-2r)$ $X=(L)\left[r(P-\pi r-2r)\right]+\left(\frac{1}{3}L\right)\left(\frac{1}{2}\pi r^2\right)$ $=Lr\left[P-r\left(\frac{5}{6}\pi+2\right)\right]$ (AG)
(ii)	$\frac{dX}{dr}=L\left[-r\left(\frac{5}{6}\pi+2\right)+P-r\left(\frac{5}{6}\pi+2\right)\right]$ $=L\left[P-r\left(\frac{5}{3}\pi+4\right)\right]$ Let $\frac{dX}{dr}=0$, $P-r\left(\frac{5}{3}\pi+4\right)=0$ $r=\frac{P}{\left(\frac{5}{3}\pi+4\right)}=\frac{3P}{(5\pi+12)}$ $\frac{d^2X}{dr^2}=(-L)\left(\frac{5}{3}\pi+4\right)<0$ $\therefore$ X is maximum when $r=\frac{3P}{(5\pi+12)}$ (Shown) (AG)
11(i)	$\int_0^x \frac{1}{\sqrt{1-4x^2}}dx=\frac{1}{2}\left[\sin^{-1}(2x)\right]_0^x=\frac{1}{2}\sin^{-1}(2x)$

11(ii)	$\left(1-4x^2\right)^{-\frac{1}{2}}$ $=1+\left(-\frac{1}{2}\right)\left(-4x^2\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}\left(-4x^2\right)^2+\ldots$ $=1+2x^2+\frac{3}{8}\left(16x^4\right)+\ldots$ $=1+2x^2+6x^4+\ldots$ Expansion is valid for $\left\|4x^2\right\|<1 \Rightarrow x^2<\frac{1}{4} \Rightarrow \frac{-1}{2}<x<\frac{1}{2}$ Set of values of x for which expansion is valid = $\left(-\frac{1}{2},\frac{1}{2}\right)$
(iii)	$\frac{1}{2}\sin^{-1}(2x)=\int_0^x \frac{1}{\sqrt{1-4x^2}}\,\mathrm{d}x$ $\sin^{-1}(2x)=2\int_0^x 1+2x^2+6x^4+\ldots\,\mathrm{d}x$ $\therefore \sin^{-1}(2x)=2\left[x+\frac{2x^3}{3}+\frac{6x^5}{5}+\ldots\right]_0^x$ $=2x+\frac{4x^3}{3}+\frac{12x^5}{5}+\ldots$ (proved)
(iv)	Let $x=\frac{1}{4}$. $\sin^{-1}\left(\frac{1}{2}\right)\approx 2\left(\frac{1}{4}\right)+\frac{4}{3}\left(\frac{1}{4}\right)^3+\frac{12}{5}\left(\frac{1}{4}\right)^5$ $\frac{\pi}{6}\approx\frac{1}{2}+\frac{1}{48}+\frac{3}{1280}$ $\frac{\pi}{6}\approx\frac{2009}{3840}$
12(i)	Equation of l_1 is $\mathbf{r}=\begin{pmatrix}1\\2\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\-1\end{pmatrix}$, $\lambda\in\mathbb{R}$ Equation of l_2 is $\mathbf{r}=\begin{pmatrix}-2\\1\\7\end{pmatrix}+\mu\begin{pmatrix}-3\\1\\5\end{pmatrix}$, $\mu\in\mathbb{R}$

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12(ii)

If $\quad 1+\lambda=-2-3\mu \quad$ ----(1)

and $\quad 2-\lambda=1+\mu \quad$ ----(2)

and $\quad -\lambda=7+5\mu \quad$ ----(3),

we have from (1) & (2), $\lambda=3$ and $\mu=-2$ which satisfies (3) since

R.H.S of (3) = 7 + 5(− 2) = − 3 = L.H.S of (3) .

$\therefore l_1$ and l_2 intersect and the coordinates of E is (4, –1, –3).

(iii)

$$\cos\theta=\frac{\left|\begin{pmatrix}-3\\1\\5\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\-1\end{pmatrix}\right|}{\sqrt{35}\sqrt{3}}=\frac{9}{\sqrt{35}\sqrt{3}}=\frac{3\sqrt{3}}{\sqrt{35}}$$

$$\therefore \sin\theta=\sqrt{1-\cos^2\theta}=\sqrt{1-\frac{27}{35}}=\frac{2\sqrt{2}}{\sqrt{35}}$$

$\sin\theta=\dfrac{p}{AE}$, where p is the shortest dist from A to l_1 and $AE=\left|\begin{pmatrix}-6\\2\\10\end{pmatrix}\right|$

$$\therefore p=\frac{2\sqrt{2}}{\sqrt{35}}\times\sqrt{140}=4\sqrt{2}$$

13

Let S_n and H_n denote the amount of savings that Selina and Hebe have at the end of the n^{th} year respectively.

(i)

$S_n=1000+200n$

(ii)

$H_n=1000+100+100(1.5)+100(1.5)^2+...+100(1.5)^{n-6}$ where $n>5$

$$=1000+\frac{100\left(1.5^{n-5}-1\right)}{1.5-1}$$

$$=1000+200\left(1.5^{n-5}-1\right)$$

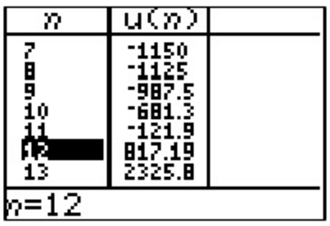
(iii)

When $H_n>S_n$,

$1000+200\left(1.5^{n-5}-1\right)>1000+200n$

i.e. $200\left(1.5^{n-5}-1-n\right)>0$

From the GC, the value of k is 12.



	Let u_k denote the interest that Hebe gets in the k^{th} year. Then $u_{12}=100\left(1.5^{12-5-1}\right)=1139.0625$, so Hebe receives \$1140 in interest at the end of the 12^{th} year. Then for $n>12$, $u_n=\left(\frac{1}{3}\right)^{n-12}u_{12}$. The total investment yield from the 13^{th} year onwards therefore cannot exceed $\frac{\frac{1}{3}u_{12}}{1-\frac{1}{3}}=\frac{1}{2}u_{12}$. Hence, the maximum value of Hebe's investment is $H_{12}+\frac{1}{2}u_{12}=\$4790$ (to 3 s.f.).
14(a)	$x=ct,\ y=\frac{c}{t}$, where $c>0$ $\frac{\mathrm{d}x}{\mathrm{d}t}=c,\ \frac{\mathrm{d}y}{\mathrm{d}t}=\frac{-c}{t^2}$ $\therefore \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-1}{t^2}$ At $P\left(cp,\frac{c}{p}\right)$, $t=p$, $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-1}{p^2}$ Equation of tangent at the point $t=p$ is $y-\frac{c}{p}=\frac{-1}{p^2}(x-cp)$ $p^2y-cp=-x+cp$ $x+p^2y-2cp=0$ (proved)
(a)(i)	When the tangent meets the x-axis at A: coordinates of A is $(2cp, 0)$ When the tangent meets the y-axis at B: coordinates of B is $\left(0,\frac{2c}{p}\right)$. Area of triangle $AOB=\frac{1}{2}\times\left(\frac{2c}{p}\right)\times(2cp)=2c^2$

(ii)	$q=-p$ Coordinates of $Q=\left(-cp\,,\dfrac{-c}{p}\right)$ Gradient of line BQ is $\dfrac{\frac{2c}{p}+\frac{c}{p}}{cp}=\dfrac{3}{p^2}$ Equation of line BQ is $y=\dfrac{3}{p^2}x+\dfrac{2c}{p}$ At L, $y=0$, coordinates of L is $\left(\dfrac{-2cp}{3},0\right)$ Area of triangle QOL is $=\dfrac{1}{2}\times\left(\dfrac{2cp}{3}\right)\times\left(\dfrac{c}{p}\right)=\dfrac{c^2}{3}$
(b)	Gradient of tangent at R is $\dfrac{-1}{r^2}$. Gradient of normal at $R=\ r^2$ Equation of normal at R is $y-\dfrac{c}{r}=r^2(x-cr)$, i.e. $ry-c=r^3(x-cr)$ ----(1) When the curve meets the curve again at $S\left(cs\,,\dfrac{c}{s}\right)$, substitute $x=cs$ and $y=\dfrac{c}{s}$ into equation (1), $\dfrac{rc}{s}-c=r^3(cs-cr)$ $r-s=r^3s(s-r)$ $-1=\ r^3s$ $\therefore r^3s+1=0$ (proved) **Alternative solution to (b)** Gradient of normal at R is r^2. Gradient of the line $RS=\dfrac{\frac{c}{s}-\frac{c}{r}}{cs-cr}=\dfrac{\frac{cr-cs}{sr}}{cs-cr}=-\dfrac{1}{sr}$ Since $r^2=-\dfrac{1}{sr}$, therefore $r^3s=-1\ \Rightarrow\ r^3s+1=0$

❄❄❄❄❄❄❄❄ **End** ❄❄❄❄❄❄❄❄

RAFFLES JUNIOR COLLEGE

JC2 Preliminary Examination 2007

MATHEMATICS
Higher 2

9740/02

Paper 2

19 September 2007

3 hours

Additional materials : Answer Paper
List of Formulae (MF15)

READ THESE INSTRUCTIONS FIRST

Write your name and CT group on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer all the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
At the end of the examination, fasten all your work securely together.

This document consists of 6 printed pages.

Section A: Pure Mathematics [40 marks]

1 **(a)** The complex numbers z and w are such that $z = \dfrac{3a-5\mathrm{i}}{1+2\mathrm{i}}$ and $w = 1+13b\mathrm{i}$, where a and b are real. Given that $z^* = w$, find the exact values of a and b. **[3]**

(b) Show that

$$\left(w^n - \mathrm{e}^{\mathrm{i}\theta}\right)\left(w^n - \mathrm{e}^{-\mathrm{i}\theta}\right) = w^{2n} + a\cos\theta\, w^n + b,$$

where a and b are real numbers to be determined.

Hence find the roots of the equation

$$z^6 - \sqrt{3}\,z^3 + 1 = 0$$

in the form of $r\mathrm{e}^{\mathrm{i}\theta}$. **[6]**

2 The following system of linear equations is given:

$$\begin{aligned} 2x-2y+z &= -4 \quad \text{-----------(1)} \\ 2x+3y-4z &= 1 \quad \text{-----------(2)} \\ 4x-3y+z &= 2 \quad \text{-----------(3)} \end{aligned}$$

(a) By solving the system of linear equations, comment on the solution of this set of equations and the geometrical representation of the equations. **[3]**

(b) The Cartesian equations of planes π_1 and π_2 are given by equation (1) and (2) respectively. Find a vector perpendicular to both the normals of π_1 and π_2. **[2]**

Hence, find

(i) a vector equation of l, the line of intersection of π_1 and π_2. **[2]**

(ii) in the form $\mathbf{r}.\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 1$, the equation of the plane π which passes through the point $\left(1,\ -1,\ \dfrac{2}{3}\right)$ and is perpendicular to both π_1 and π_2. **[2]**

3 The functions f and g are defined by

$$\mathrm{f}: x \to -2 + \frac{1}{x^2 + 2x + 3}, \quad x \in \mathbb{R},$$

$$\mathrm{g}: x \to \mathrm{e}^x, \qquad x \in \mathbb{R},\ x \le 0.$$

(i) Sketch the graph of $y = \mathrm{f}(x)$, stating the equations of any asymptotes, the coordinates of any stationary points and any intersection with the axes. **[2]**

(ii) Show that the composite function fg exists. Express fg in a similar form and state its range, giving your answer exactly. **[4]**

(iii) Show that f does not have an inverse. The function f has an inverse if its domain is restricted to $x \le k$. Find the largest possible value of k and define the inverse function f^{-1} corresponding to this domain for f. **[5]**

4 Water is being pumped into a filtration device at a constant rate of 5 litres per minute. The filtration device processes the water and discharges it at a rate proportional to the volume of water currently in it. At time t minutes, the volume of water in the device is v litres. The rate of discharge is always less than 5 litres per minute.

(i) Write down a differential equation representing the above model involving $\frac{\mathrm{d}v}{\mathrm{d}t}$ and v. **[1]**

(ii) By solving the differential equation in (i), show that the general solution is $v = \frac{1}{k}\left(5 - A\mathrm{e}^{-kt}\right)$ where A and k are positive constants. **[3]**

(iii) Sketch the family of solution curves representing the variation of v with respect to time in this context. **[3]**

The filtration device is initially empty. When $t = 5$, the volume of water in the device is 2 litres. Find the volume of water processed by the device when $t = 1$. **[4]**

Section B: Statistics [60 marks]

5 Find the number of different ways in which 'CONNOISSEUR' can be arranged, if

(i) there are no restrictions, **[2]**

(ii) a vowel must be separated from another with exactly one consonant. **[3]**

6 Alfred and Ben are involved in a duel. The rules of the duel state that they are to pick up a pie and throw at each other simultaneously. If one or both are hit, then the duel is over. If both miss, then they repeat the process. Suppose that the results of the throws are independent and that each throw of Alfred hitting Ben has probability p_A, and each throw of Ben hitting Alfred has probability p_B.

Find the probability that

(a) Alfred is not hit after the first round, **[1]**

(b) both duelists are hit after the first round, **[1]**

(c) the duel ends at the n^{th} round. **[3]**

7 A medical statistician wishes to carry out a test to see if there is any correlation between the head circumference and body length of newly-born babies. A random sample of 10 newly-born babies have their head circumference, c cm, and body length, l cm measured. This bivariate sample is illustrated in the table below.

c	31	32	33.5	34	34	51	35	36	36.5	37.5
l	45	49	49	47	50	34	50	53	51	51

One particular data point has been recorded incorrectly with its values of c and l interchanged. Identify the point. **[1]**

(i) Make the necessary correction and use a suitable regression line to estimate the length of the baby whose head has the circumference of

(a) 34.5 cm,

(b) 45 cm. **[3]**

(ii) Give a reason why the estimation found in **(i)** part **(b)** may not be a good one. **[1]**

8 As part of the Christmas promotion, Bishan Gift Shop is offering lucky dips to customers. A customer can choose to dip for either a small or a big prize. On average, a customer who dips for a small prize has a one in five chance of winning and one who dips for a big prize has a one in ten chance of winning.

In a particular day, 80 customers dipped for a small prize and 60 customers dipped for a big prize. Find the approximate distribution of the total number of prizes won on that day and hence find an approximate probability that at most 20 prizes were won on that day. **[6]**

9 The student population of 2,500 in a junior college is made up of male and female students who are either in JC1 or JC2 as follows:

	JC1 Students	JC2 Students
Male	750	650
Female	550	550

A sample of 50 students is to be taken from the population to investigate the number of hours of sleep a student from this junior college has on average per night.

(a) State and describe in detail a suitable method of sampling. **[3]**

(b) The students are now listed in alphabetical order and they are numbered from 1 to 2500 consecutively. A number between 1 and 50, inclusive, is selected at random. The corresponding student and every 50^{th} student thereafter are included in this sample.

(i) Identify this method of sampling. **[1]**

(ii) Give an advantage of using the method of sampling you have described in **(a)** rather than this method of sampling. **[1]**

10 Defects are found in toys manufactured by Toy Company. An average of 20 defects is found per 1000 toys. If a toy contains one or more defects, it will be discarded.

(i) It is required to estimate the percentage of toys that have to be discarded. Consider the following statement: Since 1000 toys contain 20 defects, approximately 2% will have to be discarded. State, giving a reason, whether you agree or disagree with the statement. **[1]**

Find the probability that

(ii) a toy will be discarded; **[2]**

(iii) one out of three toys will be discarded; **[2]**

(iv) there are more than the average number of defects in 1000 toys. **[2]**

11 An aptitude test for deep-sea divers produces scores which are normally distributed on a scale from 0 to 100. A random sample of 160 divers were assessed, and each of their individual test scores, x, was recorded. The results are summarized by

$$\sum x = 9120, \qquad \sum \left(x - \bar{x}\right)^2 = 35775.$$

(i) Calculate unbiased estimates of the population mean and variance. **[2]**

(ii) Assuming that the unbiased estimate of the population mean obtained above is equal to the population mean, find the value of c such that there is a probability of 0.05 that a sample of 160 scores has a sample mean that differs from its mean score by more than c. **[3]**

(iii) Comment on the validity of the calculations if the scores had *not* been known to be normally distributed. **[2]**

12 A hospital sees a number of babies born. For each gender, it may be assumed that the weights of newborn babies are normally distributed, with average weights and standard deviations as given in the following table.

Gender	Average weight	Standard deviation
Boy	2.9 kg	0.6 kg
Girl	2.6 kg	0.4 kg

(a) For each of the following cases, state, with a reason, whether or not a normal model is likely to be appropriate.

(i) The weight of a baby chosen at random from all the babies born in the hospital. **[2]**

(ii) The total weight of a randomly chosen baby boy and two randomly chosen baby girls born in the hospital. **[2]**

(b) One baby boy and two baby girls are chosen at random from the hospital. Find the probability that the average weight of the two baby girls is at least 500 grams less than the weight of the baby boy. **[4]**

13 The widths of a sample of eight beetles of the species '*genus prometheus*', found in Bishan Street 21, were measured and found to be 10, 15, 11, a, 13, 14, b, and 17 mm, where a and b are both positive integers, and $a > b$. In addition, it is also known that the unbiased estimates for population mean and variance of this sample are given by $\bar{x} = 13$, and $s^2 = \frac{54}{7}$ respectively.

(i) Find the values of a and b. **[4]**

Previous extensive measurements of '*genus prometheus*' beetles had shown the width to be normally distributed with mean 11.5 mm.

(ii) Carry out an appropriate test at the 5% level to find out whether the beetles from Bishan Street 21 have a different mean width from the main population. **[4]**

(iii) Entomologists have now updated the population standard deviation of widths of the '*genus prometheus*' beetles to be 6 mm. A test is carried out to find out if the beetles from Bishan Street 21 have a larger width than beetles from the main population. Find the smallest level of significance (to 4 decimal places) which would result in rejection of the null hypothesis. **[4]**

END

2007 JC2 Preliminary Exam
H2 Mathematics Paper 2 (9740/02)
Solutions

Section A: Pure Mathematics (40 marks)

1(a)	Method 1 $z^* = w = 1 + 13bi$ $z = 1 - 13bi$ Given: $z = \dfrac{3a-5i}{1+2i}$ $(1 + 2i)\, z = 3a - 5i$ $(1 + 2i)(1 - 13bi) = 3a - 5i$ $(1 + 26b) + (2 - 13b)i = 3a - 5i$ Compare real and imaginary parts: $1 + 26b = 3a$ --- (1) and $2 - 13b = -5$ --- (2) From (2): $b = \dfrac{7}{13}$ # From (1): $a = \dfrac{1+26\left(\frac{7}{13}\right)}{3} = 5$ # Method 2 $z = \dfrac{3a-5i}{1+2i} \Rightarrow z^* = \dfrac{3a+5i}{1-2i}$ Since $z^* = w$ $\dfrac{3a+5i}{1-2i} = 1 + 13bi$ $3a+5i = (1-2i)(1+13bi)$ $3a+5i = (1+26b)+(13b-2)i$ Compare real and imaginary parts: $1 + 26b = 3a$ --- (1) and $13b - 2 = 5$ --- (2) From (2): $b = \dfrac{7}{13}$ # From (1): $a = \dfrac{1+26\left(\frac{7}{13}\right)}{3} = 5$ #

Method 3

$$z=\frac{3a-5\mathrm{i}}{1+2\mathrm{i}}=\frac{3a-5\mathrm{i}}{1+2\mathrm{i}}\times\frac{1-2\mathrm{i}}{1-2\mathrm{i}}=\frac{(3a-10)-(6a+5)\mathrm{i}}{5}\pi$$

Given: $z^*=w$

$$\frac{(3a-10)+(6a+5)\mathrm{i}}{5}=1+13b\mathrm{i}$$

$$(3a-10)+(6a+5)\mathrm{i}=5+65b\mathrm{i}$$

Equate the real and imaginary parts:

$3a-10=5$ --- (1) and $6a+5=65b$ --- (2)

From (1): $a=5_{\#}$

Sub into (2): $b=\frac{6(5)+5}{65}=\frac{7}{13}_{\#}$

(b)

$$\left(w^n-\mathrm{e}^{\mathrm{i}\theta}\right)\left(w^n-\mathrm{e}^{-\mathrm{i}\theta}\right)$$

$$=w^{2n}-w^n\left(\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{-\mathrm{i}\theta}\right)+1$$

$$=w^{2n}-w^n\left(\cos\theta+\mathrm{i}\sin\theta+\cos(-\theta)+\mathrm{i}\sin(-\theta)\right)+1 \text{ where } a=-2 \text{ and } b=1.$$

$$=w^{2n}-w^n\left(\cos\theta+\mathrm{i}\sin\theta+\cos\theta-\mathrm{i}\sin\theta\right)+1$$

$$=w^{2n}-2\cos\theta\, w^n+1 \quad \text{(shown)}$$

Hence,

$$z^6-\sqrt{3}z^3+1=0$$

$$z^6-2\left(\frac{\sqrt{3}}{2}\right)z^3+1=0$$

$$z^6-2\left(\cos\frac{\pi}{6}\right)z^3+1=0$$

$$\left(z^3-\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}\right)\left(z^3-\mathrm{e}^{-\mathrm{i}\frac{\pi}{6}}\right)=0$$

$$z^3=\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{6}+2k\pi\right)} \quad \text{or} \quad z^3=\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{6}+2k\pi\right)}$$

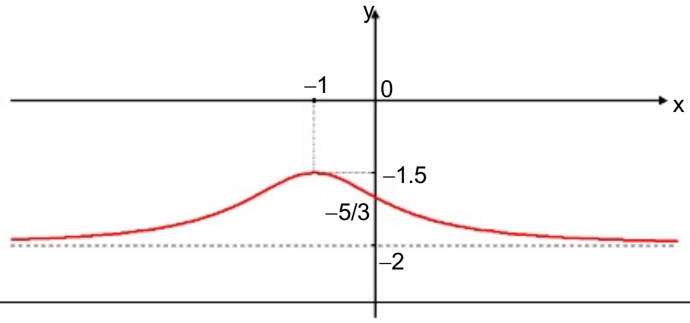
$$z=\mathrm{e}^{\mathrm{i}\left(\frac{12k+1}{18}\right)\pi} \quad \text{or} \quad z=\mathrm{e}^{\mathrm{i}\left(\frac{12k-1}{18}\right)\pi}, \quad k=0,\ 1,\ 2$$

The roots are $\mathrm{e}^{\frac{\pi}{18}\mathrm{i}}$, $\mathrm{e}^{\frac{13\pi}{18}\mathrm{i}}$, $\mathrm{e}^{\frac{25\pi}{18}\mathrm{i}}$, $\mathrm{e}^{-\frac{\pi}{18}\mathrm{i}}$, $\mathrm{e}^{\frac{11\pi}{18}\mathrm{i}}$, $\mathrm{e}^{\frac{23\pi}{18}\mathrm{i}}$.

i.e. $\mathrm{e}^{\pm\frac{\pi}{18}\mathrm{i}}$, $\mathrm{e}^{\pm\frac{11\pi}{18}\mathrm{i}}$ and $\mathrm{e}^{\pm\frac{13\pi}{18}\mathrm{i}}$.

2(a)	The augmented matrix $=\left(\begin{array}{ccc:c} 2 & -2 & 1 & -4 \\ 2 & 3 & -4 & 1 \\ 4 & -3 & 1 & 2 \end{array}\right)$ The RREF of the augmented matrix $=\left(\begin{array}{ccc:c} 1 & 0 & -0.5 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$ The final row of the RREF shows that $0x+0y+0z=1$, which implies that the system of equations is inconsistent. $\therefore$ The equations do not have a solution. The 3 planes represented by the 3 equations do not intersect in a point or line. Furthermore, since neither of the planes is parallel to any other ($\because$ neither of the normals is parallel to any other), the 3 planes form a triangular prism.
(b)	$\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \\ 10 \end{pmatrix}$ $\therefore$ a vector $\perp$ to both normals of π_1 and π_2 is $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$.
(i)	$2x-2y+z=-4$ ---------(1) $2x+3y-4z=1$ ---------(2) When $z=0$, $2x-2y=-4$ -------(4) and $2x+3y=1$ --------(5) $(5)-(4)$ gives $y=1$ and $x=-1$. $\therefore (-1,\ 1,\ 0)$ is a point on both π_1 and π_2. $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ is parallel to both π_1 and π_2. $\therefore$ a vector equation of l is $\mathbf{r} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix},\ \lambda \in \mathbb{R}$.
(ii)	A vector normal to π is $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$. A vector equation of π is $\mathbf{r} \cdot \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2/3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \frac{1}{3}$ $\therefore\ \mathbf{r} \cdot \begin{pmatrix} 3 \\ 6 \\ 6 \end{pmatrix} = 1$.

3(i)



(ii)

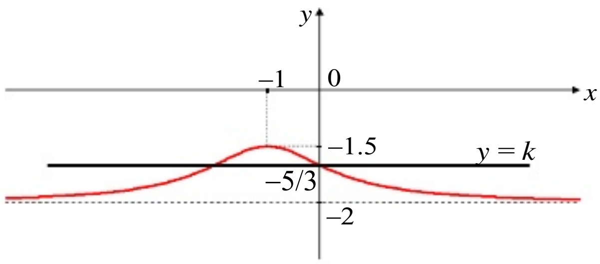
Range of g, $R_g = (0,1]$ and Domain of f, $D_f = (-\infty, \infty)$.
Since $R_g \subseteq D_f$, composite function fg exists.

$$fg(x) = f(e^x) = -2 + \frac{1}{e^{2x} + 2e^x + 3}, \quad x \in \mathbb{R}, x \le 0.$$

$$fg: x \mapsto -2 + \frac{1}{e^{2x} + 2e^x + 3_{\#}}, \quad x \in \mathbb{R},\ x \le 0_{\#}$$

Range of fg $= [-\frac{11}{6}, -\frac{5}{3})_{\#}$.

(iii)

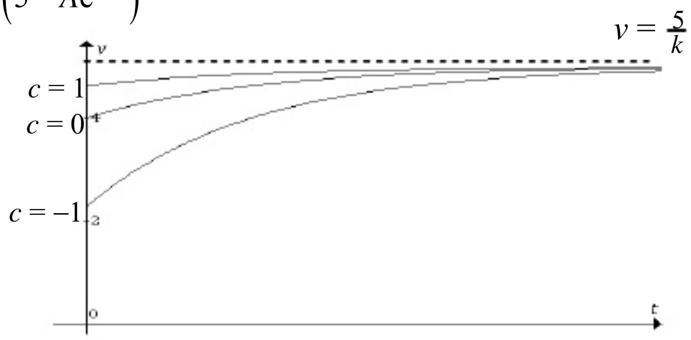


Since any horizontal line $y = k$ where $k \in (-2, -1.5)$ cuts the graph of $y = f(x)$ at more than 1 point, f is not one-one. Hence f does not have an inverse.

OR Since the horizontal line $y = -\frac{5}{3}$ (or any other value of y such that $y \in (-2, -1.5)$) cuts the graph of f at more than 1 point, f is not one-one. Hence f does not have an inverse.

OR $f(0) = -\frac{5}{3} = f(-2)$ but $-2 \neq 0$, hence f is not one-one and thus f does not have an inverse.

Largest possible value of $k = -1$ #

	Let $y = \mathrm{f}(x) = -2 + \dfrac{1}{x^2+2x+3}, \quad x \le -1$ $x^2+2x+3 = \dfrac{1}{y+2}$ $(x+1)^2 + 2 = \dfrac{1}{y+2}$ $x+1 = -\sqrt{\dfrac{1}{y+2} - 2}$ (since $x \le -1$) $x = -1 - \sqrt{\dfrac{1}{y+2} - 2}$ So $\mathrm{f}^{-1}(x) = -1 - \sqrt{\dfrac{1}{x+2} - 2}, \quad -2 < x \le -1.5_{\#}$
4(i)	$\dfrac{\mathrm{d}v}{\mathrm{d}t} = 5 - kv, \ k > 0$
(ii)	$\displaystyle\int \frac{1}{5-kv}\,\mathrm{d}v = \int 1\,\mathrm{d}t$ $\Rightarrow -\dfrac{1}{k}\ln\lvert 5-kv\rvert = t + c$ $\Rightarrow \ln\lvert 5-kv\rvert = -kt - kc$ $\Rightarrow 5 - kv = A\mathrm{e}^{-kt}$ (since $5 > kv$, $5 - kv > 0$ so $\lvert 5-kv\rvert = 5-kv$) $\Rightarrow v = \dfrac{1}{k}\left(5 - A\mathrm{e}^{-kt}\right)$
(iii)	 For the device, when $t = 0$, $v = 0$. Hence, $0 = \dfrac{1}{k}(5 - A) \Rightarrow A = 5$. $t = 5$, $v = 2$. So $2 = \dfrac{1}{k}\left(5 - 5\mathrm{e}^{-5k}\right) \Rightarrow k = 2.500$ (4 s.f., from GC) Amount of water in the device after the first minute $= \dfrac{1}{2.500}\left(5 - 5\mathrm{e}^{-2.500}\right) = 1.836$ litres Amount of water poured in after the first minute = 5 litres Hence, amount of water processed = 5 – 1.836 litres = 3.16 litres (3 s.f.)

Section B: Statistics (60 marks)

5(i) Number of ways $= \dfrac{11!}{(2!)^3} = 4989600$

(ii) Number of ways $= \dfrac{6!}{(2!)^2} \times \dfrac{5!}{2!} \times 3 = 32400$

6(a) P(Alfred is not hit) = $1 - p_B$

(b) P(both are hit) = $p_A \times p_B$

(c) P(either Alfred is hit or Ben is hit or both are hit)

$= p_A(1-p_A)^{n-1}(1-p_B)^n + p_B(1-p_A)^n(1-p_B)^{n-1} + (1-p_A)^{n-1}(1-p_B)^{n-1}p_Ap_B$

7

(i) The point is $c = 51$, $l = 34$

A suitable regression line is $l = 19.722 + 0.8698\,\mathrm{l}c$.

(a) When $c = 34.5$, $l = 49.7$.

(b) When $c = 45$, $l = 58.9$.

(ii) The extrapolation has gone beyond the interval of the provided data. Thereby, relationship between the variables may not follow the same pattern as the given set of data.

8 Let X denote the random variable representing the number of customers who won a small prize, out of 80.

$X \sim \mathrm{B}(80, 0.2)$

Since $n = 80$ is large, $np = 16 > 5$, $n(1-p) = 64 > 5$,

$X \sim \mathrm{N}(16, 12.8)$ approximately.

Let Y denote the random variable representing the number of customers who won a small prize, out of 60.

$Y \sim \mathrm{B}(60, 0.1)$

Since $n = 60$ is large, $np = 6 > 5$, $n(1-p) = 54 > 5$,

$Y \sim \mathrm{N}(6, 5.4)$ approximately.

$X + Y \sim \mathrm{N}(22, 18.2)$ approximately

$\mathrm{P}(X + Y \le 20) = \mathrm{P}(X + Y \le 20.5)$ by continuity correction

$= 0.363$

9(a) Stratified sampling.

	JC1 Students	JC2 Students
Male	15	13
Female	11	11

Within each strata, carry out random sampling, or systematic sampling to obtain the required numbers of students for the sample of 50.

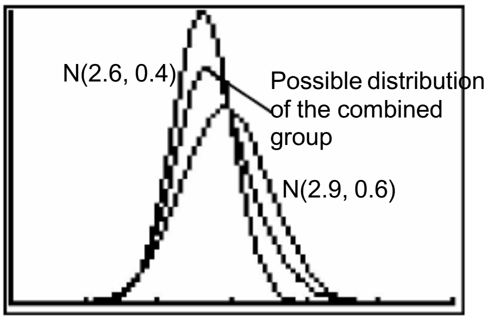
(b)(i) Systematic Sampling

(ii) Stratified sampling in (a) is able to give a good representative sample of population.

10(i)	Disagree as each toy may contain more than one defect.
(ii)	Let X denote the random variable representing the number of defects in a toy. $X \sim \text{Po}(0.02)$ Required probability = $\text{P}(X \geq 1) = 1 - \text{P}(X = 0) = 0.0198$ (to 3 s.f)
(iii)	Let Y denote the random variable representing the number of toys, out of 3, that will be discarded. $Y \sim B(3, 0.0198)$ Required probability = $\text{P}(Y = 1) = {}^3\text{C}_1\ \text{P}(Y \geq 1)\ \text{P}(Y = 0)^2 = 0.0571$ (to 3 s.f)
(iv)	Let W denote the random variable representing the number of defects in 1000 toys. $W \sim \text{Po}(20)$ $\text{P}(W > 20) = 1 - \text{P}(W \leq 20) = 0.441$(to 3 s.f)
11(i)	Unbiased estimate of the population mean μ, $\bar{x} = \dfrac{\sum x}{n} = \dfrac{9120}{160} = 57$. Unbiased estimate of the population variance σ^2, $s^2 = \dfrac{\sum\left(x - \bar{x}\right)^2}{n-1} = \dfrac{35775}{159} = 225$.
(ii)	Let $\overline{X}$ denote the random variable representing the sample mean aptitude test scores of 160 randomly chosen divers. Then $\overline{X} \sim \text{N}(\mu, \dfrac{\sigma^2}{160})$. Assuming $\mu = \bar{x} = 57$ (from **(i)**) and estimate σ^2 by s^2, we have $\overline{X} \sim \text{N}(57, \dfrac{225}{160})$ approximately. Now, $\text{P}\left(\left\|\overline{X} - 57\right\| > c\right) = 0.05$ $\Rightarrow \text{P}(\overline{X} - 57 < -c) + \text{P}(\overline{X} - 57 > c) = 0.05$ $\Rightarrow \text{P}(\overline{X} < 57 - c) + \text{P}(\overline{X} > 57 + c) = 0.05$ $\Rightarrow 2\text{P}(\overline{X} < 57 - c) = 0.05$ (by symmetry) $\Rightarrow \text{P}\left(\overline{X} < 57 - c\right) = \dfrac{0.05}{2} = 0.025$ $\Rightarrow 57 - c = 54.67576863$ (using invNorm(0.025,57,$\sqrt{225/160}$)) $\Rightarrow c = 57 - 54.67576863 = 2.324231373$ Hence, $c = 2.32$. (3 s.f.)
(iii)	If the scores had not been known to be normally distributed, we can still apply the Central Limit Theorem to establish the normality of $\overline{X}$ since $n = 160 \geq 50$ is large. Hence the calculations in **(ii)** would still be valid.

<table>
<tr><td>12(i)
(a)</td><td>Normal model inappropriate as weights of babies born belong to two different distributions.

OR
Normal model appropriate as the weights of babies belong to normal distributions with means and standard deviations such that there is a huge overlap of the probability density functions as shown in the graph. The required distribution would be uni-modal and symmetrical. A normal model would thus be appropriate for the weight of a randomly chosen baby.


</td></tr>
<tr><td>12(b)</td><td>Normal as the sum of independent normal distributions is normal</td></tr>
<tr><td>(ii)</td><td>Let G and B denote the random variables representing the weight of a baby girl and boy respectively.
Then $G \sim \mathrm{N}(2.6, 0.16)$ and $B \sim \mathrm{N}(2.9, 0.36)$
$B - \frac{G_1 + G_2}{2} \sim \mathrm{N}(0.3,\ 0.36 + \frac{0.16}{2})$
i.e. $B - \frac{G_1 + G_2}{2} \sim \mathrm{N}(0.3, 0.44)$
$\mathrm{P}\left(B - \frac{G_1 + G_2}{2} \geq 0.5\right) = 0.382$</td></tr>
<tr><td>13(i)</td><td>$\bar{x} = 13 \quad \Rightarrow \sum x = 13(8) = 104$
$\Rightarrow 10 + 15 + 11 + a + 13 + 14 + b + 17 = 104$
$\Rightarrow a + b = 24$.......................................(1)</td></tr>
</table>

	$s^2 = \frac{54}{7} \quad \Rightarrow \frac{\sum x^2 - \frac{1}{8}\left(\sum x\right)^2}{8-1} = \frac{54}{7}$ $\Rightarrow \sum x^2 - \frac{1}{8}\left(\sum x\right)^2 = 54$ $\Rightarrow \left\{10^2 + 11^2 + 15^2 + a^2 + 13^2 + 14^2 + b^2 + 17^2\right\} - \frac{(104)^2}{8} = 54$ $\Rightarrow a^2 + b^2 + 1100 - 1352 = 54$ $\Rightarrow a^2 + b^2 = 306$................................(2) Substituting (1) into (2) yields: $(24-b)^2 + b^2 = 306$ $\Rightarrow b^2 - 24b + 135 = 0$ $\Rightarrow (b-15)(b-9) = 0$ $\Rightarrow b = 15$ or $b = 9$ Since $a > b$, thus we have $a = 15$, $b = 9$.
(i)	Since the population width is normally distributed, we carry out a two-tailed t-test at 5% level of significance. Let X be the random variable representing the width, in mm, of a beetle of *'genus prometheus'* species. $H_0: \mu = 11.5$ $H_1: \mu \neq 11.5$ Since σ^2 is unknown, and n is small, we estimate σ^2 by s^2. Significance level is $\alpha = 0.05$. Under H_0, $T = \frac{\overline{X} - \mu_0}{S/\sqrt{n}} \sim t(7)$, where $\overline{X}$ is the sample mean, and s^2 is the unbiased estimate of the population variance, $\mu_0 = 11.5$, $s = 2.777$, $n = 8$, and $\overline{x} = 13$. Using a **T-test**, ***p*-value** = 0.170 (using GC) Conclusion: Since the p-value = 0.170 > 0.05, we do not reject H_0. There is sufficient evidence to support the claim $\mu \neq 11.5$ at the 5% significance level.
(iii)	With known population parameters, $X \sim N(11.5, 6^2)$. Then we have $\overline{X} \sim N(11.5, \frac{36}{8})$. $H_0: \mu = 11.5$ $H_1: \mu > 11.5$ Significance level is α (unknown).

	Under $H_0, \overline{X} \sim N\left(\mu_0, \dfrac{\sigma^2}{n}\right)$, where $\overline{X}$ is the sample mean and $\mu_0 = 11.5$, $\sigma = 6$, $n = 8$, and $\overline{x} = 13$. Using a **Z-test**, $z = \dfrac{\overline{x} - 11.5}{6/\sqrt{8}} = \dfrac{13 - 11.5}{6/\sqrt{8}} = 0.707$ (3 s.f.) ***p*-value** $= 0.23975$ (using GC) In order to reject H_0, p-value $= 0.23975 < \alpha$. Thus, the least value of $\alpha = 0.2398$ (4 d. p.)

❄❄❄❄❄❄❄❄ **End** ❄❄❄❄❄❄❄❄